Inelastic electron scattering at low momentum transfer has been found to be a particularly useful source of model-independent data in cases where the first Born approximation could be used. In most cases, however, this approximation is not good enough and the distortion of the electron waves in the Coulomb field of the nucleus has to be taken into account. This effect seemed to destroy the model independence of the data, since the Coulomb corrections could only be calculated for a given nuclear model.

The new representation of the Coulomb corrections that is presented in this paper offers a possibility of determining the corrections in a model-independent way. It is demonstrated that this can only be done by measuring the differential cross section at low momentum transfer for different electron energies. Under these conditions the Coulomb corrections can be determined directly from experiment. The model independence, which is the main purpose of our representation, is fully restored.

Although this method has been primarily devised for a model-independent analysis, it can also be used for the calculation of the Coulomb corrections for any given nuclear model. It provides a simple way of finding these corrections without making compulsory the use of a digital computer.

The results given in this paper deal with the Coulomb corrections for monopole and quadrupole excitations and for low momentum transfer. The same method can be used for dipole as well as higher multipole transitions, but it cannot be applied to inelastic electron scattering at high momentum transfer. In this case a different approach such as outlined in Sec. IV D seems to be the appropriate extension of this method.

ACKNOWLEDGMENTS

The author would like to thank Professor J. Weneser, Professor P. A. Brix, and Professor C. K. Bockelman for stimulating and helpful discussions. An encouraging suggestion by Professor T. D. Lee, which was communicated to me by J. Weneser, is also gratefully acknowledged.

PHYSICAL REVIEW VOLUME 171, NUMBER 4 20 JULY 1968

Influence of Breakup Channels in Elastic Scattering of Composite Particles*

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(Received 12 October 1967; revised manuscript received 28 February 1968)

We treat d-α elastic scattering, using a simple, physically motivated approximation to include the effects of the $He^4(d,np)He^4$ (breakup) channel. This article extends and amplifies preliminary results published earlier. A treatment of the breakup amplitude is given, in addition to a more complete derivation of the dispersion relation for the elastic amplitude. We are able to obtain a closed-form solution for the elastic particle-wave amplitudes in the one-pole approximation. We compare the elastic phase shifts parametrized in this way with experiment, obtaining good agreement. We thus estimate with some confidence the contribution arising from the inclusion of the breakup channel. Since we treat the (tightly bound) He^4 nucleus as an elementary particle, our model may be regarded as representing scattering on a polarizable target. From this viewpoint we discuss the implications our findings may have for several nuclear structure and reaction theories.

INTRODUCTION

MANY mathematical techniques have been developed in an effort to understand the enormous complexity of nuclei and the bewildering variety of their reactions. Despite numerous attempts to provide a unified picture of nuclear structure and reactions, there is as yet no general theory.

In this paper, we present an intrinsically rather simple calculation that nevertheless has qualitative implications for several structure and reaction theories. These theories are the optical model,[1] R-matrix theory,[2]

* This work made use of Princeton University Computer facilities supported in part by the National Science Foundation Grant No. GP 579.

† Formerly at Princeton University, supported in part by the Air Force Office of Scientific Research and the U. S. Atomic Energy Commission.

‡ Supported in part by the National Science Foundation.

[1] G. E. Brown, Rev. Mod. Phys. **31**, 893 (1959).

[2] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958).

the extended shell model,[3] dispersion theory,[4] and the cluster model of light nuclei.[5,6] We shall discuss these qualitative implications at some length.

In a previous article,[7] we constructed a simple, exactly soluble model of deuteron-He^4 elastic scattering that included approximately the deuteron breakup channel. We decided to study the d-He^4 system for several reasons. First, its three-body (breakup) threshold lies very close to its elastic threshold, whereas the next (He^3+H^3) inelastic threshold lies relatively far ($\sim$12 MeV) away. Our original interest was oriented toward the three-body problem, and this seemed a relatively straightforward way to examine three-body effects isolated from the spin and symmetry difficulties that beset the three-nucleon problem,[8] and from the inelastic effects characterizing heavier systems. Secondly, the Li^6 ground state ($J^\pi=1^+$, $T=0$) is only weakly bound with respect to the $d+He^4$ channel, and so it seemed reasonable that a unitarized pole approximation[9,10] should be moderately successful, at least in the $J^\pi=1^+$, $L'=L=0$ channel. Finally, the profusion of experimental data[11] on this system enabled us to convince ourselves of the consistency and essential validity of our approach by direct comparison with experiment, and therefore lent weight to our estimates of the effects of including the breakup channel.

Our model is constructed by the following procedure: We treat the He^4 plus deuteron system as three elementary particles ($n+p+\alpha$) and assume that the coupling to the He^3+H^3 channel is weak. We then write down the (exact) unitarity relation, including only elastic and breakup thresholds. A rather simple, physically motivated approximation allows the decoupling of the equations for the elastic and breakup amplitudes, permitting us to write a partial-wave dispersion relation for, say, the elastic [$He^4(d,d)He^4$] on-shell amplitude alone. This dispersion relation will (for the $J=1^+$, $T=0$ channel) contain either a left-hand cut[4] alone or a pole (representing the Li^6 ground state) in addition to the left-hand cut. The residue of the bound-state pole may be determined either by insisting that the formulation without it be equivalent to the formulation with it or by requiring that the bound-state renormalization constant of the Li^6 ground state be zero, in accordance with our prejudices that Li^6 is a composite rather than an elementary particle. It turns out that both of these procedures yield identical results and so the choice is a matter of taste and convenience.

