with

$$\mathrm{Im}P_j(M^\nu L,M^\nu L)$$
$$=(-)^{\nu+1}\frac{k^{2L}}{2[(2L-1)!!]^2}\frac{L+1}{L}\sum_n(-)^{I_i+I_n}$$
$$\times\begin{Bmatrix}L & L & j\\ I_i & I_i & I_n\end{Bmatrix}|\langle I_i\|M^\nu(L)\|I_n\rangle|^2\Gamma_n$$
$$\times[EE_n(1+(-)^j)+\tfrac{1}{2}(E_n^2+E^2+\tfrac{1}{4}\Gamma_n^2)(1-(-)^j)]$$
$$\times[(E_n^2-E^2+\tfrac{1}{4}\Gamma_n^2)^2+(\Gamma_nE)^2]^{-1},\quad(36)$$

and

$$\mathrm{Re}P_j(M1,E2)=-(k^3/2\sqrt{3})\sum_n(-)^{I_i+I_n}$$
$$\times\begin{Bmatrix}1 & 2 & j\\ I_i & I_i & I_n\end{Bmatrix}\langle I_i\|M(1)\|I_n\rangle\langle I_i\|E(2)\|I_n\rangle\Gamma_n$$
$$\times[EE_n(1+(-)^j)+\tfrac{1}{2}(E_n^2+E^2+\tfrac{1}{4}\Gamma_n^2)(1-(-)^j)]$$
$$\times[(E_n^2-E^2+\tfrac{1}{4}\Gamma_n^2)^2+(\Gamma_nE)^2]^{-1}.\quad(37)$$

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Concept of Ideal Collective Coordinate as the Foundation for a Phenomenological Theory of Nuclear Collective Motion: Basic Ideas and Relation to Other Phenomenological Methods*

ABRAHAM KLEIN, M. DREIZLER, AND ROBERT E. JOHNSON†
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
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The concept of an ideal collective coordinate is introduced by means of the following example: Consider a one-dimensional vibration of a many-body system in the sense that a large subset of states $|n\rangle$ of the system exhibits an energy spectrum and relative transition probabilities following the laws of the (in general anharmonic) oscillator described by $\mathcal{H}$ (p_α,α) $(\alpha|n)=\omega_n(\alpha|n)$. We suppose the set of many-body states $|n\rangle$ to extend indefinitely, and we take the transform $|\alpha\rangle=\sum|n\rangle(n|\alpha)$ to define a many-body generating state of the band which is precisely localized in α space. The basic assumption of collectivity, that changing the state of at most a few particles cannot much alter the value of α, is shown to be sufficient to derive a phenomenological theory from the many-body starting point. The phenomenological aspects of a recent theory of rotations due to Villars is seen to be contained in the above formulation as a special case. A brief review is given of the generator coordinate and similar projection methods in order to exhibit their relationship with the present method.

I. INTRODUCTION

IN the traditional phenomenological approach to the nuclear collective Hamiltonian,[1,2] one assumes that somehow the fundamental microscopic Hamiltonian can be written approximately as the sum of a collective and an intrinsic part, the wave function having, in first approximation, essentially a product form. *Per contra*, it is remarkable that *none* of the existing versions of fully quantum-mechanical microscopic theories of collective motion[3–17] is able to exploit this suggestion. The reason for this "failure" is physically clear: In contradistinction to the case of molecules, there are no generally valid

* Supported in part by the U. S. Atomic Energy Commission.
† Present address: McMaster University, Hamilton, Ont., Canada.

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operator collective coordinates, clearly separable from the particle degrees of freedom, such that the coupling between these and the "intrinsic" degrees of freedom is small. Since the microscopic theory has thus been unable to take any useful lead from the older phenomenology, it appears to us appropriate to attempt to invert the relationship and to construct the phenomenology to accord with the available microscopic formulations.

The plural character of the latter raises some problems of its own, however. There are, at present, two clearly distinguishable classes of microscopic theories capable of dealing from a unified viewpoint with both rotations and vibrations. The first of these is the method of generator coordinates[3–8,14,15,17] based on the idea of an intrinsic or generating state depending on both particle and collective coordinates. The actual physical states (members of a band) are obtained by projection, involving an average over the collective coordinates; the true wave functions thus contain no redundant variables. If the form of the generating state is assumed, the variational principle for the energy can be utilized to obtain the form of the projection operators on to the physical states.[3–7] Where the latter are known from invariance considerations (as for translations and rotations), the method can be used to determine the best generating state from an assumed trial class.[8,14,15] The generating state thus found generally varies *with the band member to be projected.*

We are aware of only one effort, which is quite successful as far as it goes, to derive something akin to the usual phenomenological theory from the method of generator coordinates, namely, that due to Verhaar.[18,19] This work assumes a prescribed generating state[6,7] as well as known projection operators and does not, therefore, formally include the more recent microscopic versions.[8,15] It cannot, as it stands, be applied at all to vibrations where the projection operators are not known *a priori.* For rotations, however, it is, from a purely phenomenological standpoint, as general as the customary phenomenological theory and quite close to the method to be developed in the present paper. (Quite recently a new and unconventional phenomenological analysis, which appears quite powerful, has been extracted from the generator coordinate method.[20] So far only the case of rotations has been treated.)

