Wave Equations on a Hyperplane*

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The Dirac equation, the Weaver-Hammer-Good wave equations, and the Weinberg wave equations are written in a manifestly covariant form in terms of hyperplane parameters according to Fleming's hyperplane formalism. A Fody-Wouthuysen-type wave equation is developed for the Weinberg theory and it, along with the usual Foldy-Wouthuysen wave equation and transformation, is also written in a manifestly covariant form for all spin. Fleming's formalism is extended to include the case where the hyperplane parameters are operators as well as c numbers. As a consequence, a hyperplane observer which corresponds to the particle rest frame is considered, with the result that wave equations are obtained in the usual manifestly covariant form for all spin with no auxiliary conditions or unphysical solutions.

I. INTRODUCTION

IN the past, one of the problems of theoretical physics has been to find a manifestly covariant theory for describing particles of arbitrary mass and spin. For the spin-$\frac{1}{2}$ case, the Dirac theory[1] is the most successful in that (a) the theory is manifestly covariant, (b) the wave functions have the necessary $2(2s+1)$ components to describe the particle state, and (c) minimal electromagnetic-field interactions are introduced by replacing the canonical momenta p_μ with $\pi_\mu = p_\mu - (e/c)A_\mu$. In developing a theory for arbitrary mass and spin, it is desirable that these conditions be satisfied. In this sense, no completely satisfactory theory for arbitrary mass and spin has been formulated as yet.

One of the earliest attempts to formulate a theory was made by Fierz, Pauli, and Dirac[2] in 1939. Their theory is manifestly covariant in spinor form, but is cumbersome in that the wave functions contain many more than the necessary $2(2s+1)$ components. Similar theories of Proca[3] and of Bargmann and Wigner[4] have the same difficulty. In order to remedy this, auxiliary conditions on the wave functions are imposed. In fact, Pursey[5] has shown that for a given mass and spin, an infinite number of different wave equations, together with auxiliary conditions when necessary, may be constructed. Although all of these formulations are equivalent for free particles, this equivalence is broken in the presence of interactions. Indeed, it is an open question whether any such system of equations is consistent when minimal electromagnetic coupling is introduced. For example, with p_μ replaced with π_μ in the theory of Fierz, Pauli, and Dirac, some of the polarization states are completely eliminated.[2] As a result, Fierz and Pauli were led to add more components to the wave functions in an arbitrary way.

Similarly, a later theory due to Weinberg[6] is manifestly covariant, but admits unphysical solutions if the Klein-Gordon equation is not required as an auxiliary condition on the wave function. As in the Fierz-Pauli-Dirac case, the equations become inconsistent when interactions are included.

More recently, Weaver, Hammer, and Good[7] (WHG) have formulated a theory in which the wave functions have no redundant components. However, this theory is not manifestly covariant, so it is not clear how interactions are to be included.

Toward this end, the recent hyperplane formalism introduced by Fleming[8] is promising, in that it provides a geometrical construct which generalizes the idea of manifest covariance. In this paper, the hyperplane formalism is generalized and used to derive two manifestly covariant equations without auxiliary conditions for free particles of arbitrary spin and mass. The first is based on the WHG theory, and is seen to be particularly simple for half-integral spin states. The second is based on the formulation of Weinberg, and is simple for integral spin. These equations are novel in that in the absence of interactions, they contain both massive and massless particle solutions.

In Sec. II, the notation and matrix representation used herein are given. In Sec. III, the basic ideas of Fleming's hyperplane formalism are developed, with special emphasis being given to the equivalence of spacelike hyperplanes and equivalent inertial frames. A Schrödinger equation is derived for the hyperplane system for $2(2s+1)$ component wave functions. In Sec. IV, the Dirac equation is written for hyperplane observers, and a Foldy-Wouthuysen type of transformation is derived. The same general procedure is followed in Secs. V and VI for deriving the two manifestly covariant equations for arbitrary spin and mass.

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II. NOTATION

A Euclidean metric is used, so that $x_\mu=(\mathbf{x},it)$, units being chosen such that c and $\hbar$ are unity. Latin indices run from one to three, Greek indices run from one to four, and the summation convention is used throughout.

The matrices used are the $2(2s+1)$-dimensional matrices defined by

$$\boldsymbol{\alpha}=\frac{1}{s}\begin{pmatrix}\mathbf{s} & 0\\ 0 & -\mathbf{s}\end{pmatrix},\quad \mathbf{S}=\begin{pmatrix}\mathbf{s} & 0\\ 0 & \mathbf{s}\end{pmatrix},\quad \beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \gamma_5=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \tag{1}$$

where $\mathbf{s}$ is the usual $(2s+1)$-dimensional spin matrix discussed by Schiff.[9] The covariantly defined 4-vector γ_μ is given by $\gamma_\mu=(-i\beta\boldsymbol{\alpha},\beta)$. The set of covariantly defined spin operators $s_{\mu\nu}$ is defined by

$$s_{ij}=\epsilon_{ijk}s_k\,,\quad s_{i4}=-s_{4i}=s\alpha_i\,,\quad s_{44}=0\,, \tag{2}$$

where $\sigma_{\mu\nu}=2s_{\mu\nu}$ for spin $\frac{1}{2}$.

The generalized $2(2s+1)$-dimensional Dirac matrices studied by Barut, Muzinich, and Williams[10] and by Weinberg[6] are used, where

$$\gamma_{[\mu]}\equiv\gamma_{\mu_1\mu_2\cdots\mu_{2s}}\,, \tag{3}$$

with $\gamma_{[4]}\equiv\beta$. The notation $a_{[\mu\nu]}$ for the matrix elements $a_{\mu\nu}$ of the Lorentz transformation A is defined by

$$a_{[\mu\nu]}\equiv a_{\mu_1\nu_1}a_{\mu_2\nu_2}\cdots a_{\mu_{2s}\nu_{2s}}\,. \tag{4}$$

This notation is extended to other indexed quantities $\Theta_{\alpha\beta\cdots\gamma}$ by

$$\Theta_{[\alpha\beta\cdots\gamma]}\equiv\Theta_{\alpha_1\beta_1\cdots\gamma_1}\Theta_{\alpha_2\beta_2\cdots\gamma_2}\cdots\Theta_{\alpha_{2s}\beta_{2s}\cdots\gamma_{2s}}\,. \tag{5}$$

III. HYPERPLANE FORMALISM

One of the basic postulates of physics is that measurements made on a physical system with respect to equivalent inertial frames are equivalent. By equivalent inertial frames one means Lorentz frames which are related by Lorentz transformations continuous with the identity, i.e., transformations belonging to the proper orthochronous Lorentz group. Frequently, one thinks of an observer being associated with each Lorentz frame. Entities, such as energy or momentum, used by an observer to describe a system relate both to the system and to the observer's inertial frame. Thus what one observer would interpret as the energy of the system would not be so interpreted by a different observer, but instead would be related through a Lorentz transformation to both the energy and momentum of the system in the second observer's interpretation.