Although the preceding description of our formalism sounds formidable, this is really the result of the specialized jargon that has grown up in the field of dispersion theory. Actually, the methods we employ are extremely simple at our level of sophistication. Since the result we shall ultimately derive is a generalization of the usual effective-range approximation, we begin with a brief review of the one-pole approximation in two-body scattering and its equivalence to the usual effective-range expansion. We then generalize this formalism by using the three-body unitarity relation appropriate to our model of $d+\alpha$. Section II describes in some detail our approximation scheme for eliminating the breakup channel and introduces experimental evidence[12] for the validity of this approximation scheme. In Sec. III, we show how the breakup amplitude may be calculated in terms of the elastic amplitude by solving the Omnes-Mushkelishvili equation.[4,13] Section IV contains a description of our comparison with experiment, together with some remarks on various approximation schemes such as the Born and impulse approximations. In Sec. V, we describe our estimates of the effect of the three-body channel and discuss the implications of our model calculation for the various nuclear theories enumerated above.

I. DISPERSION THEORETIC APPROACH TO LOW-ENERGY SCATTERING

A. Review of Results in the Two-Particle Case

We first briefly review the familiar case of two-particle (single-channel) elastic scattering. The partial-wave amplitude $f_l^{(+)}(E)$ is regarded as the boundary value

$$f_l^{(+)}(E)=\lim_{\eta\to 0^+}F(E+i\eta) \tag{1}$$

of a real analytic function $F(z)$. $F(z)$ has singularities only on the real z axis[4]; these are the right-hand cut (arising from unitarity) which runs from 0 to $+\infty$, the left-hand cut (arising from the interaction) which runs, in the case of the Yukawa potential $V(r)=ge^{-\mu r}/\mu r$, from $E=-\mu^2/8m$ to $-\infty$, and finally, poles (representing bound states of the system) which may lie anywhere to the left of $z=0$. We remind the reader that the "left-hand cut" need not be a cut: The Hulthén potential $V(r)=g[\exp(\mu r)-1]^{-1}$ and the exponential potential $V(r)=g\exp(-\mu r)$ both produce infinite sequences of poles beginning at $E=-\mu^2/8m$.

[3] W. MacDonald, Nucl. Phys. **54**, 393 (1963); H. A. Weidenmuller, *ibid.* **75**, 189 (1966); C. Bloch, in *Many-Body Description of Nuclear Structure and Reactions*, edited by C. Bloch (Academic Press Inc., New York, 1966), p. 394.

[4] I. S. Shapiro, *Selected Topics in Nuclear Theory* (International Atomic Energy Commission, Vienna, 1963), p. 85.

[5] D. R. Thompson and Y. C. Tang, Phys. Rev. Letters **19**, 87 (1967).

[6] D. Brink, in *Many-Body Description of Nuclear Structure and Reactions*, edited by C. Bloch (Academic Press Inc., New York, 1966), p. 247.

[7] P. M. Fishbane and J. V. Noble, Phys. Rev. **160**, 880 (1967); hereinafter called I.

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[9] S. Weinberg, Phys. Rev. **137**, B672 (1965).

[10] S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963), p. 7.

[11] L. C. McIntyre and W. Haeberli, Nucl. Phys. **A91**, 382 (1967).

[12] K. Nagatani, T. A. Tombrello, and D. A. Bromley, Phys. Rev. **140**, B824 (1965).

[13] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), pp. 533 and 908.

$F(z)$ is most easily studied using the Weiner-Hopf[10,14] factorization. Quite generally, $F(z)$ may be written as a ratio of two functions,

$$F(z)=N(z)/D(z). \tag{2}$$

The function $D(z)$ is taken to have only the right-hand cut, whereas $N(z)$ has the left-hand cut. [The poles at the bound-state energies $z=-z_n$, $n=1,\cdots$, may be put into $N(z)$; it is also possible to make these poles appear as zeros of $D(z)$. We shall soon see why zeros of $D(z)$ have traditionally been connected with the idea of compositeness.] Since $D(z)$ has only the right-hand cut and is nonsingular, we may immediately write it as

$$D(z)=1+\frac{1}{\pi}\int_0^\infty dx\frac{\mathrm{Im}D(x+i\epsilon)}{x-z}, \tag{3}$$

where we have used Cauchy's theorem together with the conventional normalization $\lim_{|z|\to\infty}D(z)=1$. At this point, we employ the elastic unitarity relation (m is the reduced mass)

$$\mathrm{Im}f_l^{(+)}(E)=(2mE)^{1/2}|f_l^{(+)}(E)|^2. \tag{4}$$

We note that [since $N(z)$ is real on the right-hand cut]

$$\lim_{\epsilon\to0}\mathrm{Im}F(x+i\epsilon)=-N(x)\mathrm{Im}D(x+i\epsilon)/|D(x+i\epsilon)|^2$$
$$=(2mx)^{1/2}N^2(x)/|D(x+i\epsilon)|^2.$$

Thus,

$$\lim_{\epsilon\to0}\mathrm{Im}D(x+i\epsilon)=-(2mx)^{1/2}N(x), \tag{5}$$

leading to

$$D(z)=1+\frac{1}{\pi}\int_0^\infty dx\frac{(2mx)^{1/2}N(x)}{z-x}. \tag{6}$$

Similarly (we suppose here that there are no poles in N),

$$N(z)=-\frac{1}{\pi}\int_{\mathrm{lhc}}dx\frac{D(x)\,\mathrm{Im}F(x+i\epsilon)}{x-z}. \tag{7}$$

