General Formulas for Describing the Absorption of Polarized Photons by Oriented Nuclei

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The cross section for the absorption of polarized photons by oriented nuclei is given in terms of the nuclear orientation parameters and the photon polarization parameters for linear and circular polarization. All quantities are given in terms of the nuclear polarizabilities, which contain the electric and magnetic transition matrix elements. The polarizabilities of higher than zero rank contribute to the absorption cross section only if the nucleus is oriented. Without photon polarization, only the even-rank polarizabilities affect the absorption cross section. Photon polarization effects show up only with oriented nuclei. In that case the polarizabilities of odd rank are connected with circular photon polarization, while those of even rank are connected with linear photon polarization. The latter is not true if time-reversal invariance is violated. Finally, the contributions of the various multipole transitions to the polarizability of given rank cannot be separated experimentally.

RECENTLY, photonuclear experiments with oriented nuclei have become feasible,[1] and compared with experiments with unoriented nuclei they yield more detailed information about nuclear polarizabilities of higher than zero rank. This can be used as a sensitive test for nuclear models. Furthermore, it seems that monochromatic polarized photon beams will become available in the near future. Therefore it might be worthwhile to give the general expression of the cross section for the absorption of polarized photons by oriented nuclei in terms of the nuclear orientation parameters and the parameters describing linear and circular photon polarization. This way it will become apparent what kind of information one can obtain with such experiments.

We start from the optical theorem, which gives the total absorption cross section in terms of the imaginary part of the elastic forward scattering amplitude:

$$\sigma_a = (4\pi/k)\,\mathrm{Im}(\mathrm{Tr}\rho\bar{K}) = (4\pi/k)\,\mathrm{Im}\Big(\sum_{\lambda,\lambda',M_i,M_i'} \rho_{\lambda\lambda' M_i M_i'} \bar{K}_{M_i' M_i}{}^{\lambda'\lambda}\Big). \quad (1)$$

$\bar{K}$ is the scattering amplitude for elastic forward scattering and ρ is the density matrix of the initial photon and nuclear states. The general scattering amplitude in terms of the nuclear polarizabilities $P_j{}^{LL'\lambda\lambda'}$ is[2,3]

$$K_{M_f M_i}{}^{\lambda'\lambda} = -\sum_{L,M,L',M',j} (-)^{\lambda'+I_f-M_i+L+L'}(2j+1) \times\begin{pmatrix} I_f & j & I_i \\ -M_f & m & M_i \end{pmatrix}\begin{pmatrix} L & L' & j \\ M & M' & -m \end{pmatrix} \times P_j{}^{LL'\lambda\lambda'} D_{M,\lambda}{}^{L}(R) D_{M',-\lambda'}{}^{L'}(R'). \quad (2)$$

λ and λ' denote the circular polarization ($=\pm1$) of the incoming and outgoing photons, respectively. The nucleus changes its state from $|I_i M_i\rangle$ to $|I_f M_f\rangle$ during the scattering event, where I_i, M_i and I_f, M_f are spin and its projection of the initial and the final nuclear states, respectively. R and R' denote the rotations that transform the nuclear quantization axis into the direction of the incoming and outgoing photons. The rotation matrices are in the convention of Rose.[4] In terms of the nuclear electric and magnetic multipole matrix elements the polarizabilities are given by[2]

$$P_j{}^{LL'\lambda\lambda'} = P_j(EL,EL') - \lambda\lambda' P_j(ML,ML') - i\lambda P_j(ML,EL') - i\lambda' P_j(EL,ML') = \sum_{\nu,\nu'=0,1} (-i\lambda)^{\nu}(-i\lambda')^{\nu'} P_j(M^{\nu}L, M^{\nu'}L'), \quad (3)$$