Perhaps the most significant aspect of Fleming's hyperplane formalism is that it gives us a manifestly covariant way of describing an arbitrary observer's interpretation of physical quantities. Fleming introduces families of hyperplanes defined by

$$\eta_\mu x_\mu=-\tau\,,\quad \eta_\mu\eta_\mu=-1\,.$$

The family is specified by the unit normal η_μ, and the parameter τ defines a particular member of the family. An "observer" is associated with each family of hyperplanes—such an observer will interpret τ as his time. At this point, it helps in maintaining clarity to conceive of a godlike "superobserver" who simultaneously sees both the system under study and all the observers associated with hyperplane families. The component labels in vectors and tensors, such as in η_μ, p_μ, x_μ, etc., refer to the superobserver's frame. Thus operators such as p_μ relate the observed system to the superobserver, while η_μ relates one particular observer to the superobserver. If, as is the case with Fleming, the observer frame is defined in some way independent of the system under study, then η_ν and τ will be c numbers. However, in this paper, the properties of the system itself will also be used to define an observer frame analogous to the "instantaneous rest frame" of unquantized relativistic mechanics, in which case η_ν is an operator.

Simultaneous Lorentz transformations both of the system observed and of all observers relative to the superobserver can now be considered. (Here an active interpretation in which events and observers are transformed is preferred to a passive interpretation in which the superobserver's frame changes.) Thus, if x_μ are space-time coordinates of a point on the world line of a particle, the Lorentz transformation A is defined by

$$A:\ x_\mu\rightarrow x_\mu'=a_{\mu\nu}x_\nu\,, \tag{6}$$

where the elements $a_{\mu\nu}$ of the orthogonal matrix A satisfy

$$a_{\lambda\mu}a_{\lambda\nu}=a_{\nu\lambda}a_{\mu\lambda}=\delta_{\nu\mu}\,.$$

Similarly, since observers as well as events are to be transformed,

$$\eta_\mu\rightarrow\eta_\mu'=a_{\mu\nu}\eta_\nu\,.$$

Of particular interest is the Lorentz transformation without rotation which carries the "instantaneous" hyperplanes with normal $\eta_\mu{}^0=(\mathbf{0},i)$ into a general hyperplane with normal η_μ. (The observer associated with the family of instantaneous hyperplanes is that observer whose inertial frame coincides with that of the superobserver.) If this transformation is denoted by $\tilde{A}$, with matrix elements $\tilde{a}_{\mu\nu}$, then it is easily seen in particular that

$$\tilde{a}_{\mu4}=-i\eta_\mu\,. \tag{7}$$

This special Lorentz transformation will become important in what follows.

[9] L. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., Inc., New York, 1955).

[10] A. O. Barut, I. Muzinich, and D. N. Williams, Phys. Rev. **130**, 442 (1963).

This section is concerned with the transformation properties of operators acting on the state space of some physical system, both as seen by the superobserver and as seen by an observer associated with the hyperplane family (η_μ,τ). The systems considered are those described by relativistic wave equations such as the Dirac equation, which correspond to particles of definite mass and spin, and with either sign of the energy. To avoid the need for subsidiary conditions and still have a manifestly covariant theory, the wave function is taken to transform like the representation $(s,0)+(0,s)$ of the Lorentz group. A state space, with an appropriate scalar product, can be defined in terms of such wave functions, and a Lorentz transformation A may then be represented by an operator acting on states of this space. In particular, the operator corresponding to the particular Lorentz transformation $\tilde{A}$ is $\Lambda(\eta)$.

The operators representing the generators of the Poincaré group acting on wave functions of the state space are represented by p_μ and $M_{\mu\nu}$. The physical meaning of p_μ is the energy-momentum 4-vector, and of $M_{\mu\nu}$ the angular momentum and the "boost" generators. Under a general Lorentz transformation A, a 4-vector operator such as p_μ transforms like

$$\Lambda^{-1}p_\mu\Lambda=a_{\mu\nu}p_\nu\,, \tag{8}$$

while a tensor operator such as $M_{\mu\nu}$ satisfies

$$\Lambda^{-1}M_{\mu\nu}\Lambda=a_{\mu\rho}a_{\nu\sigma}M_{\rho\sigma}. \tag{9}$$

In addition to operators covariantly defined as tensors by their transformation properties as in the above examples, operators are considered which do not transform so simply. In particular, it is a common trick in classical relativistic mechanics to define a 4-vector by specifying its components in a particular frame and then defining it in an arbitrary frame by the Lorentz-transformation property; however, this brute-force technique may be inconsistent with an invariant functional dependence of the 4-vector on the dynamical variables. Thus, by analogy with Eq. (8), corresponding to an operator $\mathcal{O}_\mu$, an operator $\mathcal{O}_\mu'$ may be defined such that

$$\Lambda^{-1}\,\mathcal{O}_\mu'\Lambda=a_{\mu\nu}\,\mathcal{O}_\nu \tag{10}$$

or

$$\mathcal{O}_\mu'=a_{\mu\nu}\Lambda\,\mathcal{O}_\nu\Lambda^{-1}. \tag{11}$$

Only if $\mathcal{O}_\mu$ is a covariantly defined function of tensor operators will $\mathcal{O}_\mu'$ be identical with $\mathcal{O}_\mu$. Nevertheless, all operators such as $\mathcal{O}_\mu'$ will be tensors as seen by the superobserver, and for this reason, and to distinguish them from the usual concepts, they will be called supertensors.