Our over-all view of the method of generator coordinates is that, despite its venerable service, it is still young in development. Though it has proved a powerful tool for study of the *s-d* shell, for instance,[17,21–27] and may yet prove to be a useful tool for heavy nuclei,[8,20,28,29] no one has yet recorded by means of this method a systematic theory of *rotations, vibrations, and their coupling,* either microscopically, or phenomenologically. A partial exception to this statement must be allowed, however, for the work of Levinson *et al.*[17]

A second microscopic method has been developed in recent years, one originally christened the generalized Hartree-Fock approximation[9–11,16,30,31] but since renamed the self-consistent core-particle coupling method.[32] Though the publication of the consequences of this method is far from complete, it has at least been indicated that a *uniform* method is available for the treatment of rotations and vibrations[16] and that the method contains as a limiting case[30] the full machinery of the time-dependent self-consistent field method.[33] It is, of course, intended to develop these ideas more fully in future publications. In the course of these studies, however, it has become clear that one could, and perhaps should, abstract for quite separate consideration the phenomenological concepts of collective motion inherent in this approach. In so doing, one is enabled to give what appears to be a soundly based theory of vibrations, rotations, and their coupling.

For the present we restrict ourselves to a single idealized collective degree of freedom (Sec. II) applicable to monopole vibrations, translations, and rotations in a plane. The detailed application of this method to rotations and vibrations in the real world will be given in a separate paper which is now in preparation. The idealization consists in the assumption that the collective branch of the spectrum, though at best only partially observed experimentally, can be continued indefinitely to states of higher energy and that the resulting states can be put into perfect correspondence with some simple quantum-mechanical system; the coordinates of the latter are the collective degrees of freedom. The assumption of collective motion is realized in the statement that the values of these coordinates cannot change much in consequence of the change of state of at most a few particles. The natural exploitation of this statement leads straightforwardly to the phenomenological theory. It is argued that the description may still be useful in the real case that the collective spectrum is definitely bounded in extent, and that in this case the description still differs from that given by the method of generator coordinates.

Villars has recently given a new theory of rotations.[12,13] In Sec. III it is shown that its macroscopic

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aspects can be deduced from the results of Sec. II. Insofar as it has been developed, microscopically, it yields anew the self-consistent cranking theory. The way in which this comes about[34] is reviewed briefly.

In the attempt to understand the accomplishments to date of the generator-coordinate method, we have, of course, studied only the published literature in this area. In Sec. IV we have reviewed only a few basic elements relevant to the present discussion. A summary discussion in Sec. V contrasts the two methods.

II. IDEALIZED THEORY OF A COLLECTIVE DEGREE OF FREEDOM

We illustrate our basic point of view by an instructive though academic example. We suppose that observation on our system has brought to light a sequence of levels of zero angular momentum with the uniform spacing of a simple harmonic oscillator with energy differences ω_0. Assuming that certain conditions on transition probabilities (a matter to which we shall return) are met, it is then natural to suppose that we can associate with this subset of states a collective oscillator coordinate α and its conjugate momentum p_α in the sense that the simple Schrödinger equation

$$\mathcal{H}(\alpha,p_\alpha)(\alpha|n)=(n+\tfrac{1}{2})\omega_0(\alpha|n) \tag{2.1}$$

is satisfied and

$$\mathcal{H}(\alpha,p_\alpha)=(p_\alpha{}^2/2B)+\tfrac{1}{2}C_2\alpha^2, \tag{2.2}$$

where the inertial parameter B and the force constant C_2 must somehow be related to many-body dynamics. We shall envisage a "breathing mode" with α some measure of the fractional change in radius of the nuclear system.

[We are begging several questions, but only temporarily. Thus Eqs. (2.1) and (2.2) can never be given empirically because they imply an infinite number of states. We shall be required finally to provide answers to the following queries: Even supposing an infinite number of states, can any realistic many-body system have a collective Hamiltonian as simple as (2.2)? What modifications are necessary if at best a finite number of states can be associated with the particular notions under discussion?]

In Eq. (2.1) it is fundamental to emphasize the dual significance of the "wave functions" $(\alpha|n)$. They are trivially the wave functions of a one-dimensional harmonic oscillator, but the basic physics of our approach is that they are simultaneously and more vitally the overlap between the many-body state $|n\rangle$ and an idealized many-body generating state $|\alpha\rangle$. This is expressed by the equation

$$|n\rangle=\int d\alpha\ |\alpha\rangle(\alpha|n), \tag{2.3}$$

where, with our suppositions, the $|\alpha\rangle$ form a complete set of *localized* states

$$\langle\alpha|\alpha'\rangle=\delta(\alpha-\alpha'). \tag{2.4}$$

Equations (2.1) to (2.4) may be understood as the expression of the assumption that our many-body system possesses a collective degree of freedom.