with

$$P_j(M^{\nu}L,M^{\nu'}L') = (-)^{L'+j+I_i+I_f} \frac{i^{L+L'}k^{L}k'^{L'}}{2(2L-1)!!(2L'-1)!!}\left(\frac{(L+1)(L'+1)}{LL'}\right)^{1/2} \times\sum_n \left(\begin{Bmatrix} L & L' & j \\ I_i & I_f & I_n \end{Bmatrix} \frac{\langle I_f\|M^{\nu}(L)\|I_n\rangle\langle I_n\|M^{\nu'}(L')\|I_i\rangle}{E_n+E'+\frac{1}{2}i\Gamma_n} + (-)^{L+L'+j} \begin{Bmatrix} L' & L & j \\ I_i & I_f & I_n \end{Bmatrix} \frac{\langle I_f\|M^{\nu'}(L')\|I_n\rangle\langle I_n\|M^{\nu}(L)\|I_i\rangle}{E_n-E-\frac{1}{2}i\Gamma_n} \right) - \delta_{L1}\delta_{L'1}\delta_{j0}\delta_{kk'}\delta_{if}[3(2I_i+1)]^{1/2}Z^2e^2/AMc^2. \quad (4)$$

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E, k and E', k' denote the energies and the wave numbers of the incoming and the outgoing photons, respectively. The summation n in (4) goes over all nuclear states of energy E_n and width Γ_n. The ground state is normalized to zero energy. In (3) and (4) we have used the notation $M^0L \equiv EL$ to describe electric transitions of order L, i.e., parity change $(-)^L$, with the multipole operator $M^0(L,m) \equiv E(L,m)$ and $M^1L \equiv ML$ to describe magnetic transitions of order L, i.e., parity change $(-)^{L+1}$, with the multipole operator $M^1(L,m) \equiv M(L,m)$. The electric and magnetic multipole operators include charge and spin contributions and they are defined in the convention of Brink and Satchler.[5] The reduced matrix elements that occur in (4) are defined in the following way:

$$\langle I_f M_f | T_{LM} | I_i M_i \rangle = (-)^{I_f - M_f} \begin{pmatrix} I_f & L & I_i \\ -M_f & M & M_i \end{pmatrix} \langle I_f \| T_L \| I_i \rangle, \quad (5)$$

where T_{LM} is an irreducible tensor of rank L. In the convention of Ref. 5 for the multipole operators and with (5) we have

$$\langle I_i \| M^\nu(L) \| I_n \rangle^* = (-)^{I_n - I_i + \nu + 1} \langle I_n \| M^\nu(L) \| I_i \rangle. \quad (6)$$

If we assume time-reversal invariance all transition matrix elements can be made real or pure imaginary depending on whether $L+\nu$ is even or odd.[6] For a given initial and final nuclear state the polarizability $P_j(M^\nu L, M^{\nu'} L')$ contributes only if $(-)^{L+L'+\nu+\nu'} = \pi_i \pi_f$ is fulfilled, where π_i and π_f are the parities of the initial and the final nuclear states.

For elastic forward scattering ($E=E'$, $R=R'$), the scattering amplitude becomes

$$\bar{K}_{M_i' M_i}{}^{\lambda'\lambda} = -\sum_{L,L',j} (-)^{\lambda + I_i - M_i' + L + L'} (2j+1) \times \begin{pmatrix} I_i & j & I_i \\ -M_i' & m & M_i \end{pmatrix} \begin{pmatrix} L & L' & j \\ \lambda & -\lambda' & \lambda' - \lambda \end{pmatrix} \times P_j{}^{LL'\lambda\lambda'} D_{m,\lambda-\lambda'}{}^j(R). \quad (7)$$

Here we have made use of the Clebsch-Gordan series of the rotation matrices and the orthogonality property of the $3j$ symbols. As a shorthand we introduce the notation

$$\hat{P}_j{}^{\lambda'\lambda} = \sum_{L,L'} (-)^{L+L'+\lambda} \begin{pmatrix} L & L' & j \\ \lambda & -\lambda' & \lambda' - \lambda \end{pmatrix} P_j{}^{LL'\lambda\lambda'}. \quad (8)$$

Later we will need the relation

$$\hat{P}_j{}^{-\lambda'-\lambda} = (-)^j \hat{P}_j{}^{\lambda'\lambda}, \quad (9)$$

which one easily proves from (3) and (8) taking into account $P_j(M^\nu L, M^{\nu'} L') = 0$ if $(-)^{L+L'+\nu+\nu'} = -1$.