Each observer, associated with his own family of hyperplanes (η_μ,τ), will make a different separation of p_μ into a Hamiltonian and a spatial momentum vector. Fleming has developed a description of this process which is covariant from the viewpoint of the superobserver. Fleming's results will be derived first by the construction of tensors $\mathcal{O}_\mu'$ as described above. These will then be reinterpreted in terms of the superobserver idea.

All tensors will be initially defined in the instantaneous family of hyperplanes and then transformed to an arbitrary family via Eq. (11), with the general Lorentz transformation Λ replaced with $\Lambda(\eta)$. The hyperplane Hamiltonian will be a "superscalar" defined as $-ip_4$ in the instantaneous frame. This yields

$$\begin{aligned}\mathcal{H}&=-i\Lambda p_4\Lambda^{-1}\\&=-i\tilde{a}_{\mu4}p_\mu\\&=-\eta_\mu p_\mu.\end{aligned} \tag{12}$$

From the space components of momentum, a 4-vector $\mathbf{P}_\mu$ (the hyperplane momentum) can be constructed by

$$\begin{aligned}\mathbf{P}_\mu&=\tilde{a}_{\mu i}\Lambda p_i\Lambda^{-1}\\&=\tilde{a}_{\mu i}\tilde{a}_{\nu i}p_\nu\\&=(\delta_{\mu\nu}-\tilde{a}_{\mu4}\tilde{a}_{\nu4})p_\nu\\&=p_\mu+\eta_\mu(\eta_\nu p_\nu)\,,\end{aligned} \tag{13}$$

where, in the last line but one, the orthogonality of the Lorentz-transformation matrix has been used. These operators $\mathcal{H}$ and $\mathbf{P}_\mu$ are Fleming's H and K_μ. The notation used here for $\mathbf{P}_\mu$ emphasizes its physical meaning as a hyperplane momentum operator. Fleming calls H the hyperplane mass operator; however, the term "hyperplane Hamiltonian" appears to be more appropriate, as is illustrated below.

The wave function for the system as seen by the observer on the hyperplane family (η_ν,τ) is related to that seen by the superobserver through

$$\psi'(\eta,\tau)=\Lambda(\eta)\psi. \tag{14}$$

Then, since $\eta_\nu x_\nu=-\tau$, a translation normal to the hyperplane, of amount $\Delta\tau$, is produced by the operator

$$T(\Delta\tau)=e^{i\Delta\tau(\eta_\mu p_\mu)}=\exp(-i\Delta\tau\mathcal{H}). \tag{15}$$

The time-dependent Schrödinger equation on the hyperplane family (η_μ,τ) now follows directly, and is

$$\mathcal{H}\psi'(\eta,\tau)=i\partial\psi'(\eta,\tau)/\partial\tau. \tag{16}$$

With the identification of $\mathcal{H}$ as the hyperplane Hamiltonian and $\mathbf{P}_\mu$ as the hyperplane momentum, it is not surprising to find the hyperplane analog of the relativistic relation between energy and momentum. From Eqs. (12) and (13) it is easily seen that

$$\mathbf{P}_\mu\mathbf{P}_\mu+m^2=\mathcal{H}^2. \tag{17}$$

If one now specializes to the Dirac theory, one finds the hyperplane analogs of the various Dirac matrices. First, the matrix β is just γ_4; consequently, $\beta(\eta)$ is constructed as the superscalar

$$\beta(\eta)=-i\eta_\mu\gamma_\mu. \tag{18}$$

The space components of γ_μ become the supervector

$$\begin{aligned}\gamma_\mu &= \gamma_\mu + \eta_\mu(\eta_\nu\gamma_\nu)\\ &= \gamma_\mu + i\eta_\mu\beta(\eta).\end{aligned} \tag{19}$$

The matrices $\boldsymbol{\alpha}$ generalize either by generalizing the rule $\boldsymbol{\alpha}=i\beta\boldsymbol{\Upsilon}$ or by constructing a super 4-vector from the mixed space and time components of the spin tensor $s_{\mu\nu}=-\frac{1}{4}i[\gamma_\mu,\gamma_\nu]\equiv\frac{1}{2}\sigma_{\mu\nu}$. In either case,

$$\alpha_\mu = i\beta(\eta)\gamma_\mu - \eta_\mu \tag{20}$$

$$= i\eta_\nu\sigma_{\nu\mu}. \tag{21}$$

Similarly, for systems of arbitrary spin, the operators β and $\boldsymbol{\alpha}$ may be given hyperplane generalizations. For $\boldsymbol{\alpha}$, following the definitions in Sec. II, a supervector is constructed from the mixed space-time components of $s_{\mu\nu}$ to obtain

$$\alpha_\mu = (1/s)i\eta_\nu s_{\nu\mu}. \tag{22}$$

For β, one must obtain a superscalar. The techniques developed above obviously yield

$$\begin{aligned}\beta(\eta) &= (-i)^{2s}\eta_{[\mu]}\gamma_{[\mu]}\\ &\equiv (-i)^{2s}\eta_{\mu_1}\eta_{\mu_2}\cdots\eta_{\mu_{2s}}\gamma_{\mu_1\mu_2\cdots\mu_{2s}}.\end{aligned} \tag{23}$$

The preceding results have been obtained by a transformation of operators from the instantaneous set of hyperplanes to the family (η_μ,τ). An alternative derivation is based upon the superobserver's viewpoint. Consider again Lorentz transformations of both the observed event and of the observer [associated with the hyperplane family (η_μ,τ)] relative to the superobserver. Such transformations of the event are generated by $M_{\mu\nu}$, which in Fleming's interpretation, however, commutes with η_μ. In order to transform the observer as well, $M_{\mu\nu}$ may be replaced with

$$\mathfrak{M}_{\mu\nu} = M_{\mu\nu} - i(\eta_\mu\partial/\partial\eta_\nu - \eta_\nu\partial/\partial\eta_\mu). \tag{24}$$

A transformation generated by $\mathfrak{M}_{\mu\nu}$ will leave the relationship between the frames of the event and of the observer unaltered.