Though the basic distinction between Eqs. (2.3) and (2.4) and superficially similar equations of the generator-coordinate method[3–7] will hopefully be fully clarified in consequence of the further content of this paper, it is perhaps necessary to warn the reader explicitly at this point to avoid the possible confusion of the two methods. In the traditional approach one builds on an explicit many-body generating state $|\alpha\rangle$, which is necessarily approximate (Slater determinant or BCS wave function) depending on the parameters α. Only through utilization of such an explicit form does one ultimately obtain information about $(\alpha|n)$ and $|n\rangle$. The number of states $|n\rangle$ is then always *finite* and the overlap (2.4), though peaked, is not a δ function.

In the present approach $|\alpha\rangle$ and $|n\rangle$ are idealizations whose explicit form is never needed (in fact cannot be given). The property (2.4) requires only the correspondence between an infinite subset of many-body states $|n\rangle$ and the harmonic oscillator (in this case). We now describe the assumed properties of this correspondence which allows us to do physics, even though we never assume explicit wave functions for the nucleus.

Consider a general observable Θ, which we can think of in all practical cases as a one- or two-body operator. For the computation of Θ in the basis of states $|n\rangle$, we write

$$\langle n|\Theta|n'\rangle=\int d\alpha d\alpha'(n|\alpha)\langle\alpha|\Theta|\alpha'\rangle(\alpha'|n'). \tag{2.5}$$

We then consider the matrix element $\langle\alpha|\Theta|\alpha'\rangle$. Since α must depend in some symmetrical way on the coordinates of all the particles, whereas Θ is an operator that changes, at most, the state of a few particles, it is difficult to imagine other than that $\langle\alpha|\Theta|\alpha'\rangle$ is a distribution peaked in the neighborhood of $\alpha=\alpha'$. In the literature, such a behavior has often been verified for specific examples,[7,13,15,17] where it is found that the width of the distribution narrows as the number of particles increases. This distribution therefore admits an expansion in moments

$$\langle\alpha|\Theta|\alpha'\rangle=\sum_{k=0}^{\infty}\Theta^{(k)}(\alpha)(\partial^k/\partial\alpha^k)\delta(\alpha-\alpha'), \tag{2.6}$$

where

$$\Theta^{(k)}(\alpha)=\int d\alpha'[(\alpha'-\alpha)^k/k!]\langle\alpha|\Theta|\alpha'\rangle. \tag{2.7}$$

The remainder of our discussion in this section involves the straightforward exploitation of (2.5)–(2.7).

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We shall first see how these equations may be used to obtain formulas for B and C_2 of Eq. (2.2), or more generally, to obtain the collective Hamiltonian $\mathcal{H}(\alpha,p_\alpha)$. In this procedure we suppose the state $|\alpha\rangle$ to exist according to (2.3) and (2.4), but we assume a momentary skepticism about $\langle\alpha|n\rangle$ possessing properties as simple as those expressed by (2.1) and (2.2); i.e., we allow for a more general $\mathcal{H}(\alpha,p_\alpha)$. By a straightforward symmetrization we can write for matrix elements of the many-body Hamiltonian H, as an application of (2.6) and (2.7),

$$\langle n|H|n'\rangle=\int d\alpha(n|\alpha)[H^{(0)\prime}(\alpha)+\tfrac{1}{2}\{H^{(1)\prime}(\alpha),(\partial/\partial\alpha)\}$$
$$+\tfrac{1}{4}\{\{H^{(2)\prime}(\alpha),(\partial/\partial\alpha)\},(\partial/\partial\alpha)\}+\cdots](\alpha|n'), \quad (2.8)$$

where, e.g.,

$$H^{(0)\prime}(\alpha)=H^{(0)}(\alpha)-\tfrac{1}{2}(\partial/\partial\alpha)H^{(1)}(\alpha)$$
$$+\tfrac{1}{4}(\partial^2/\partial\alpha^2)H^{(2)}(\alpha)+\cdots, \quad (2.9)$$

$$H^{(1)\prime}(\alpha)=H^{(1)}(\alpha)-(\partial/\partial\alpha)H^{(2)}(\alpha)+\cdots, \quad (2.10)$$

$$H^{(2)\prime}(\alpha)=H^{(2)}(\alpha)+\cdots. \quad (2.11)$$

It is supposed that, in every case, there are rapidly converging series; for illustrative purposes, we shall therefore retain only the first term of each and drop the primes. (The nature of the convergence is discussed among other questions in an accompanying paper.[35]) Furthermore, if it is not already zero, the term in $H^{(1)}$ can be eliminated in the usual way by a canonical transformation