With the aid of (6) and applying time-reversal invariance, the polarizabilities of (4) for elastic scattering become

$$P_j(M^\nu L, M^{\nu'} L') = -\frac{i^{L+L'} k^{L+L'}}{2(2L-1)!!(2L'-1)!!} \times [(L+1)(L'+1)/(LL')]^{1/2} \sum_n (-)^{I_n + I_i} \times \begin{Bmatrix} L & L' & j \\ I_i & I_i & I_n \end{Bmatrix} \langle I_i \| M^\nu(L) \| I_n \rangle \langle I_i \| M^{\nu'}(L') \| I_n \rangle \times \left(\frac{(-)^j}{E_n + E + \frac{1}{2} i \Gamma_n} + \frac{1}{E_n - E - \frac{1}{2} i \Gamma_n} \right) - \delta_{L1} \delta_{L'1} \delta_{j0} [3(2I_i+1)]^{1/2} Z^2 e^2 / AMc^2. \quad (10)$$

From this expression one obtains immediately

$$P_j(M^\nu L, M^{\nu'} L') = P_j(M^{\nu'} L', M^\nu L). \quad (11)$$

Then we get from (3)

$$P_j{}^{LL'\lambda\lambda'} = (-)^{(L+L')(\lambda-\lambda')/2} P_j{}^{L'L\lambda\lambda'}, \quad (12)$$

and $\hat{P}_j{}^{\lambda'\lambda}$ of (8) becomes

$$\hat{P}_j{}^{\lambda'\lambda} = \sum_{L \le L'} \frac{(-)^{\lambda+L+L'}}{1+\delta_{LL'}} \left[\begin{pmatrix} L & L' & j \\ \lambda & -\lambda' & \lambda' - \lambda \end{pmatrix} + (-)^{(L+L')(\lambda-\lambda')/2} \begin{pmatrix} L' & L & j \\ \lambda & -\lambda' & \lambda' - \lambda \end{pmatrix} \right] P_j{}^{LL'\lambda\lambda'}. \quad (13)$$

In more detail,

$$\hat{P}_j{}^{11} = -\sum_{L \le L'} \frac{2}{1+\delta_{LL'}} \begin{pmatrix} L & L' & j \\ 1 & -1 & 0 \end{pmatrix} \times [P_j(EL,EL') - P_j(ML,ML') + iP_j(ML,EL') + iP_j(EL,ML')], \quad (14)$$

$$\hat{P}_j{}^{-11} = -(1+(-)^j) \sum_{L \le L'} \frac{1}{1+\delta_{LL'}} \begin{pmatrix} L & L' & j \\ 1 & 1 & -2 \end{pmatrix} \times [P_j(EL,EL') + P_j(ML,ML') + iP_j(ML,EL') - iP_j(EL,ML')]. \quad (15)$$

Using the shorthand of (8) we get now

$$\bar{K}_{M_i' M_i}{}^{\lambda'\lambda} = -\sum_j (-)^{I_i - M_i'} (2j+1) \times \begin{pmatrix} I_i & j & I_i \\ -M_i' & m & M_i \end{pmatrix} \hat{P}_j{}^{\lambda\lambda'} D_{m,\lambda-\lambda'}{}^j(R). \quad (16)$$

The density matrix ρ is a direct product of the density matrices σ of the photon states and τ of the nuclear

[5] D. M. Brink and G. R. Satchler, *Angular Momentum* (Clarendon Press, Oxford, England, 1962).

[6] L. C. Biedenharn and M. E. Rose, Rev. Mod. Phys. **25**, 729 (1953).

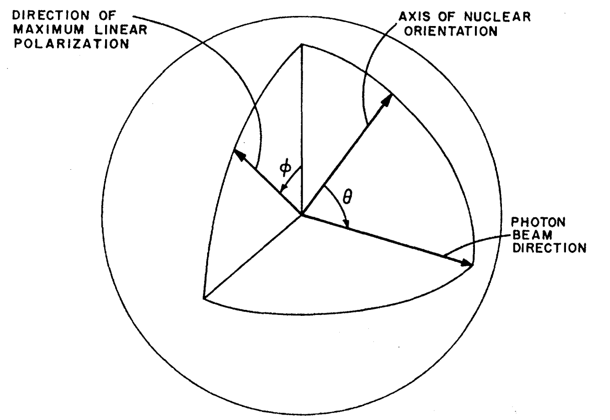

FIG. 1. The geometrical meaning of the angles θ and φ occurring in Eqs. (29)–(35).