Let us represent the left-hand cut by one simple pole: this amounts to the replacement $N(z)\to\lambda/(z+E_0)$.[10] We can immediately perform the integration in (6) and obtain

$$D(z)=1-(2m)^{1/2}\lambda/(E_0^{1/2}+(-z)^{1/2}). \tag{8}$$

The s-wave scattering amplitude is thus

$$f_0^{(+)}(E)=\{[E_0/\lambda-(2mE_0)^{1/2}]+E/\lambda-i(2mE)^{1/2}\}^{-1}, \tag{9}$$

which corresponds exactly with the usual effective-range approximation $(2mE)^{1/2}\cot\delta_0(E)\simeq1/a_0+r_e mE$.[15] As has been pointed out many times before,[10] the effective-range approximation and the one-pole approximation lead to identical forms for the scattering amplitude. We see that by taking $\lambda=(E_0/2m)^{1/2}+(B/2m)^{1/2}$, we can make Eq. (8) have a zero at $z=-B$; the binding energy can be made arbitrarily large by making λ sufficiently large. Also, by making λ small or negative we can prevent Eq. (8) from having a zero, and so there will be no bound state. The fact that this sort of zero of $D(z)$ can be produced by increasing the strength of the two-particle interaction indicates that the bound state of the two particles is composite, rather than elementary.

Let us examine in detail the connection between the moving zero of D and the composite nature of the bound state. Following Weinberg[9] we suppose that the two particles are described by a Hamiltonian

$$H=H_0+V, \tag{10}$$

where H_0 describes the noninteracting particles and V is their interaction. The noninteracting Hamiltonian H_0 may have a discrete state ψ_0 with the same quantum numbers as the bound state ψ_B of the interacting system. This latter state is defined by

$$(H+B)|\psi_B\rangle=0. \tag{11}$$

The spectrum of H_0 is composed of continuum states $|\mathbf{k}\rangle$ given by

$$(H_0-E_k)|\mathbf{k}\rangle=0 \tag{12}$$

and the discrete state ψ_0 given by

$$(H_0+B_0)|\psi_0\rangle=0. \tag{13}$$

The overlap $|\langle\psi_0|\psi_B\rangle|^2=Z$ is called the renormalization constant of ψ_B. If $Z=0$, then there is no discrete state of H_0 (i.e., $\psi_0\equiv0$) and so the bound state ψ_B is entirely a result of the interaction, i.e., it is a composite particle. Thus, Z measures "how composite" the bound state is. Since ψ_0 and ψ_B have unit norm we find using the Hölder inequality,[16]

$$|\langle\psi_0|\psi_B\rangle|^2\leq\|\psi_0\|^2\|\psi_B\|^2=1,$$

that $0\leq Z\leq1$. In terms of the eigenfunction spectrum of H_0, we can find an explicit representation of Z:

$$1=\langle\psi_B|\psi_B\rangle\equiv Z+\int d^3k\frac{|\langle\psi_B|V|\mathbf{k}\rangle|^2}{(B+E_k)^2}. \tag{14}$$

We see (restricting ψ_B to be an $l=0$ state for simplicity) that if B is small,[9] the major contribution to the integral in (15) comes from the pole at $E_k=-B$; thus we may write

$$Z\simeq1-|\langle\psi_B|V|i(2mB)^{1/2}\rangle|^2\int_0^\infty\frac{dk\,k^2}{(B+E_k)^2}. \tag{15}$$

Suppose we take $Z=0$; then

$$|\langle\psi_B(l=0)|V|i(2mB)^{1/2}\rangle|^2\simeq(2mB)^{1/2}/\pi m^2. \tag{16}$$

[14] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Co., New York, 1953), p. 978.
[15] Reference 13, p. 286.
[16] N. Kazarinoff, *Analytic Inequalities* (Holt, Rinehart and Winston, Inc., New York, 1961), p. 71.

On the other hand, it is easily seen[17] that the residue of the bound-state pole in the scattering amplitude is $-\pi m|\langle\psi_B|V|i(2mB)^{1/2}\rangle|^2\simeq-(2B/m)^{1/2}$.

Comparing this result with the residue of $f_0^{(+)}(E)$ [from Eq. (9), with $\lambda=(E_0/2m)^{1/2}+(B/2m)^{1/2}$], we see that the residues are in agreement to order $(B/E_0)^{1/2}$. [This is actually good agreement: B/E_0 must be $\ll1$ in order that the one-pole approximation be physically meaningful, and in order to write Eq. (15).] Weinberg[9] has applied this procedure in reverse. Using the experimentally determined low-energy n-p scattering parameters he has demonstrated the compositeness of the deuteron (i.e., that $Z=0$ for the deuteron, within the experimental errors).

It is conventional in high-energy physics, where the poles of the scattering amplitudes may not represent purely composite states (i.e., $Z>0$) to insert these singularities into $N(E)$, with residues $-|g|^2=-\pi m|\langle\psi_B|V|(2mB)^{1/2}\rangle|^2D(B)=-(1-Z)|g_0|^2$, where $-|g_0|^2$ is the residue when $Z=0$.