If now an operator which transforms tensorially as seen by the superobserver is defined, and if in the instantaneous frame it has a particular physical interpretation, then the meaning of this operator will be the same to all observers. For example, $\mathcal{H}=-\eta_\mu p_\mu$ is the energy in the instantaneous frame and is a scalar to the superobserver. Consequently, $\mathcal{H}$ will also be interpreted as the energy by the observer associated with the hyperplane family (η_μ,τ). Similarly, $\mathbf{P}_\mu$ is the momentum to the instantaneous observer, and therefore $\mathbf{P}_\mu$ defined by Eq. (13) will be interpreted as the momentum by the observer on the hyperplane family (η_μ,τ). In fact, since $\eta_\mu\mathbf{P}_\mu=0$, it is clear that $\mathbf{P}_\mu$ generates translations which lie entirely within the observer's hyperplane, and this is the characterizing feature of linear momentum.

At this stage, the superobserver concept permits a generalization of Fleming's hyperplane concepts. If the frame defined by η_μ is to be defined by the observed system, as, for example, the instantaneous rest frame of the system, then η_μ will be an operator rather than a c number, as in Fleming's work. In this case, it is unnecessary to extend $M_{\mu\nu}$ to $\mathfrak{M}_{\mu\nu}$ as in Eq. (24), since all the arguments of the last few paragraphs still hold, provided only that the operator η_ν has vector-commutation properties with $M_{\mu\nu}$ and commutes with all other relevant operators.

IV. GENERALIZED DIRAC THEORY

The direct hyperplane generalization of the Dirac equation is obvious from the results of Sec. III. Instead of the conventional

$$H\psi = (\boldsymbol{\alpha}\cdot\mathbf{p}+m\beta)\psi = i\partial\psi/\partial t, \tag{25}$$

one obtains

$$\mathcal{H}\psi' = i\partial\psi'/\partial\tau, \tag{26}$$

with

$$\mathcal{H} = \alpha_\mu\mathbf{P}_\mu + \beta(\eta)m. \tag{27}$$

Indeed, this latter form can be very readily derived by substituting $p_\mu=\mathbf{P}_\mu+\eta_\mu\mathcal{H}$ into the conventional form $(\gamma_\mu p_\mu - im)\psi=0$ of the Dirac equation.

The extension of the hyperplane formalism to higher spins is most simply illustrated by applying the technique to the Foldy-Wouthuysen wave equation[11]

$$E\beta\varphi = i\partial\varphi/\partial t, \tag{28}$$

where

$$E = (p^2+m^2)^{1/2},$$

and to the Foldy-Wouthuysen transformation[12]

$$\varphi = S_{\mathrm{FW}}\psi, \tag{29}$$

$$S_{\mathrm{FW}} = (E+m-\beta\boldsymbol{\alpha}\cdot\mathbf{p})/[2E(E+m)]^{1/2}. \tag{30}$$

By using the results of Sec. III, these equations can be written as

$$|\mathcal{H}|\beta(\eta)\varphi'(\eta) = i\partial\varphi'(\eta)/\partial\tau, \tag{31}$$

where

$$|\mathcal{H}| = [\mathbf{P}_\mu\mathbf{P}_\mu + m^2]^{1/2} \tag{32}$$

and

$$\varphi'(\eta) = S_{\mathrm{FW}}(\eta)\psi'(\eta), \tag{33}$$

$$S_{\mathrm{FW}}(\eta) = \frac{|\mathcal{H}|+m-i\gamma_\mu\mathbf{P}_\mu}{[2|\mathcal{H}|(|\mathcal{H}|+m)]^{1/2}}. \tag{34}$$

Just as H can be obtained by transforming $E\beta$ according to

$$\begin{aligned}H &= S_{\mathrm{FW}}{}^\dagger E\beta S_{\mathrm{FW}}\\ &= \boldsymbol{\alpha}\cdot\mathbf{p}+m\beta,\end{aligned} \tag{35}$$

[11] L. L. Foldy, Phys. Rev. **102**, 568 (1956).

[12] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

the hyperplane Hamiltonian $\mathcal{H}$ can be obtained from

$$\begin{aligned}\mathcal{H}&=S_{\mathrm{FW}}^{\dagger}(\eta)\,|\mathcal{H}|\,\beta(\eta)S_{\mathrm{FW}}(\eta)\\&=\boldsymbol{\alpha}_\mu\mathbf{P}_\mu+m\beta(\eta).\end{aligned}\tag{36}$$

Other hyperplane operators can be found in a similar fashion. For example, the rest-frame polarization supertensor operator is defined by

$$W_\mu{}^0=(\beta\boldsymbol{\sigma},0),\tag{37}$$

or, in terms of covariant quantities, by

$$W_\mu{}^0=(i\gamma_5\gamma_i,\,0).\tag{38}$$

Under the Lorentz transformation $\Lambda(\eta)$

$$W_\mu=\tilde{a}_{\mu\nu}\Lambda W_\nu{}^0\Lambda^{-1},\tag{39}$$

Eq. (38) becomes

$$\begin{aligned}W_\mu(\eta)&=i\gamma_5[\gamma_\mu+(\eta_\nu\gamma_\nu)\eta_\mu]\\&=i\gamma_5\gamma_\mu.\end{aligned}\tag{40}$$

Besides the instantaneous hyperplane family, which contains the observer's Lorentz frame, there is another physically significant hyperplane family. This is the family in which the particle, or antiparticle, is at rest. This special family has the normal

$$\eta_\nu=\hat{\epsilon}p_\nu(-p_\mu p_\mu)^{-1/2},\tag{41}$$

where $\hat{\epsilon}p_\nu$ is the physical momentum and $\hat{\epsilon}$ is the operator corresponding to the sign of the energy. In the rest hyperplane family, Eqs. (32), (13), and (31) become (when $i\partial/\partial\tau$ is replaced with $-\eta_\mu p_\mu=\mathcal{H}$)

$$|\mathcal{H}|=m,\tag{42}$$

$$\mathbf{P}_\mu=0,\tag{43}$$

and

$$im\gamma_\mu p_\mu\varphi'(\eta_R,\tau_R)=p_\mu p_\mu\varphi'(\eta_R,\tau_R).\tag{44}$$