states:

$$\rho_{\lambda\lambda' M_i M_i'} = \sigma_{\lambda\lambda'} \tau_{M_i M_i'} . \tag{17}$$

It is convenient to choose the nuclear quantization axis such that τ becomes diagonal. This is the definition of the orientation axis. Then the density matrix τ is specified by the $2I_i+1$ diagonal elements $\tau_{M_iM_i}$. However, it is more useful to deal with the orientation parameters f_α,[7] which are defined by

$$f_\alpha = g(I_i,\alpha)^{-1} \sum_{M_i} (-)^{I_i-M_i} \begin{pmatrix} I_i & I_i & \alpha \\ M_i & -M_i & 0 \end{pmatrix} \tau_{M_iM_i} , \tag{18}$$

with

$$g(I_i,\alpha) = I_i^{\alpha} \binom{2\alpha}{\alpha} [(2I_i-\alpha)!/(2I_i+\alpha+1)!]^{1/2} . \tag{19}$$

The inverse relation is

$$\tau_{M_iM_i} = (-)^{I_i-M_i} \sum_\alpha (2\alpha+1) \times \begin{pmatrix} I_i & I_i & \alpha \\ M_i & -M_i & 0 \end{pmatrix} g(I_i,\alpha) f_\alpha , \tag{20}$$

and as is seen from (18) the f_α are the irreducible components of the nuclear density matrix. If the nuclei are aligned then f_α vanishes for odd α. For unoriented nuclei we have $f_\alpha = \delta_{\alpha 0}$.

The density matrix σ of the photon polarization states is[7]

$$\sigma = \tfrac{1}{2}(\mathbf{1} + \mathbf{P}\cdot\mathfrak{S}) , \tag{21}$$

$\mathbf{1}$ is the 2×2 unit matrix, $\mathfrak{S}$ is the Pauli spin vector, and $\mathbf{P}$ is the Poincaré vector. The components are

$$\begin{aligned} P_{\bar{x}} &= \sigma_{1-1} + \sigma_{-11} , \\ P_{\bar{y}} &= i(\sigma_{1-1} - \sigma_{-11}) , \\ P_{\bar{z}} &= \sigma_{11} - \sigma_{-1-1} . \end{aligned} \tag{22}$$

Furthermore, we give the matrix elements in terms of these components:

$$\begin{aligned} \sigma_{\pm1\pm1} &= \tfrac{1}{2}(1 \pm P_{\bar{z}}) , \\ \sigma_{\pm1\mp1} &= \tfrac{1}{2}(P_{\bar{x}} \mp iP_{\bar{y}}) . \end{aligned} \tag{23}$$

The degree of polarization s is given by the length of the Poincaré vector:

$$s = |\mathbf{P}| . \tag{24}$$

If $s=1$ we have full polarization and if $s<1$ we have only partial polarization. $P_c = P_{\bar{z}}$ is the degree of circular polarization (right handed if $P_c>0$ and left handed if $P_c<0$). In other words $P_c = (I_R - I_L)/(I_R + I_L)$, where I_R and I_L are the intensities measured by a detector that is sensitive to right- or left-handed circular polarization, respectively. The degree of linear polarization is $P_l = (P_{\bar{x}}^2 + P_{\bar{y}}^2)^{1/2}$. $P_{\bar{x}}$ is the difference of the probabilities for finding a photon linear polarized in the y or x axis, if the z axis coincides with photon beam direction, which is assumed throughout this paper. That means $P_{\bar{x}} = (I_y - I_x)/(I_x + I_y)$, where I_x and I_y are the intensities measured by a detector that is sensitive to photons linear polarized in the x or y axis, respectively.