$$(\partial/\partial\alpha)\rightarrow(\partial/\partial\alpha)+\chi(\alpha), \quad (2.12)$$

with

$$\chi=-\tfrac{1}{2}[H^{(1)}/H^{(2)}]. \quad (2.13)$$

It is evident from (2.5) and the special case (2.8) why (2.1) and (2.2) are too naive and must be generalized. From (2.5) alone we should write

$$\int\mathcal{H}(\alpha,\alpha')\langle\alpha'|n\rangle d\alpha'=\omega_n\langle\alpha|n\rangle, \quad (2.14)$$

with

$$\mathcal{H}(\alpha\alpha')=\langle\alpha|H|\alpha'\rangle-W_0\delta(\alpha-\alpha') \quad (2.15)$$

and W_0 the ground-state energy. If we now admit (2.8), we have, more particularly,

$$\mathcal{H}(\alpha,p_\alpha)\langle\alpha|n\rangle=\omega_n\langle\alpha|n\rangle, \quad (2.16)$$

where

$$\mathcal{H}(\alpha,p_\alpha)+W_0=H^{(0)}(\alpha)$$
$$+\tfrac{1}{4}\{\{H^{(2)}(\alpha),(\partial/\partial\alpha)\},(\partial/\partial\alpha)\}+\cdots. \quad (2.17)$$

There are certainly circumstances where one of the more general forms just given must be utilized. Such circumstances are best considered within the framework of a microscopic theory. If, on the other hand, one wishes to describe a nearly harmonic spectrum, this implies that one may introduce the further expansion

$$H^{(n)}(\alpha)=\sum_{m=0}^{\infty}\frac{\alpha^m H^{(n,m)}}{m!}. \quad (2.18)$$

Supposing the linear term to be absent (as can always be arranged), one then has upon comparison of (2.17) and (2.18) with (2.2)

$$W_0=H^{(00)}, \quad (2.19)$$

$$(2B)^{-1}=-H^{(20)}, \quad (2.20)$$

$$C_2=H^{(02)}. \quad (2.21)$$

In this procedure we are, moreover, hardly constrained to remain within the harmonic approximation; the extension to higher-order terms is straightforward and will be considered for physically more interesting cases in the paper under preparation. It is also illustrated for a microscopic model in the paper immediately following this one.[35]

It is to be remarked further that the analysis given here applies not only to one-dimensional vibrations but also to translations and to rotations in a plane if, in these cases, we set $C_2=0$, and reinterpret the collective coordinates. Most of the comparison with other work will be done for these latter cases since this is where the most effort has been concentrated by previous authors.

Next we consider briefly the "experimental test" of the vibrational model within the harmonic approximation. We have, first of all, the occurrence of a fundamental energy difference

$$\omega_0=(C/B)^{1/2}. \quad (2.22)$$

Next, let Θ be an operator (monopole moment, for example) connecting successive states of our "system." We compute

$$\langle n|\Theta|n+1\rangle=\int d\alpha\,\langle n|\alpha\rangle[\Theta^{(0)}(\alpha)+\tfrac{1}{2}\{\Theta^{(1)}(\alpha),(\partial/\partial\alpha)\}$$
$$+\cdots]\langle\alpha|n+1\rangle$$
$$\cong\int d\alpha\,\langle n|\alpha\rangle[\Theta^{(01)}\alpha$$
$$+\Theta^{(10)}(\partial/\partial\alpha)+\cdots]\langle\alpha|n+1\rangle. \quad (2.23)$$

Introducing the creation and annihilation operators $b^\dagger$, b,

$$\omega_0 b^\dagger=[(\tfrac{1}{2}C)^{\frac{1}{2}}\alpha-(2B)^{-1/2}(\partial/\partial\alpha)],$$
$$\omega_0 b=[(\tfrac{1}{2}C)^{\frac{1}{2}}\alpha+(2B)^{-1/2}(\partial/\partial\alpha)], \quad (2.24)$$

we find

$$|\langle n|\Theta|n+1\rangle|^2=(n+1)|[\Theta^{(01)}(2B)^{-1/2}$$
$$+\Theta^{(10)}(\tfrac{1}{2}C)^{1/2}]|^2+\cdots. \quad (2.25)$$

[35] A. Klein and R. E. Johnson, following paper, Phys. Rev. 171, 1224 (1968).

Of course the factor $(n+1)$ is the oscillator signature, the possible presence of higher-order terms signifying the breakdown of the simple picture. If, however, one can fit the energies with an anharmonic oscillator Hamiltonian where the anharmonic terms are small, and, at the same time one finds that higher-order corrections to formulas like (2.25) are small, one can consider the model to be a viable one. One would, obviously, require that the data overdetermine the parameters, in order to have a reasonable consistency check.