Finally we should note that for convenience, the preceding discussion was restricted to $l=0$. In this case, the approximation of the left-hand cut by a simple pole led to a convergent integral in Eq. (6). In general, the amplitude $f_l^{(+)}(E)$ has the dependence E^l near $E=0$: If we insert a simple factor of E^l in $N(E)$ to give the correct threshold dependence, then the integral in (6) will not converge in the one-pole approximation for $l>0$. One way out of this dilemma is to take subtractions in the dispersion relation (6). That is, one employs identities like

$$g(z)=\int_0^\infty\frac{dx\,f(x)}{z-x}=(z_0-z)\int_0^\infty\frac{dx\,f(x)}{(z-x)(z_0-x)}+g(z_0)$$

and then treat $g(z_0)$ as an arbitrary constant. We prefer the equivalent approach of inserting the threshold factor $[E/(E_0+E)]^l$ and shall use this method in Sec. II.

B. Generalization to Three-Particle Case

We now consider the system of $n+p+\alpha$ mentioned in the Introduction. With the α treated as elementary, the only channels are $d+\alpha$ and $n+p+\alpha$.[18] Following the notation of I, we write the elastic unitarity relation

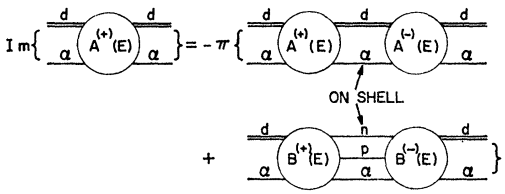

FIG. 1. Schematic representation of elastic unitarity relation expressed by Eq. (17).

$$\begin{aligned}&\mathrm{Im}\langle\hat K'|A^+(E)|\hat K\rangle\\&=-\tfrac12\pi(2M)^{3/2}(E+\epsilon)^{1/2}\theta(E+\epsilon)\\&\times\int d\hat K''\langle\hat K'|A^-(E)|\hat K''\rangle\langle\hat K''|A^+(E)|\hat K\rangle\\&-\pi\int d\mathbf K''\int d\mathbf k''\langle\hat K'|B^-(E)|\mathbf K''\mathbf k''\rangle\\&\times\delta(E-k''^2/2\mu-K''^2/2M)\langle\mathbf K''\mathbf k''|B^+(E)|\hat K\rangle.\quad(17)\end{aligned}$$

Figure 1 shows a schematic representation of this relation. In Eq. (17), $\mathbf k''$ and $\mathbf K''$ are the relative n-p momentum and the n-p total momentum relative to α, respectively, in the intermediate states. The reduced masses are

$$M=M_dM_\alpha/(M_d+M_\alpha),\quad(18a)$$

$$\mu=M_nM_p/(M_n+M_p).\quad(18b)$$

$\langle\hat K'|A^+(E)|\hat K\rangle$ is the elastic (d-α) on-shell scattering amplitude, and $\langle\mathbf K'\mathbf k'|B^\pm(E)|\hat K\rangle$ are, respectively, the normal or time-reversed amplitudes for the breakup reaction, in which the particle d, incident on α with momentum K and energy $E=K^2/2M-\epsilon$, breaks up into $n+p$, with relative momentum $\mathbf k'$ and total momentum $\mathbf K'$ (relative to α). We note that conservation of isospin also implies the conservation of $S_{np}(\mathbf S_{np}=\boldsymbol\sigma_n+\boldsymbol\sigma_p)$ and we are only interested in $T=0$, $S_{np}=1$. Expanding (17) in partial waves leads to[19]

$$\begin{aligned}&\mathrm{Im}\langle L'1|A_J^{(+)}(E)|L1\rangle\\&=-\tfrac12\pi(2M)^{3/2}(E+\epsilon)^{1/2}\theta(E+\epsilon)\\&\times\sum_{L''}\langle L'1|A_J^{(-)}(E)|L''1\rangle\langle L''1|A_J^{(+)}(E)|L1\rangle\\&-\pi\sum_{L''l''j''}\int_0^\infty\int_0^\infty\{\langle L'1|B_J^{(-)}(E)|K'k'L''l''j''\rangle\\&\times\delta(E-E_{K''k''})\langle K''k''L''l''j''|B_J^{(+)}(E)|L1\rangle\}\\&\times K''^2dK''k''^2dk'',\quad(19)\end{aligned}$$

where L is the orbital angular momentum associated with $\mathbf K$, l is that associated with $\mathbf k$, $\mathbf j=\mathbf l+\mathbf S_{np}$, $\mathbf J=\mathbf L+\mathbf j$, and $E_{Kk}\equiv(2M)^{-1}K^2+(2\mu)^{-1}k^2$.

[17] The scattering matrix may be written formally $T=V+V(W-H)^{-1}V$; near the pole at $-B$, $(W-H)^{-1}\simeq|\psi_B\rangle(W+B)^{-1}\times\langle\psi_B|$, so that the residue is

$$\lim_{k\to i(2mB)^{1/2}}|\langle k|V|\psi_B\rangle|^2.$$

Using our conventions for the scattering amplitude and the partial-wave analysis (see Ref. 19), we find that the residue of $f_0^{(+)}(E)$ is

$$-\pi m|\langle\psi_B|V|i(2mB)^{1/2},l=0\rangle|^2.$$

[18] Our energy scale is the same as in I, with $\epsilon=2.225$ MeV the deuteron binding energy, and with the zero of energy taken at the breakup threshold.

[19] Reference 13, p. 226.