By a special rotation around the z axis we can make $P_{\bar{y}}$ vanish. Then we have maximum linear polarization in the x or y axis depending on $P_{\bar{x}}<0$ or $P_{\bar{x}}>0$ and $P_l = |P_{\bar{x}}|$. Therefore, without loss of generality we can choose the case $P_{\bar{y}}=0$ and $P_{\bar{x}}<0$. Then we have maximum linear polarization in the x direction.

Since σ and τ are Hermitian we obtain

$$\mathrm{Im}\Big(\sum_{\lambda,\lambda',M_i,M_i'} \sigma_{\lambda\lambda'} \tau_{M_iM_i'} \bar{K}_{M_i'M_i}{}^{\lambda'\lambda} \Big) = \sum_{\lambda,\lambda',M_i,M_i'} \sigma_{\lambda\lambda'} \times \tau_{M_iM_i'} (1/2i) (\bar{K}_{M_i'M_i}{}^{\lambda'\lambda} - \bar{K}_{M_iM_i'}{}^{\lambda\lambda'*}) . \tag{25}$$

Further, from (16),

$$\begin{aligned} \bar{K}_{M_i'M_i}{}^{\lambda'\lambda} - \bar{K}_{M_iM_i'}{}^{\lambda\lambda'*} &= -\sum_j (2j+1) \Bigg[(-)^{I_i-M_i'} \begin{pmatrix} I_i & j & I_i \\ -M_i' & m & M_i \end{pmatrix} \hat{P}_j{}^{\lambda'\lambda} D_{m,\lambda-\lambda'}{}^{j} \\ &\qquad - (-)^{I_i-M_i} \begin{pmatrix} I_i & j & I_i \\ -M_i & -m & M_i' \end{pmatrix} \hat{P}_j{}^{\lambda\lambda'*} D_{-m,\lambda'-\lambda}{}^{j*} \Bigg] \\ &= -\sum_j (2j+1) (-)^{I_i-M_i'} \begin{pmatrix} I_i & j & I_i \\ -M_i' & m & M_i \end{pmatrix} D_{m,\lambda-\lambda'}{}^{j} (\hat{P}_j{}^{\lambda'\lambda} - (-)^{\lambda-\lambda'} \hat{P}_j{}^{\lambda\lambda'*}) . \end{aligned} \tag{26}$$

[7] H. A. Tolhoek and J. A. M. Cox, Physica 19, 101 (1953).

With the aid of (18) and the orthogonality property of the $3j$ symbols we get

$$\sum_{M_i,M_i'} \tau_{M_iM_i'}(\bar{K}_{M_i'M_i}{}^{\lambda'\lambda}-\bar{K}_{M_iM_i'}{}^{\lambda\lambda'*})=-\sum_j(2j+1)g(I_i,j)f_jD_{0,\lambda-\lambda'}{}^j(\hat{P}_j{}^{\lambda'\lambda}-(-)^{\lambda-\lambda'}\hat{P}_j{}^{\lambda\lambda'*}). \quad (27)$$

The use of (23) with $P_c=P_{\bar{z}}$, $P_l=-P_{\bar{x}}$, and $P_{\bar{y}}=0$ leads to

$$\mathrm{Im}\Big(\sum_{\lambda,\lambda',M_i,M_i'}\sigma_{\lambda\lambda'}\tau_{M_iM_i'}\bar{K}_{M_i'M_i}{}^{\lambda'\lambda}\Big)$$

$$=-\frac{1}{2i}\sum_j(2j+1)g(I_i,j)f_j\{P_j(\cos\theta)[\sigma_{11}(\hat{P}_j{}^{11}-\hat{P}_j{}^{11*})+\sigma_{-1-1}(\hat{P}_j{}^{-1-1}-\hat{P}_j{}^{-1-1*})]$$
$$+\sigma_{1-1}(\hat{P}_j{}^{-11}-\hat{P}_j{}^{1-1*})D_{0,2}{}^j+\sigma_{-11}(\hat{P}_j{}^{1-1}-\hat{P}_j{}^{-11*})D_{0,-2}{}^j\}$$
$$=-\tfrac{1}{2}\sum_j(2j+1)g(I_i,j)f_j\{[(1+(-)^j)+(1-(-)^j)P_c]P_j(\cos\theta)\,\mathrm{Im}\hat{P}_j{}^{11}$$
$$+P_ld_{0,2}{}^j(\theta)[-(1+(-)^j)\cos2\varphi\,\mathrm{Im}\hat{P}_j{}^{-11}+(1-(-)^j)\sin2\varphi\,\mathrm{Re}\hat{P}_j{}^{-11}]\}. \quad (28)$$