One can see that the value of (2.25) cannot depend on the absolute values or dimensions of B and C_2 separately, but rather (2.25) remains unaffected by any change $\alpha \rightarrow \lambda\alpha$ which leaves ω_0 invariant (and thus $C_2 \rightarrow \lambda^{-2}C_2$, $B \rightarrow \lambda^{-2}B$). From the purely phenomenological point of view, the transition probability (2.25) brings in new parameters not necessarily related simply to those which define the collective Hamiltonian. Only in the case of translations and rotations are there additional consistency conditions arising from the associated symmetries as, for example, the requirement that the total momentum and the translational energy define the same mass parameter.

We have thus presented, in outline for a prototype, a complete phenomenological quantum-mechanical theory of an idealized collective degree of freedom. The idealization consists in the assumption of an infinite number of states representing various modes of excitation of the given degree of freedom and consequently of the existence of a complete set of localized generating states satisfying (2.4).

In the next section, we shall show that a rather different looking formulation of Villars (restricted to rotations and translations) can be derived directly by rewriting the results of this section.

In practice, we must candidly admit the imperfection of our collective degree of freedom, in that the band must break off for sufficiently high n. By a simple and undoubtedly naive modification of our basic equations, we can argue that the phenomenology developed above should still be valid for small n. Let us suppose, for example, that as empirical fact a *finite* set of states n can be put into correspondence with the simple quantum-mechanical system described by

$$\mathcal{H}(\alpha,p_\alpha)(\alpha|n)=\omega_n(\alpha|n) \tag{2.26}$$

but that this equation holds only for $n<n_{\max}$. If we *define* the generating state $|\alpha\rangle$ by the equation

$$|\alpha\rangle=\sum_n^{n_{\max}}|n\rangle(n|\alpha), \tag{2.27}$$

then we may still expect Eqs. (2.5)–(2.7) to be reasonably correct, since $\langle\alpha|\alpha'\rangle$, though no longer a singular function, is still strongly peaked as a function of $(\alpha-\alpha')$, and the same should be true for any matrix element such as (2.5). This is sufficient to preserve the structure of the phenomenology. The only new point is that at least one additional parameter $n_{\max}$ has been introduced into the microscopic theory, i.e., the form (2.27) must be used to evaluate formulas (2.19)–(2.21). The value of $n_{\max}$ should be determined by shell-model considerations. A first step in the direction of this kind of theory has in fact been made.[31]

The generating state $|\alpha\rangle$ defined by (2.27), though sharing some formal properties with the corresponding state in the method of generator coordinates, must in reality be distinguished from the latter for the following reason: As we shall indicate in Sec. IV, the method of generator coordinates cannot yield an equation equivalent to (2.26) beyond the harmonic approximation. It remains for the microscopic theory to establish the self-consistency of (2.26).

III. RELATION TO VILLARS'S METHOD

A. Phenomenological Theory

This development has both a kinematical and a dynamic aspect. In accordance with our program, we shall emphasize the former. Work which is in part equivalent to that of Villars has been carried out by Kammuri.[34]

We shall show how the results of Sec. II can be reformulated and reinterpreted, specifically for the case of rotations, to give precisely the formulation of Villars. (The method of this section, *as it stands*, is not applicable to vibrations, the relation of the collective momentum operator to a constant of the motion proving essential. In that sense the previous phenomenology is more general than that which follows.)

Let us study the series (2.8) for the case of two-dimensional rotations. It then simplifies because $H^{(k)}$ is independent of α (which we relabel ϕ) as a consequence of the assumed rotational invariance of the Hamiltonian. We have, as a special case of (2.7),

$$H^{(k)}=\int d\phi\ (\phi^k/k!)\langle\phi=0|H|\phi\rangle$$

$$=\int d\phi\langle 0|[H,\boldsymbol{\phi}]^{(k)}|\phi\rangle, \tag{3.1}$$

where $\boldsymbol{\phi}$ is the angle operator whose eigenvalues label the "needle" states $|\phi\rangle$, and

$$[H,\boldsymbol{\phi}]^{(k)}=[\cdots[[H,\boldsymbol{\phi}],\boldsymbol{\phi}]\cdots,\boldsymbol{\phi}]/k!, \tag{3.2}$$

where k commutators are taken all together. For practical purposes the operator $\boldsymbol{\phi}$ is canonically conjugate to the angular-momentum operator $\mathbf{I}$. The series (2.8) is, for this case, trivially evaluated:

$$\langle I|H|I'\rangle=\delta_{II'}[H^{(0)}+H^{(1)}iI'+H^{(2)}(iI')^2+\cdots]. \tag{3.3}$$

This series can be given the following interpretation:

Within the space of states $|I\rangle$, we may consider the quantities $H^{(k)}$ to be the matrix elements of operators both diagonal in I and independent thereof—in short, multiples of the unit operator. We write

$$\mathbf{H}=H^{(0)}+H^{(1)}i\mathbf{I}+\cdots+H^{(k)}(i\mathbf{I})^k+\cdots \quad (3.4)$$

and the assertions made above can be correct if and only if the $H^{(k)}$ are independent of both ϕ and $\mathbf{I}$. In this sense the $H^{(k)}$ are truly intrinsic operators. Within the context of Sec. II their matrix elements were recognized as the parameters defining the collective Hamiltonian.