For this particular family the transformed Foldy-Wouthuysen transformation given by Eq. (34) is unity, which gives

$$\psi'(\eta_R,\tau_R)=\varphi'(\eta_R,\tau_R).\tag{45}$$

Hence the spinor $\psi'(\eta_R,\tau_R)$ is a solution of the manifestly covariant Eq. (44). This equation can be rewritten as

$$\gamma_\mu p_\mu(\gamma_\nu p_\nu-im)\psi'(\eta_R,\tau_R)=0,\tag{46}$$

so that ψ' is a solution of either

$$(\gamma_\nu p_\nu-im)\psi'=0\tag{47}$$

or

$$\gamma_\nu p_\nu\psi'=0.\tag{48}$$

These are the Dirac equations for massive and massless particles, and imply

$$(p_\mu p_\mu+m^2)\psi'=0\tag{49}$$

and

$$p_\mu p_\mu\psi'=0.\tag{50}$$

Similarly, the polarization operator W_μ becomes

$$W_\mu(\eta_R)=i\gamma_5[\gamma_\mu+p_\mu(p_\nu\gamma_\nu)(-p_\lambda p_\lambda)^{-1}].\tag{51}$$

For states which satisfy Eq. (44), this is

$$W_\mu(\eta_R)=\gamma_5(i\gamma_\mu-p_\mu/m),\tag{52}$$

the Bargmann-Wigner polarization operator,[4] with respect to the instantaneous hyperplane family.

V. GENERALIZED WHG THEORY

The WHG theory for particles of arbitrary spin in effect performs a generalization of the Foldy-Wouthuysen transformation backwards. The starting point is a $2(2s+1)$-component wave function $\varphi(\mathbf{x},t)$ which satisfies the equation

$$E\beta\varphi=i\partial\varphi/\partial t.\tag{53}$$

The analog of the Dirac wave function ψ is obtained by applying (to the spin indices only) that Lorentz transformation which would accelerate a particle initially at rest up to the actual momentum $\mathbf{p}$. This gives, for a particle of spin s,

$$\psi(\mathbf{x},t)=m^sE^{-1/2}S\varphi(\mathbf{x},t),\tag{54}$$

where $E=(p^2+m^2)^{1/2}$ and S is the Lorentz transformation

$$S=\exp[s\boldsymbol{\alpha}\cdot(\mathbf{q}/q)\operatorname{arctanh}(p/E)],\tag{55}$$

where the physical momentum operator $\mathbf{q}$ is $\hat{\epsilon}\mathbf{p}$. For $s=\frac{1}{2}$ the Lorentz transformation given by Eq. (55) is related to $S_{\mathrm{FW}}^{\dagger}$ by

$$S_{\mathrm{FW}}^{\dagger}=(m/E)^{1/2}S.\tag{56}$$

Equations (53)–(55) are now generalized to the hyperplane family (η_μ,τ) by

$$(\mathbf{P}_\mu\mathbf{P}_\mu+m^2)^{1/2}\beta(\eta)\varphi'(\eta,\tau)=i\partial\varphi'(\eta,\tau)/\partial\tau,\tag{57}$$

$$\psi'(\eta,\tau)=m^s(\mathbf{P}_\mu\mathbf{P}_\mu+m^2)^{-1/2}S(\eta)\varphi'(\eta,\tau),\tag{58}$$

and

$$S(\eta)=\exp[s\boldsymbol{\alpha}_\mu(\mathbf{Q}_\mu/P)\operatorname{arctanh}(P/|\mathcal{H}|)],\tag{59}$$

where, for general spin, $\boldsymbol{\alpha}_\mu$ and $\beta(\eta)$ are defined in Eqs. (22) and (23), and

$$P=(\mathbf{P}_\mu\mathbf{P}_\mu)^{1/2},\quad \mathbf{Q}_\mu=\hat{\epsilon}\mathbf{P}_\mu,\quad |\mathcal{H}|=(\mathbf{P}_\mu\mathbf{P}_\mu+m^2)^{1/2}.\tag{60}$$

The Lorentz transformation given in Eq. (55) can be more conveniently written as

$$S=\cosh[\omega s_{i4}(p_i/p)]-\gamma_{[4]}\sinh(\omega s_{i4}p_i/p),\tag{61}$$

where $\hat{\epsilon}$ has been replaced with $\gamma_{[4]}$, since S always operates on Foldy-Wouthuysen functions, and where

$$\omega=\operatorname{arctanh}(p/E).\tag{62}$$

Then the hyperplane Foldy-Wouthuysen transformation corresponding to Eq. (54) is

$$S_{\mathrm{FW}}^{-1}(\eta)=m^s|\mathcal{H}|^{-1/2}S(\eta),\tag{63}$$

with

$$S(\eta)=\cos[(\mathbf{P}_\mu/P)\eta_\nu s_{\mu\nu}\omega(\eta)] + i\beta(\eta)\sin[(\mathbf{P}_\mu/P)\eta_\nu s_{\mu\nu}\omega(\eta)], \quad (64)$$

where

$$\omega(\eta)=\operatorname{arctanh}(P/|\mathcal{H}|). \quad (65)$$

For given spin, $S_{\mathrm{FW}}^{-1}(\eta)$ can be calculated by the spin-matrix polynomial procedure given by Williams, Draayer, and Weber.[13]

For the rest family of hyperplanes, Eq. (57) becomes

$$mp_{[\mu]}\gamma_{[\mu]}\varphi'(\eta_R,\tau_R)=(i)^{2s}\hat{\epsilon}^{2s+1}(-p_\mu p_\mu)^{s+1/2}\varphi'(\eta_R,\tau_R) \quad (66)$$

and

$$S_{\mathrm{FW}}^{-1}(\eta)=m^{s-1/2},$$

so that $\psi'(\eta_R,\tau_R)=m^{s-1/2}\varphi'(\eta_R,\tau_R)$ also satisfies Eq. (66), as in the Dirac case. This is the manifestly covariant generalization of the WHG theory, and is simple for half-integral spin, since $\hat{\epsilon}^{2s+1}=1$ when operating on $\psi'(\eta_R,\tau_R)$, and $-p_\mu p_\mu$ is taken to an integral power.