We have further used (9) to obtain the last line. Since $\hat{P}_j{}^{-11}$ is proportional to $(1+(-)^j)$ the last term in (28) vanishes.[8] $P_j(\cos\theta)$ are the Legendre polynomials and $d_{0,2}{}^j(\theta)=P_j{}^2[(j-2)!/(j+2)!]^{1/2}$, $P_j{}^2$ being the associate Legendre polynomial. θ is the angle between the nuclear orientation axis and the photon beam direction, and φ is the angle between the plane defined by the orientation axis and the beam direction and the direction of maximum linear polarization (see Fig. 1). Now we can express the absorption cross section in terms of the nuclear orientation parameters and the photon linear and circular polarization degrees:

$$\sigma_a(E,\sigma,\tau,\theta,\varphi)=\sum_j f_j(\sigma_j{}^0(E,\theta)+P_l\sigma_j{}^l(E,\theta,\varphi)+P_c\sigma_j{}^c(E,\theta)), \quad (29)$$

where

$$\sigma_j{}^0=(1+(-)^j)\sigma_j, \quad (30)$$

$$\sigma_j{}^c=(1-(-)^j)\sigma_j, \quad (31)$$

$$\sigma_j=-(2\pi/k)(2j+1)g(I_i,j)\times P_j(\cos\theta)\,\mathrm{Im}\hat{P}_j{}^{11},\quad j=0,1,2,\cdots \quad (32)$$

$$\sigma_j{}^l=(2\pi/k)(2j+1)g(I_i,j)(1+(-)^j)\cos2\varphi\times d_{0,2}{}^j(\theta)\,\mathrm{Im}\hat{P}_j{}^{-11},\quad j=2,3,4,\cdots. \quad (33)$$

From these expressions one can draw the following general conclusions:

(i) For unoriented nuclei only the scalar polarizabilities contribute to the absorption cross section. In order to observe polarizabilities of higher rank than zero one needs nuclear orientation.

(ii) If the photon beam is not polarized only the even-rank polarizabilities contribute. In that case nuclear alignment is sufficient.

(iii) The absorption cross section depends on the photon polarization only if the nuclei are oriented. The odd-rank polarizabilities are connected with the circular-polarization parameter P_c, while for linear polarization the even-rank polarizabilities affect the absorption cross section depending on the direction of maximum linear polarization. Equation (33) shows that $\sigma_j{}^l$ is symmetrical to the plane that is determined by the orientation axis and the beam direction, i.e., $\sigma_j{}^l(E,\theta,-\varphi)=\sigma_j{}^l(E,\theta,\varphi)$.[9] Furthermore, $\sigma_j{}^l$ changes sign if one changes the direction of maximum linear polarization by an angle of 90°. For $\varphi=45°$, $\sigma_j{}^l$ vanishes.

(iv) In general the contributions of the various multipole transitions to the polarizability of given rank j, e.g., $P_j(E1,E1)$, $P_j(M1,M1)$, $P_j(M1,E2)$, $\cdots$, cannot be separated experimentally since they have all the same angular distribution. This can be done in a photon scattering experiment.[2,3]

Finally, we give explicit expressions for electric dipole ($E1$), electric quadrupole ($E2$), and magnetic dipole ($M1$) absorption:

$$\sigma_j=(2\pi/k)(2j+1)g(I_i,j)P_j(\cos\theta)\left\{\begin{pmatrix}1&1&j\\1&-1&0\end{pmatrix}\times[\mathrm{Im}(P_j(E1,E1)-P_j(M1,M1))]+2\begin{pmatrix}1&2&j\\1&-1&0\end{pmatrix}\mathrm{Re}P_j(M1,E2)+\begin{pmatrix}2&2&j\\1&-1&0\end{pmatrix}\mathrm{Im}P_j(E2,E2)\right\},\quad j=0,1,2,3,4; \quad (34)$$