By a further evaluation of (3.1), we have, since

$$|\phi\rangle=\exp(-i\mathbf{I}\phi)|\phi=0\rangle, \quad (3.5)$$

$$\begin{aligned}H^{(k)}&=2\pi\langle\phi=0|[\mathbf{H},\phi]^{(k)}\delta_{\mathbf{I},0}|\phi=0\rangle\\&=\sum_{II'}\langle I|[\mathbf{H},\phi]^{(k)}\delta_{\mathbf{I},0}|I'\rangle\\&=\langle I=0|[\mathbf{H},\phi]^{(k)}|I=0\rangle.\end{aligned} \quad (3.6)$$

In reaching the last form, we have utilized a fact which can be inferred from (3.4), namely, that $[H,\phi]^{(k)}$ is rotationally invariant. Equation (3.6) permits us to recognize the existence of potentially useful recursive formulas for the operators $H^{(k)}$. From (3.4) and (3.6) we have, in fact,

$$H^{(k)}=[\mathbf{H},\phi]^{(k)}-\sum_{l=k+1}^{\infty}\frac{l!}{k!(l-k)!}H^{(l)}(i\mathbf{I})^{l-k}. \quad (3.7)$$

Equations (3.4)–(3.7) and attendant discussion constitute the core of Villars's phenomenological formulation of the rotational problem. How can we, using these ideas, determine the intrinsic operators in practice?

In describing the procedure for doing calculations we shall tie together the viewpoints of Villars[12,13] and Kammuri[34] by utilizing a result of Sec. II as expressed by Eq. (3.1). Our normal prejudices require the odd terms in I of Eqs. (3.3) or (3.4) to vanish, i.e., we must expect the $H^{(k)}$ to vanish for k odd. This indeed follows from the assumption that if P_1 is the reflection, with respect to the 1 or x axis, then any rational definition of the operator ϕ yields

$$P_1|\phi\rangle=|-\phi\rangle \quad (3.8)$$

or

$$P_1\phi P_1^{-1}=-\phi, \quad (3.9)$$

to an irrelevant phase, whereas

$$P_1\mathbf{H}P_1^{-1}=\mathbf{H}. \quad (3.10)$$

More generally we may obtain the same result from time-reversal invariance.

With the help of (3.7) we now obtain a series of invaluable conditions, of which the first is

$$\begin{aligned}H^{(1)}=0&=-i\dot{\phi}-2H^{(2)}(i\mathbf{I})-3H^{(3)}(i\mathbf{I})^2-\cdots\\&=-i\dot{\phi}+i[\dot{\phi},\phi]i\mathbf{I}+\cdots,\end{aligned} \quad (3.11)$$

since

$$H^{(2)}=-\tfrac{1}{2}i[\dot{\phi},\phi]+\cdots. \quad (3.12)$$

For illustrative purposes, we shall restrict ourselves to the approximation $H^{(k)}=0$, $k>4$, k even (the rigid rotator). Then, according to (3.4), we have

$$\begin{aligned}\mathbf{H}&=H^{(0)}+H^{(2)}(i\mathbf{I})^2\\&\equiv H^{(0)}+\mathbf{I}^2/2\mathcal{I}\end{aligned} \quad (3.13)$$

or, comparing with (3.12),

$$\mathcal{I}^{-1}=i[\dot{\phi},\phi]. \quad (3.14)$$

This equation together with the implied simplification of (3.11),

$$\dot{\phi}=-i[\phi,\mathbf{H}]=\mathbf{I}/\mathcal{I}, \quad (3.15)$$

are in this formulation the fundamental equations of the rigid rotator.

B. Microscopic Theory

Thus far the theory—within the bounds of the approximations imposed—is unobjectionable. The meaning of Eqs. (3.14) and (3.15) is that they are equations for intrinsic quantities. *Their consequences should then be independent of the state of rotation in which the system finds itself.* In fact these remarks are quite academic since they merely pose the microscopic problem: We must (i) find a *realization* of the operator ϕ within (iia) a space of states representing the rotational band such that (3.14) and (3.15) are satisfied. We may take advantage of the intrinsic property of these equations to replace (iia) by the weaker requirement (iib) that (3.14) and (3.15) be satisfied by a single linear combination of the states of the band. We add as a requirement (iii) that any scheme of solution when transcribed literally for the case of uniform translational motion must yield the total mass M as the inertial parameter.