Expect for spin $\frac{1}{2}$, Eq. (66) cannot be factored directly to give the massive- and massless-particle solutions. However, $\psi'(\eta_R,\tau_R)$ does satisfy the Klein-Gordon equation, as can be seen by operating on Eq. (66) with $mp_{[\mu]}\gamma_{[\mu]}$. Since

$$\gamma_{[\mu]}p_{[\mu]}\gamma_{[\nu]}p_{[\nu]}=(p_\mu p_\mu)^{2s}, \quad (67)$$

this gives

$$(p_\mu p_\mu)^{2s}(p_\mu p_\mu+m^2)\psi'(\eta_R,\tau_R)=0. \quad (68)$$

Then, for free particles, $\psi'(\eta_R,\tau_R)$ is a solution of either

$$[p_\mu p_\mu+m^2]\psi'(\eta_R,\tau_R)=0 \quad (69)$$

or

$$p_\mu p_\mu\psi'(\eta_R,\tau_R)=0, \quad (70)$$

so that $\psi'(\eta_R,\tau_R)$ describes both a massive and a massless particle. For the massive-particle part, Eq. (66) reduces to

$$[\gamma_{[\mu]}p_{[\mu]}-(i)^{2s}\hat{\epsilon}^{2s+1}m^{2s}]\psi'(\eta_R,\tau_R)=0, \quad (71)$$

which, when taken together with the auxiliary condition

$$[p_\mu p_\mu+m^2]\psi'(\eta_R,\tau_R)=0, \quad (72)$$

is the generalization of the Dirac equation to arbitrary spin. For the massless-particle part, Eq. (66) reduces to

$$\gamma_{[\lambda]}p_{[\mu]}\psi'(\eta_R,\tau_R)=0. \quad (73)$$

If minimal electromagnetic coupling is introduced by replacing p_μ with π_μ, such a factorization is no longer possible, since $[\pi_\mu,\pi_\nu]\neq 0$. In this case, Eq. (66) for half-integral spin becomes

$$m\pi_{[\mu]}\gamma_{[\mu]}\psi'(\eta_R,\tau_R)=(i)^{2s}(-\pi_\mu\pi_\mu)^{s+1/2}\psi'(\eta_R,\tau_R), \quad (74)$$

and is not equivalent to replacing p_μ with π_μ in Eqs. (71)–(73), as is usually attempted.

[13] S. A. Williams, J. P. Draayer, and T. A. Weber, Phys. Rev. **152**, 1207 (1966).

As is the case for spin $\frac{1}{2}$, covariant hyperplane operators can be generated from rest hyperplane operators by using the hyperplane Foldy-Wouthuysen transformation of Eq. (64). The Hamiltonian for an arbitrary hyperplane family $\mathcal{H}$ is found by

$$\mathcal{H}=S_{\mathrm{FW}}^{-1}(\eta)|\mathcal{H}|\beta(\eta)S_{\mathrm{FW}}(\eta) \quad (75)$$

or directly from H as

$$\mathcal{H}=\Lambda(\eta)H\Lambda^{-1}(\eta). \quad (76)$$

In the instantaneous hyperplane, H is found to be

$$H=E\{\tanh[2\boldsymbol{\alpha}\cdot(\mathbf{p}/p)\omega]+\beta\operatorname{sech}[2\boldsymbol{\alpha}\cdot(\mathbf{p}/p)\omega]\}. \quad (77)$$

In terms of covariant quantities,

$$H=(p_ip_i+m^2)^{1/2}\{\tanh[2s_{i4}(p_i/p)\omega] +\gamma_{[4]}\operatorname{sech}[2s_{i4}(p_i/p)\omega]\}, \quad (78)$$

so that

$$\mathcal{H}=(\mathbf{P}_\mu\mathbf{P}_\mu+m^2)^{1/2}\{-i\tan[2(\mathbf{P}_\mu/P)\eta_\nu s_{\mu\nu}\omega(\eta)] +\beta(\eta)\sec[2(\mathbf{P}_\mu/P)\eta_\nu s_{\mu\nu}\omega(\eta)]\}. \quad (79)$$

For given spin, this operator may be calculated by using the spin-matrix polynomial procedure given by Williams, Draayer, and Weber.[13] For the rest hyperplane family, Eq. (79) reduces to $m\beta(\eta_R)$ as expected.

Similarly, a polarization operator W_μ may be generalized to arbitrary hyperplanes. In the instantaneous hyperplane family, the rest-system polarization operator $W_\mu=(\beta\mathbf{S},0)$ may be written in terms of covariant quantities as

$$W_\mu{}^0=(i\gamma_5\gamma_{i[4]_{-1}},0), \quad (80)$$

where $[\mu]_{-1}$ means $2s-1$ indices. Under the transformation $\Lambda(\eta)$, Eq. (80) becomes

$$-i\gamma_5W_\mu(\eta)=(-i)^{2s-1}\eta_{[\nu]_{-1}}\gamma_{\mu[\nu]_{-1}} -(-i)^{2s+1}\eta_\mu\eta_{[\nu]}\gamma_{[\nu]}. \quad (81)$$

In the rest hyperplane family, this is

$$-i\gamma_5W_\mu(\eta_R)=(-i)^{2s-1}(\hat{\epsilon})^{2s-1}p_{[\nu]_{-1}}\gamma_{\mu[\nu]_{-1}}(-p_\mu p_\mu)^{1/2-s} -(i)^{2s+1}(\hat{\epsilon})^{2s+1}p_\mu p_{[\nu]}\gamma_{[\nu]}(-p_\mu p_\mu)^{-s-1/2}, \quad (82)$$

and for particles which satisfy Eq. (66), this becomes

$$W_\mu(\eta_R)=-\gamma_5[(-i)^{2s}\hat{\epsilon}^{2s-1}\gamma_{\mu[\nu]_{-1}}p_{[\nu]_{-1}} \times(-p_\mu p_\mu)^{1/2-s}-p_\mu/m]. \quad (83)$$

This is the generalized Bargmann-Wigner[4] polarization operator on the instantaneous hyperplane family. It is clear that $W_\mu(\eta_R)p_\mu=0$, since the right-hand side of Eq. (83) then becomes equivalent to Eq. (66).