$$\sigma_j{}^l=(4\pi/k)(2j+1)g(I_i,j)d_{0,2}{}^j(\theta)\cos2\varphi\times\left[\begin{pmatrix}1&1&j\\1&1&-2\end{pmatrix}\mathrm{Im}(P_j(E1,E1)+P_j(M1,M1))+2\begin{pmatrix}1&2&j\\1&1&-2\end{pmatrix}\mathrm{Re}P_j(M1,E2)+\begin{pmatrix}2&2&j\\1&1&-2\end{pmatrix}\mathrm{Im}P_j(E2,E2)\right],\quad j=2,4, \quad (35)$$

[8] This is not true if time-reversal invariance does not apply.

[9] If time-reversal invariance is not fulfilled this would show up in an asymmetry [O. C. Kistner, Phys. Rev. Letters **19**, 872 (1967)].

with

$$\mathrm{Im}P_j(M^\nu L,M^\nu L)$$

$$=(-)^{\nu+1}\frac{k^{2L}}{2[(2L-1)!!]^2}\frac{L+1}{L}\sum_n(-)^{I_i+I_n}$$

$$\times\begin{Bmatrix}L & L & j\\ I_i & I_i & I_n\end{Bmatrix}|\langle I_i\|M^\nu(L)\|I_n\rangle|^2\Gamma_n$$

$$\times[EE_n(1+(-)^j)+\tfrac{1}{2}(E_n^2+E^2+\tfrac{1}{4}\Gamma_n^2)(1-(-)^j)]$$

$$\times[(E_n^2-E^2+\tfrac{1}{4}\Gamma_n^2)^2+(\Gamma_nE)^2]^{-1}, \quad (36)$$

and

$$\mathrm{Re}P_j(M1,E2)=-(k^3/2\sqrt{3})\sum_n(-)^{I_i+I_n}$$

$$\times\begin{Bmatrix}1 & 2 & j\\ I_i & I_i & I_n\end{Bmatrix}\langle I_i\|M(1)\|I_n\rangle\langle I_i\|E(2)\|I_n\rangle\Gamma_n$$

$$\times[EE_n(1+(-)^j)+\tfrac{1}{2}(E_n^2+E^2+\tfrac{1}{4}\Gamma_n^2)(1-(-)^j)]$$

$$\times[(E_n^2-E^2+\tfrac{1}{4}\Gamma_n^2)^2+(\Gamma_nE)^2]^{-1}. \quad (37)$$

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Concept of Ideal Collective Coordinate as the Foundation for a Phenomenological Theory of Nuclear Collective Motion: Basic Ideas and Relation to Other Phenomenological Methods*

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The concept of an ideal collective coordinate is introduced by means of the following example: Consider a one-dimensional vibration of a many-body system in the sense that a large subset of states $|n\rangle$ of the system exhibits an energy spectrum and relative transition probabilities following the laws of the (in general anharmonic) oscillator described by $\mathcal{H}\ (p_\alpha,\alpha)\ (\alpha|n)=\omega_n(\alpha|n)$. We suppose the set of many-body states $|n\rangle$ to extend indefinitely, and we take the transform $|\alpha\rangle=\sum|n\rangle\ (n|\alpha)$ to define a many-body generating state of the band which is precisely localized in α space. The basic assumption of collectivity, that changing the state of at most a few particles cannot much alter the value of α, is shown to be sufficient to derive a phenomenological theory from the many-body starting point. The phenomenological aspects of a recent theory of rotations due to Villars is seen to be contained in the above formulation as a special case. A brief review is given of the generator coordinate and similar projection methods in order to exhibit their relationship with the present method.

I. INTRODUCTION

IN the traditional phenomenological approach to the nuclear collective Hamiltonian,[1,2] one assumes that somehow the fundamental microscopic Hamiltonian can be written approximately as the sum of a collective and an intrinsic part, the wave function having, in first approximation, essentially a product form. *Per contra*, it is remarkable that *none* of the existing versions of fully quantum-mechanical microscopic theories of collective motion[3–17] is able to exploit this suggestion. The reason for this "failure" is physically clear: In contradistinction to the case of molecules, there are no generally valid

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