At this point the problem takes a familiar turn, which we note only briefly. A solution is found by representing ϕ as a one-particle operator and by supposing that the *self-consistently determined* (*deformed*) *Hartree-Fock state* is the required linear combination of the members of the band. This leads us right back to the self-consistent cranking model.[36,37]

IV. PROJECTION METHODS FOR NONIDEAL COLLECTIVE MOTION

As emphasized at the conclusion of Sec. II, Eqs. (2.3) and (2.4) represent an idealization of the actual physical situation. We there proposed Eq. (2.27) as a step toward reality, which nevertheless retained our basic approach to the definition of collective motion. Here we provide for contrast a short summary of the projection methods, which constitute an alternative and apparently *distinct* attack on this problem.

[36] D. J. Thouless, Nucl. Phys. **21**, 225 (1960).
[37] D. J. Thouless and T. G. Valatin, Nucl. Phys. **31**, 211 (1962).

Following Wheeler-Griffin[4] and Peierls-Yoccoz[6] we may modify the notion of a generating state by keeping (2.3), but suppose thereby that only a finite number of states can be projected out of $|\alpha\rangle$. In practice, now, the generating state $|\alpha\rangle$ is a definite many-body wave function, e.g., a Hartree-Fock state or a BCS state corresponding to some assumed orientation or deformation α, the specific dependence on α being required in any actual microscopic calculation. This in turn means that the overlap

$$\langle\alpha|\alpha'\rangle=N(\alpha\alpha'), \tag{4.1}$$

though a localized function in the sense of (2.6) and (2.7), is no longer infinitely localized. For the states $|n\rangle$ in question, we have

$$\delta_{nn'}=\int d\alpha d\alpha'\langle n|\alpha\rangle N(\alpha\alpha')\langle\alpha'|n'\rangle. \tag{4.2}$$

This suggests that we replace (2.1) or even (2.14) by the equation

$$\int\mathcal{H}(\alpha\alpha')d\alpha'\langle\alpha|n\rangle=\omega_n\int N(\alpha\alpha')d\alpha'\langle\alpha'|n\rangle. \tag{4.3}$$

Comparing (4.2) and (4.3) with the equation

$$\langle n|H|n'\rangle=(W_0+\omega_n)\delta_{nn'}$$

$$=\int\langle n|\alpha\rangle\langle\alpha|H|\alpha'\rangle\langle\alpha'|n\rangle, \tag{4.4}$$

we may conclude that

$$\mathcal{H}(\alpha\alpha')=\langle\alpha|H|\alpha'\rangle-W_0N(\alpha\alpha'). \tag{4.5}$$

The formulation just given may be rendered more cogent if we remark that (4.3) is essentially the expression of the conventional variational principle

$$\delta\left\{\int d\alpha d\alpha'\langle n|\alpha\rangle[\langle\alpha|H|\alpha'\rangle - W_nN(\alpha\alpha')]\langle\alpha'|n\rangle\right\}=0. \tag{4.6}$$

The quantity to be varied and determined in this principle is the transformation function $\langle\alpha|n\rangle$.

Traditionally, this method has been used in the following sense: For a given trial $|\alpha\rangle$, we can compute $\mathcal{H}$ and N. The solution of (4.3) then yields the $\langle\alpha|n\rangle$, after which the approximate physical eigenstates $|n\rangle$ can be constructed from (2.3). This is not, however, the only possibility.

Recently a different formulation of the variational principle applied to a projected wave function has been utilized, one which is especially appropriate to the problems of translation and rotation. In this formulation we write

$$|n\rangle=\int d\alpha|\alpha;n\rangle\langle\alpha|n\rangle_0, \tag{4.7}$$

where $\langle\alpha|n\rangle_0$ is a prescribed transformation function (eigenfunctions of momentum or angular momentum), and we allow the generator state $|\alpha;n\rangle$ to depend (hopefully in a weak or essentially determinable way) on the quantum numbers of $|n\rangle$. The variational principle is then applied to the determination of $|\alpha;n\rangle$. Examples of this approach include the work of Peierls and Thouless[8] and Rouhaninejad and Yoccoz.[15] It is by no means evident in what sense these methods, conceived as *microscopic* approaches, can be reduced to simple and general phenomenological terms. Only for the first of the methods are general remarks, independent of the detailed nature of the generating state, possible. For example, we seek to reduce this approach to equivalent oscillator form by the consistent use of (2.6) and (2.7). Because in so doing it is necessary finally to return to the standard norm

$$\int\langle n|\alpha\rangle\langle\alpha|n'\rangle d\alpha=\delta_{nn'}, \tag{4.8}$$

the reduction should be carried out on the norm-independent structure

$$\omega_n\delta_{nn'}=\int\langle n|\alpha\rangle\{\langle\alpha|H|\alpha'\rangle-W_0N(\alpha\alpha')\}\langle\alpha'|n'\rangle \times\left[\int\langle n|\alpha\rangle N(\alpha\alpha')\langle\alpha'|n\rangle\right]^{-1/2} \times\left[\int\langle n'|\alpha\rangle N(\alpha\alpha')\langle\alpha'|n'\rangle\right]^{-1/2}. \tag{4.9}$$