VI. GENERALIZED WEINBERG THEORY

For bosons, Eq. (66) is complicated by the appearance of the sign operator $\hat{\epsilon}$ and the operator $-p_\mu p_\mu$ taken to a half-integral power. However, an equivalent equation can be found such that for integral spin, the sign operator does not appear. Toward this end, one may consider the $2(2s+1)$-component functions $\tilde{\psi}$ studied by Weinberg[6] which satisfy

$$[\gamma_{[\mu]}p_{[\mu]}-(i)^{2s}m^{2s}]\tilde{\psi}=0\,, \qquad (84)$$

together with the auxiliary condition

$$(p_\mu p_\mu+m^2)\tilde{\psi}=0 \qquad (85)$$

in the instantaneous hyperplane family. Sankaranarayanan and Good[14] and Nelson and Good[15] have shown that the Weinberg function $\tilde{\psi}$ is related to the spinor ψ by

$$\tilde{\psi}=[\tfrac{1}{2}(1-\gamma_5)+\tfrac{1}{2}(1+\gamma_5)\hat{\epsilon}]\psi\,, \qquad (86)$$

where ψ satisfies

$$H\psi=i\partial\psi/\partial t. \qquad (87)$$

Corresponding to $\tilde{\psi}$, a Foldy-Wouthuysen function $\tilde{\varphi}$ can be defined by

$$\tilde{\psi}=m^sE^{-1/2}S\tilde{\varphi}. \qquad (88)$$

Since γ_5 and $\hat{\epsilon}$ commute with $E^{-1/2}S$,

$$\tilde{\varphi}=[\tfrac{1}{2}(1-\gamma_5)+\tfrac{1}{2}(1+\gamma_5)\beta]\varphi\,, \qquad (89)$$

where $\hat{\epsilon}$ operating on φ has been replaced with β.

The matrix operator in Eq. (89) has no inverse; thus $\tilde{\varphi}$ satisfies no Hamiltonian-type equation as does φ. However, $\tilde{\varphi}$ does satisfy a second-order equation in t. Combining Eq. (89) with Eq. (53) yields

$$E[\tfrac{1}{2}(1-\gamma_5)+\tfrac{1}{2}(1+\gamma_5)\beta]\beta\varphi=i\partial\tilde{\varphi}/\partial t. \qquad (90)$$

Commuting β to the left and then applying $i\partial/\partial t$ gives

$$2E(1+\beta)\times\tfrac{1}{2}(1+\gamma_5)i\partial\varphi/\partial t=-(1+\beta)\partial^2\tilde{\varphi}/\partial t^2\,, \qquad (91)$$

or, equivalently,

$$2E^2\tilde{\varphi}=-(1+\beta)\partial^2\tilde{\varphi}/\partial t^2. \qquad (92)$$

This equation contains the condition

$$\beta\tilde{\varphi}=\tilde{\varphi}\,, \qquad (93)$$

as can be seen by operating on Eq. (92) with $1-\beta$, which corresponds to the Foldy-Wouthuysen form of Eq. (84). In addition, Eq. (93) with Eq. (92) implies the Klein-Gordon equation, which is of course equivalent to Eq. (85). Consequently, Eq. (92) gives rise to Weinberg functions without any auxiliary conditions. Under the transformation $\Lambda(\eta)$, Eq. (92) becomes

$$2(\mathbf{P}_\mu\mathbf{P}_\mu+m^2)\tilde{\varphi}'(\eta,\tau)=[1+(-i)^{2s}\eta_{[\mu]}\gamma_{[\mu]}]\times(\eta_\nu p_\nu)^2\tilde{\varphi}'(\eta,\tau)\,, \qquad (94)$$

where

$$\tilde{\varphi}'(\eta,\tau)=\Lambda(\eta)\tilde{\varphi}. \qquad (95)$$

The Weinberg function $\tilde{\psi}'(\eta,\tau)$ is then given by

$$\tilde{\psi}'(\eta,\tau)=m^s(\mathbf{P}_\mu\mathbf{P}_\mu+m^2)^{-1/4}S(\eta)\tilde{\varphi}'(\eta,\tau). \qquad (96)$$

[14] A. Sankaranarayanan and R. H. Good, Jr., Nuovo Cimento **36**, 1303 (1965).

[15] T. J. Nelson and R. H. Good, Jr., Rev. Mod. Phys. **40**, 508 (1968).

As before, for the rest family of hyperplanes,

$$\tilde{\psi}'(\eta_R,\tau_R)=m^{s-1/2}\tilde{\varphi}'(\eta_R,\tau_R)\,, \qquad (97)$$

since $S(\eta_R)=1$. Then Eq. (94) becomes

$$(-p_\mu p_\mu)^{s-1}(p_\mu p_\mu+2m^2)\tilde{\psi}'(\eta_R,\tau_R)=(-i)^{2s}\hat{\epsilon}^{2s}\gamma_{[\mu]}p_{[\mu]}\tilde{\psi}'(\eta_R,\tau_R). \qquad (98)$$

This is the manifestly convariant generalization of the Weinberg theory for arbitrary spin and mass. It is particularly simple for integral spin, since $\hat{\epsilon}^{2s}=1$ when operating on $\tilde{\psi}$, and $-p_\mu p_\mu$ is taken to an integral power. Equation (98) can be obtained directly from Eq. (66) by using the relationship

$$\tilde{\psi}'(\eta_R,\tau_R)=[\tfrac{1}{2}(1-\gamma_5)+\tfrac{1}{2}(1+\gamma_5)\hat{\epsilon}]\psi'(\eta_R,\tau_R)\,, \qquad (99)$$

but only if the Klein-Gordon equation is also used.