In I we introduced the approximation that only states with small k'' contribute importantly in the integral in (19). We shall give detailed arguments justifying this approximation in Sec. II. For the moment, we remark that it leads to the replacement of the integral in (19) by

$$\tfrac{1}{2}(4\mu M)^{3/2}E^2\theta(E)\sum_{L''}\int_0^{\pi/2}d\lambda''\sin^2\lambda''\cos^2\lambda''$$
$$\times\{\langle L',1|B_J^{(-)}(E)|(2ME)^{1/2}\cos\lambda'',0,L'',0,1\rangle$$
$$\times\langle(2ME)^{1/2}\cos\lambda'',0,L'',0,1|B_J^{(+)}(E)|L,1\rangle\}\quad(20)$$

and that the weak binding of the deuteron makes it reasonable to replace

$$\langle(2ME)^{1/2}\cos\lambda'',0,L'',0,1|B_J^{(\pm)}(E)|L,1\rangle$$

by

$$(4\pi)^{1/2}\xi\langle(2ME)^{1/2}\cos\lambda'',L''1|$$
$$\times A_J^{(\pm)}(E)|[2M(E+\epsilon)]^{1/2},L,1\rangle,$$

where $\langle K'L'1|A_J^{(\pm)}(E)|KL1\rangle$ is the off-shell partial-wave elastic amplitude, ξ is a constant, and we have made the transformation to cylindrical coordinates:

$$|\mathbf{K}''|=(2ME'')^{1/2}\cos\lambda'',\quad|\mathbf{k}''|=(2\mu E'')^{1/2}\sin\lambda''.$$

The above approximation for the breakup amplitude uncouples the partial-wave elastic unitarity relation from this amplitude. This procedure retains approximately the coupling to the breakup channel without actually requiring us to solve for the breakup amplitude. We note that conservation of total angular momentum and parity require that the matrix

$$\langle K'L'1|A_J^{(\pm)}(E)|KL1\rangle$$

be 3×3. We explicitly remove the threshold dependence of this matrix by writing

$$\langle K'L'1|A_J^{(\pm)}(E)|KL1\rangle$$
$$=\beta_{L'}(K')\mathfrak{A}_{J;L'L}^{(\pm)}(E;K',K)\beta_L(K).\quad(21)$$

The type of approximation used in the above has also been suggested by Wong and Zambotti[20] in a slightly different context. The uncoupled approximate unitarity relation is thus

$$\mathrm{Im}\mathfrak{A}_{J;L'L}^{(+)}(E)$$
$$\simeq-\tfrac{1}{2}\pi(2M)^{3/2}(E+\epsilon)^{1/2}\theta(E+\epsilon)$$
$$\times\sum_{L''}\mathfrak{A}_{J;L'L''}^{(-)}(E)\beta_{L''}{}^2[(2M(E+\epsilon))^{1/2}]\mathfrak{A}_{J;L''L}^{(+)}(E)$$
$$-2\pi^2(4\mu M)^{3/2}E^2\theta(E)|\xi|^2$$
$$\times\sum_{L''}\mathfrak{A}_{J;L'L''}^{(-)}(E)\gamma_{L''}(E)\mathfrak{A}_{J;L''L}^{(+)}(E),\quad(22)$$

where

$$\gamma_{L''}(E)=\int_0^{\pi/2}d\lambda''\sin^2\lambda''\cos^2\lambda''\beta_{L''}{}^2((2ME)^{1/2}\cos\lambda'').$$

Exactly as before, we write the matrix

$$\mathfrak{A}_J^{(+)}(E)=\lim_{\epsilon\to0}\mathbf{N}_J(E+i\epsilon)\mathbf{D}_J^{-1}(E+i\epsilon).\quad(23)$$

[There are obviously many ways[21] to introduce the Weiner-Hopf factorization in the case of coupled channels: Since our ultimate interest is to truncate the matrix equation (22), the form of Eq. (23) is irrelevant.] As in I, we find

$$\mathbf{D}_J(Z)=\mathbf{1}+\int_{-\epsilon}^{\infty}ds\,\boldsymbol{\varrho}_J(s)\cdot\mathbf{N}_J(s)(s-z)^{-1},\quad(24)$$

where

$$[\boldsymbol{\varrho}_J(E)]_{L'L}$$
$$=\delta_{L'L}\{\tfrac{1}{2}(2M)^{3/2}(E+\epsilon)^{1/2}\theta(E+\epsilon)\beta_L{}^2[(2M(E+\epsilon))^{1/2}]$$
$$+2\pi(4\mu M)^{3/2}E^2\theta(E)|\xi|^2\gamma_L(E)\}.\quad(25)$$

The analogy of the effective-range approximation (9) now becomes clear: We need merely replace $\mathbf{N}_J(E)$ by $\mathbf{n}_J(E+E_0)^{-1}$, where $\mathbf{n}_J$ is a constant matrix, and perform the integral in (24). This formalism differs from the usual effective-range theory in that in addition to satisfying two-body unitarity exactly, it approximately satisfies three-body unitarity. In general, for any $\mathbf{N}_J(s)$ analytic for $\mathrm{Re}(s)\geq-\epsilon$, we may write

$$\lim_{\eta\to0}[\mathbf{D}_J(E+i\eta)]_{L'L}$$
$$=[\mathbf{R}_J(E)]_{L'L}+i\pi[\boldsymbol{\varrho}_J(E)\cdot\mathbf{N}_J(E)]_{L'L}$$
$$-2\pi(4\mu M)^{3/2}|\xi|^2E^2\gamma_{L'}(E)\ln|E|[\mathbf{N}_J(E)]_{L'L}.\quad(26)$$