Applying the expansion (2.6) to $\langle\alpha|H|\alpha'\rangle$ and to $\langle\alpha|\alpha'\rangle$ we now find to quadratic terms that $\mathcal{H}(\alpha,p_\alpha)$ is of oscillator form, with

$$W_0=(H^{(00)}/N^{(00)}), \tag{4.10}$$

$$(2B)^{-1}=\{[N^{(20)}H^{(00)}/(N^{(00)})^2]-[H^{(20)}/N^{(00)}]\}, \tag{4.11}$$

$$C_2=\{[H^{(02)}/N^{(00)}]-[N^{(02)}H^{(00)}/(N^{(00)})^2]\}. \tag{4.12}$$

These formulas are well known.[4,6] They may be seen to reduce, in the limit of an idealized degree of freedom, to the results of Sec. II, Eqs. (2.19)–(2.21).

The considerations of the previous paragraph require further discussion, however, in order to delimit their range of application. Thus Eq. (4.8) is generally compatible (involving only a renormalization) with (4.2) only in the case of translations and rotations. For the case of vibrations they are compatible only in the harmonic-oscillator approximation. Further examin-

ation of the expansion shows that there is no anharmonic equivalent to (4.2) and (4.3) because the effective Hamiltonian depends on the state n, and this would have to be compensated for by a dependence of $|\alpha\rangle$. Thus the demand that a model based on (4.2) and (4.3) be equivalent to an anharmonic oscillator requires of the generating state $|\alpha\rangle$ properties not possessed by the usual simple choices (Slater determinants, etc.) made for these. It thus appears that the simple phenomenological vibrational picture and the simple projection scheme are complementary rather than coincidental; where the phenomenological picture applies, the projection scheme cannot be too simple, and where the projection scheme applies in its most naive form, the phenomenological picture will probably fail. In practice, the projection scheme has been most useful for light nuclei.[21–27]

V. SUMMARY AND DISCUSSION

We have assumed that the relationship among a certain *finite* set of nuclear states of a given nucleus can be understood by supposing these to behave with respect to each other as the states of an oscillator or a rotator, albeit in each case an imperfect one. We have pointed out essentially two different phenomenological viewpoints toward this state of affairs, each of which has its microscopic counterpart.

In the first method we assume that the set of states can be described in the sense of a correspondence by a collective Hamiltonian. This leads to the limiting concept of an ideal collective variable and to a self-contained, although oversimplified, phenomenological theory. In this theory it is straightforward to introduce coupling between modes, although we have not discussed this question in the present paper. The corresponding microscopic theory has been partially developed[16] and appears, together with the phenomenological concept, to be the most natural basis for the study of collective effects in heavy nuclei, especially deformed ones.[11] Within the microscopic version two extensions have been studied. In the first, one considers states of high quantum numbers, where the collective effects are still apparent but the expansion of the collective Hamiltonian and other operators in a straight power series in the collective variables begins to fail. Here an altered phenomenology based on the cranking model has been introduced with remarkable success[38] and subsequently justified from the microscopic version[30] within the framework of the ideas of the ideal collective coordinate. Ultimately one must also consider the way in which the collective states cut off. The work done here[31] is only a beginning.

In the second method, we consider the technique of generator coordinates as originally introduced[3–7] and as subsequently modified.[8,14,15,29] In contrast to the previous method where one starts from the energy levels or, equivalently, from the collective Hamiltonian and the explicit form of the generator state *never* enters, here, in most attempts, one starts from the assumption that the generator state is simple and can therefore either be given *a priori* or calculated from the variational principle. There is no guarantee that any given generator state is equivalent to a simple phenomenological description; in general, this is not the case, even formally, except for the problems of rotations and translations where the invariance properties intervene. For the problem of rotations, Vehaar[18,19] has shown that the method of generator coordinates can be used to derive a completely general-looking phenomenological theory of rotations, not essentially different from some of the results of the method of ideal collective coordinates, as we shall show in a subsequent paper. The main objection to this approach is that the microscopic version of this theory, corresponding to the work of Peierls and Yoccoz,[6,7] is not in accord with experiment for heavy nuclei where the derivations might apply, and the wave function of Peierls and Thouless,[8] which can be said to correspond to experiment at least for a ground-state rotational band, does not fall within the framework of the phenomenological model. Nevertheless, as has been mentioned, the method of projected wave functions has so far proved to be the most valuable simple way of studying collective motion in light nuclei.

Since our present interest is in heavy nuclei, we shall concentrate our future efforts on applying the method of ideal collective coordinates as far as it can be stretched. In an accompanying article[35] we show by means of a simple model how the concepts of the present paper can be applied in the most direct fashion possible to a microscopic case.

[38] S. M. Harris, Phys. Rev. 138, B509 (1965).