If Eq. (98) is rewritten as

$$2m^2(-p_\mu p_\mu)^{s-1}\tilde{\psi}'(\eta_R,\tau_R)=[(-i)^{2s}\hat{\epsilon}^{2s}\gamma_{[\mu]}p_{[\mu]}+(-p_\mu p_\mu)^s]\tilde{\psi}'(\eta_R,\tau_R)\,, \qquad (100)$$

then operating from the left with $(-i)^{2s}\hat{\epsilon}^{2s}\gamma_{[\mu]}p_{[\mu]}-(-p_\mu p_\mu)^s$ gives

$$2m^2(-p_\mu p_\mu)^{s-1}[(-i)^{2s}\hat{\epsilon}^{2s}\gamma_{[\mu]}p_{[\mu]}-(-p_\mu p_\mu)^s]\tilde{\psi}'(\eta_R,\tau_R)=0\,, \qquad (101)$$

where Eq. (67) has been used to eliminate the right-hand side. It is then clear that for spin greater than 1, $\tilde{\psi}'(\eta_R,\tau_R)$ contains both the massive- and massless-particle solutions

$$[\hat{\epsilon}^{2s}\gamma_{[\mu]}p_{[\mu]}-(-p_\mu p_\mu)^s]\tilde{\psi}'(\eta_R,\tau_R)=0 \qquad (102)$$

and

$$(-p_\mu p_\mu)^{s-1}\tilde{\psi}'(\eta_R,\tau_R)=0. \qquad (103)$$

Taken with Eq. (100), Eq. (102) implies the Klein-Gordon equation. On the other hand, Eq. (102), together with the Klein-Gordon equation, is the Weinberg theory for integral spin.

For spin 1, Eq. (98) is

$$(\gamma_{\mu\nu}p_\mu p_\nu+p_\mu p_\mu+2m^2)\tilde{\psi}'(\eta_R,\tau_R)=0\,, \qquad (104)$$

which is the spin-1 free-particle equation studied by Shay and Good.[16] It is apparent that the spin-1 case admits only the massive-particle solution.

For spin 0, Eq. (98) is

$$-2m^2\tilde{\psi}'(\eta_R,\tau_R)=p_\mu p_\mu(1+\beta)\tilde{\psi}'(\eta_R,\tau_R). \qquad (105)$$

However, $\tilde{\psi}(\eta_R,\tau_R)=m^{-1/2}\tilde{\varphi}'(\eta_R,\tau_R)$ and $\beta\tilde{\varphi}=\tilde{\varphi}$, since $S=\Lambda=1$ for spin 0. Therefore Eq. (105) is just the Klein-Gordon equation

$$(p_\mu p_\mu+m^2)\tilde{\psi}'(\eta_R,\tau_R)=0. \qquad (106)$$

Minimal electromagnetic coupling may be introduced by replacing p_μ with π_μ in Eq. (98). For integral spin

[16] D. Shay and R. H. Good, Jr. (unpublished).

this gives

$$(-\pi_\mu\pi_\mu)^{s-1}(\pi_\mu\pi_\mu+2m^2)\tilde{\psi}'(\eta_R,\tau_R) = (-1)^s\gamma_{[\mu]}\pi_{[\mu]}\tilde{\psi}'(\eta_R,\tau_R). \quad (107)$$

As is the case for Eq. (74), Eq. (107) is no longer factorable into massive- and massless-particle parts. Furthermore, since the sign operator $\hat{\epsilon}$ is not well defined when interactions are present, $\tilde{\psi}(\eta_R,\tau_R)$ is not related to $\psi(\eta_R,\tau_R)$ in a simple way.

VII. CONCLUSIONS AND SUMMARY

The hyperplane formalism presented in this paper has made it possible to write for any spin a manifestly covariant Foldy-Wouthuysen transformation, Eq. (63), and Foldy-Wouthuysen wave function which satisfies a manifestly covariant wave equation given by either Eq. (57) or Eq. (94). In the same sense, manifestly covariant Hamiltonians and polarization operators, Eqs. (79) and (81), can also be obtained. By specializing to the particular hyperplane observer which corresponds to the particle rest system, the usual manifestly covariant equations are obtained for spin 0, $\frac{1}{2}$, and 1, and for higher spins new wave equations without auxiliary conditions are obtained.

Furthermore, as pointed out by Mathews,[17] the various wave equations and operators presented here are unique in the sense that they are obtained by a continuous Lorentz transformation from well-defined rest-system wave equations and operators.

It is appropriate to use Eq. (66) only for half-integral spin and Eq. (98) for integral spin since, besides the simplicity of these equations for this case, only then in the second quantized theory are local anticommutation and commutation relationships for the fields obtained.[15]

The fact that, for free particles, zero-mass solutions as well as massive solutions are contained in the same wave equation is interesting since there seems to be no basis on physical grounds for rejecting these solutions.[18] Furthermore, when interactions are present, the wave equation no longer separates into two distinct solutions. The spin-$\frac{1}{2}$ case is an exception. For example, before putting in the electromagnetic interaction, Eq. (44) can be left unaltered, written as Eq. (46), or separated as Eqs. (47) and (48). Then Eq. (44) becomes

$$im\gamma_\mu\pi_\mu\psi' = \pi_\mu\pi_\mu\psi' \quad (108)$$

or

$$[\gamma_\mu\pi_\mu + (e/2m)\sigma_{\mu\nu}F_{\mu\nu}]\psi' = (-i/m)(\gamma_\mu\pi_\mu)^2\psi',$$
$$F_{\mu\nu} = (\partial/\partial x_\mu)A_\nu - (\partial/\partial x_\nu)A_\mu, \quad (109)$$

an equation with an anomalous magnetic-moment term.

On the other hand, Eq. (46) becomes

$$\gamma_\mu\pi_\mu(\gamma_\nu\pi_\nu - im)\psi' = 0. \quad (110)$$

The exact Green's function then separates into two terms:

$$G = G_e/im - G_\nu/im, \quad (111)$$

where

$$G_e = [\gamma_\mu(p_\mu - eA_\mu) - im]^{-1}$$

and

$$G_\nu = [\gamma_\mu(p_\mu - eA_\mu)]^{-1}.$$

The theories described by Eqs. (109) and (110) are different from the usual Dirac equation with fields which can be obtained directly from Eq. (47). Both Eqs. (109) and (110) are sufficiently interesting to bear further investigation.

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[17] P. M. Mathews, Phys. Rev. **143**, 978 (1966).

[18] One could argue by Eq. (41) that η_μ is restricted to values that correspond to instantaneous hyperplanes in the rest frame of the particles and that this restriction makes sense only if the particle has a rest frame. This would then provide a physical reason for rejecting the massless solutions. The authors, however, take the alternative view that Eqs. (66) and (98) are of interest independently of their derivation. The advance represented by these equations is that they are manifestly covariant without any auxiliary conditions. Rejection of the zero-mass solutions is equivalent to reintroducing the Klein-Gordon equation as an auxiliary condition, as in Weinberg's work.