Equation (26) is the generalization of a result we gave in I. We derive this equation in Appendix A. The function $\mathbf{R}_J(E)$ is analytic for $\mathrm{Re}(E)\geq-\epsilon$, and its radius of analyticity at $E=-\epsilon$ is the distance between the elastic threshold ($E=-\epsilon$) and the beginning of the left-hand cut. In addition to the term $i\pi[\boldsymbol{\varrho}_J(E)\cdot\mathbf{N}_J(E)]$ appearing in (26), which follows explicitly from the unitarity relation (12), we find that the three-body threshold contributes a nonanalytic real term, proportional to $\ln|E|$. This singularity is therefore quite different in its effect from the two-body threshold.

In exact analogy with Sec. I A, we may identify a zero of $\det[\mathbf{D}_J(E)]$ with a bound state of a purely composite system. We show in Appendix B that a zero of $\mathbf{D}_J$ is equivalent to setting the renormalization constant to zero.

It is known experimentally that the ground state of Li^6 (viewed as a weakly bound state of $d+He^4$ 3.697 MeV below He^4+n+p) has quantum numbers $J^\pi=1^+$, $T=0$. Since this state is just below the elastic scattering threshold, and since it is principally an $l=0$ state, it will strongly influence low-energy d-He^4 scattering. The other important low-lying $T=0$ Li^6 state is the sharp (23 keV) 3^+ resonance at -1.513 MeV. Since we intend to compare our results with experiment, we prefer to

[20] D. Y. Wong and G. Zambotti, Phys. Rev. 154, 1540 (1967).

[21] J. B. Hartle and J. R. Taylor, J. Math. Phys. 8, 651 (1967).

preclude a plethora of phenomenological parameters. We do this by using the one-pole approximation described above, and we truncate the 3×3 matrix Eq. (24), keeping only the $L'=L=J-1$ amplitude. Since even $\rho_{1;00}(E)$ increases as E^2 for large E, it is clear that the integral in (24), with $N_{1;00}(E)=n_{1;00}/(E+E_0)$, will diverge. It turns out that two subtractions are necessary to ensure the convergence of this integral. We arbitrarily choose them such that $D_{1;00}(-E_0)=1$, $D_{1;00}'(-E_0)=0$, giving

$$D_{1;00}(E)=1-(E+E_0)^2\times\int_{-\epsilon}^{\infty}ds\,\rho_{1;00}(s)\frac{n_{1;00}}{(s+E_0)^3(s-E-i\eta)}. \quad (27)$$

Clearly $n_{1;00}$ should be chosen so that $D(-3.697\text{ MeV})=0$. Doing the integral by elementary methods and letting $\beta_0(k)=1$, we obtain

$$D_{1;00}(E)=1-\tfrac{1}{2}n_{1;00}\left[\frac{c_1\pi}{(E_0-\epsilon)^{1/2}(E+E_0)}\left(E_0-E-2\epsilon+2i[(E+\epsilon)(E_0-\epsilon)]^{1/2}-\frac{(E+E_0)^2}{4(E_0-\epsilon)}\right)-\frac{c_2}{E+E_0}\{2E^2\ln(-E/E_0)-(3E+E_0)(E+E_0)\}\right], \quad (28)$$

where

$$c_1=\tfrac{1}{2}(2M)^{3/2},\quad c_2=\tfrac{1}{8}\pi^2(4\mu M)^{3/2}|\xi|^2.$$

The analogous result for the $L'=L=2$, $J^\pi=3^+$ channel requires a specific form for $\beta_L(k)$. Again for simplicity, we take

$$\beta_2^2[(2m(E+\epsilon))^{1/2}]=(\epsilon+E)^2/(E+E_0)^2$$

and similarly,

$$c_2\gamma_2(E)=E^2/(E+E_0)^2(5\pi^2/128)(4\mu M)^{3/2}|\xi|^2.$$

This leads to (with the same subtraction convention and a single pole for N)

$$D_{3;22}(E)=1-(E+E_0)^2n_{3;22}\left(\int_0^\infty ds\frac{c_1s^{5/2}}{(s+E_0-\epsilon)^5(s-E-\epsilon-i\eta)}-\int_0^\infty ds\,\tfrac{5}{16}c_2\frac{s^4}{(s+E_0)^5(s-E-i\eta)}\right)$$

$$=1-\frac{n_{3;22}}{(E+E_0)^3}\left\{c_1(E_0-\epsilon)^{5/2}\pi\left[1-\left(-\frac{E+\epsilon}{E_0-\epsilon}\right)^{5/2}-\frac{5}{2}\left(\frac{E+E_0}{E_0-\epsilon}\right)+\frac{15}{8}\left(\frac{E+E_0}{E_0-\epsilon}\right)^2-\frac{5}{16}\left(\frac{E+E_0}{E_0-\epsilon}\right)^3-\frac{5}{128}\left(\frac{E+E_0}{E_0-\epsilon}\right)^4\right]\right.$$

$$-\tfrac{5}{16}c_2E_0^4[(25/12)(1+E/E_0)^4-(13/3)(1+E/E_0)^3+\tfrac{7}{2}(1+E/E_0)^2$$

$$\left.-(1+E/E_0)-(E/E_0)^4\ln(-E/E_0)]\right\}. \quad (29)$$

The parameters $n_{1;00}$ and E_0 may be chosen to fit the binding energy and scattering length in the 1^+ channel; similarly $n_{3;22}$ and E_0 may be chosen to fit the position and width of the resonance in the 3^+ channel. (Note that E_0 will in general be different for the two cases, although *a priori* we would not expect them to differ greatly, as we show in Appendix C.) We shall discuss these parameters in more detail in Sec. IV.

II. ELIMINATION OF THE BREAKUP CHANNEL

In deriving the approximate unitarity relation (22), we have made the crucial assumption that the major contribution to the integral over three-body intermediate states comes from states in which the neutron and proton are at low relative momentum. The reader will recall that this assumption allowed us to make the approximation (20), which in turn led to the soluble form (22). Let us now examine what is involved in this approximation. First, the fact that the intermediate states are on the energy shell means that $0\leq k''^2\leq 2\mu E$, so that for small energies k''^2 must perforce be small. At larger energies, what we have essentially assumed is that over the range $0\leq k''^2\leq 2\mu E$ the variation of the breakup amplitude

$$\langle\mathbf{K}''\mathbf{k}''|B^{(\pm)}(E)|\mathbf{K}\rangle$$

is so slow that the major variations in the breakup cross section result from the phase space. Once this assumption has been made, there is no reason not to evaluate the breakup amplitude at any value of k'' that happens to be convenient, such as $k''=0$.

The work of Nagatani, Tombrello, and Bromley[12] on the α-induced breakup of deuterium indicates that our assumption is quite reasonable. Figure 2 shows one of their proton singles spectra, together with the phase space normalized to the same area: It is clear that the major variation of $d^2\sigma/dE_pd\Omega_p$ with proton energy is already given by the phase space. There is a small bump at the upper end of the proton spectrum, corresponding to the He^5 resonance; there is also an excess of protons with energy near 13.2 MeV, about half the maximum proton energy. This latter effect probably arises from the fact that the breakup amplitude has a weak, broad

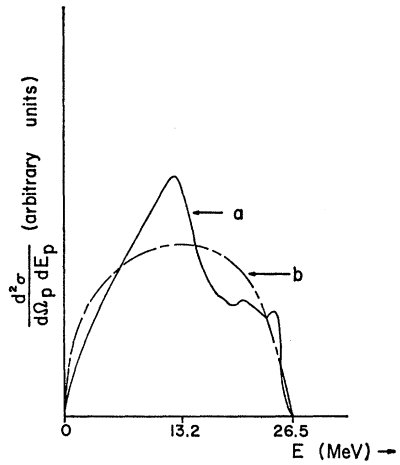

FIG. 2. Comparison of breakup proton spectrum of Nagatani *et al.* (see Ref. 12) at 12.6° (a) and with phase space normalized to same total area (b).

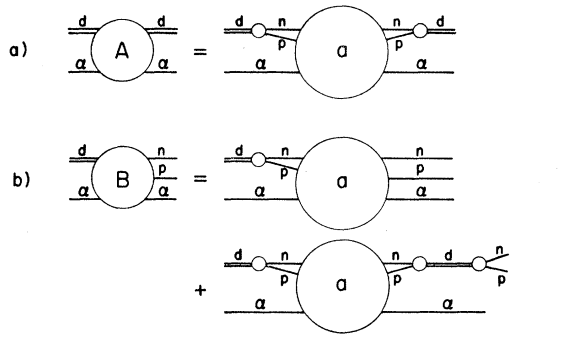

FIG. 3. (a) Schematic representation of elastic scattering as a matrix element of the formal operator A. (b) Schematic representation of the breakup amplitude as a matrix element of the formal operator B.

maximum at $k''=0$ (which corresponds to this point on the spectrum). We can see this by noting that the impulse approximation and even the plane-wave Born approximation (PWBA) already give proton spectra agreeing in shape (although *not* in normalization) with those observed experimentally.[12] That is, the major momentum dependence of the breakup amplitude is essentially determined by the PWBA. Assuming isotopic invariance, the PWBA amplitude has the form

$$\langle \mathbf{K}'\mathbf{k}'|B^{(\pm)}(E)|\mathbf{K}\rangle_{\mathrm{PWBA}} = V(|\mathbf{K}'-\mathbf{K}|)\{\phi_d(\mathbf{k}'+\tfrac{1}{2}(\mathbf{K}'-\mathbf{K})) + \phi_d(\mathbf{k}'-\tfrac{1}{2}(\mathbf{K}'-\mathbf{K}))\},$$

where $V(Q)$ is the Fourier transform of the nucleon-α potential, and presumably has a range ~ 1.4 fm. Given this sort of range, the potential only localizes $|\mathbf{K}'-\mathbf{K}|^2$ to within $\sim m_\pi^2 c^2$. Thus, even though the deuteron wave function $\phi_d(k)$ is strongly peaked at $k=0$, k'^2 in the above PWBA amplitude is only restricted to be $\lesssim m_\pi^2 c^2$ also. In energy terms, we see that $k'^2/2\mu$ is only restricted to be $\lesssim m_\pi c^2/7 = 20$ MeV. That is, one must go to rather large barycentric energies before the variation of the breakup amplitude with k' becomes important: As we have already seen in Eqs. (27) and (29), however, the dispersion integrals emphasize the low-energy behavior of the amplitudes, so that the end result is not sensitive to the details of the approximation.

The above assumption effected a considerable simplification in the three-body contribution to elastic unitarity; the weak binding of the deuteron allows a further simplification which, by decoupling the elastic and breakup amplitudes, makes the solution of the resulting dispersion relation straightforward. To see how this simplification comes about, we note that from the definitions of the scattering amplitudes as matrix elements of formal operators

$$\lim_{|\mathbf{k}'|\to 0}\langle \mathbf{K}'\mathbf{k}'1\nu'|B^{(\pm)}(E)|\mathbf{K}1\nu\rangle = \int \langle \mathbf{K}'01\nu'|\mathfrak{A}(E\pm i\eta)|\mathbf{K}\mathbf{k}1\nu\rangle\phi_d(\mathbf{k})d\mathbf{k} + \sum_{\nu''}\int \frac{\langle 0,\nu'|t_{np}{}^1(E+i\eta-K'^2/2M)|\mathbf{k}''\nu''\rangle}{(E+i\eta-K'^2/2M-k''^2/2\mu)} \times \langle \mathbf{K}',\mathbf{k}''1\nu''|\mathfrak{A}(E+i\eta)|\mathbf{K}\mathbf{k}1\nu\rangle\phi_d(\mathbf{k})d\mathbf{k}''d\mathbf{k}, \quad (30)$$

where $\mathfrak{A}(W)$ is the elastic formal scattering operator. Figures 3(a) and 3(b) show schematic representations of the matrix elements of $\mathfrak{A}$. [See Appendix D.] In (30), $t_{np}{}^1$ is the n-p off-shell scattering matrix with $S_{np}=1$. This scattering matrix is well approximated at low energies by the separable form[22]

$$\langle \mathbf{k}'\nu'|t_{np}{}^1(W)|\mathbf{k}\nu\rangle \simeq -\delta_{\nu'\nu}v(k')\tau_d(W)v^*(k) \quad (31)$$

and clearly,[22] the deuteron wave function is given by

$$\phi_d(\mathbf{k}) = -v(k)(\epsilon+k^2/2\mu)^{-1}. \quad (32)$$

Thus we find, on the energy shell,

$$\langle \mathbf{K}'01\nu'|B^{(\pm)}(K'^2/2M)|\mathbf{K}1\nu\rangle = \int d\mathbf{k}\langle \mathbf{K}'01\nu'|\mathfrak{A}(K'^2/2M\pm i\eta)|\mathbf{K}\mathbf{k}1\nu\rangle\phi_d(\mathbf{k}) - v(0)\tau_d(0) \times \int\left(\frac{2\mu\epsilon+k''^2}{k''^2}\right)\phi_d{}^*(\mathbf{k}'')\langle \mathbf{K}'\mathbf{k}''1\nu'|\mathfrak{A}(K'^2/2M\pm i\eta)|\mathbf{K}\mathbf{k}1\nu\rangle\phi_d(\mathbf{k})d\mathbf{k}''d\mathbf{k}. \quad (33)$$

[22] J. V. Noble, Phys. Rev. **157**, 939 (1967).

But $2\mu\epsilon=\kappa_d^2$ is extremely small compared to any other momentum in the problem, and so $(\kappa_d^2+k''^2)/k''^2\simeq 1$; furthermore, the smallness of κ_d^2 implies that $k^2\phi_d(\mathbf{k})$ is peaked at small k, so that

$$\langle \mathbf{K}'1\nu'|A^{(\pm)}(K'^2/2M)|\mathbf{K}1\nu\rangle \equiv \int \phi_d^*(\mathbf{k}')\langle \mathbf{K}'\mathbf{k}'1\nu'|\mathfrak{A}(K'^2/2M\pm i\eta)|\mathbf{K}\mathbf{k}1\nu\rangle\phi_d(\mathbf{k})d\mathbf{k}'d\mathbf{k}$$
$$\simeq \int \phi_d^*(\mathbf{k}')d\mathbf{k}'\left[\int \langle \mathbf{K}'01\nu'|\mathfrak{A}(K'^2/2M\pm i\eta)|\mathbf{K}\mathbf{k}1\nu\rangle\phi_d(\mathbf{k})d\mathbf{k}\right] \quad (33')$$

[Fig. 4(a) shows a schematic representation of this equation] and therefore we may rewrite Eq. (33) as

$$\langle \mathbf{K}'01\nu'|B^{(\pm)}(K'^2/2M)|\mathbf{K}1\nu\rangle \simeq \left\{\left[\int d\mathbf{k}\phi_d^*(\mathbf{k})\right]^{-1} - v(0)\tau_d(0)\right\}\langle \mathbf{K}'1\nu'|A^{(\pm)}(K'^2/2M)|\mathbf{K}1\nu\rangle. \quad (34)$$

[See Fig. 4(b).] The energy-independent quantity in braces in Eq. (34) is what we have denoted by ξ. The extra factor of $(4\pi)^{1/2}$ appearing in our previous use of this approximation [Eq. (34)] comes from our convention for defining partial-wave amplitudes.

III. DISPERSION THEORY OF THE BREAKUP AMPLITUDE

We include this section for completeness, even though we have not yet attempted any comparison with experiment. As we have mentioned in Sec. II, experimental studies of the breakup reaction indicate a strong preference for intermediate states of low n-p relative momentum. We have already used this fact in constructing our model of the elastic scattering amplitude, and we shall apply it once more to obtain the soluble truncated unitarity relation for the breakup reaction. Our approximation will yield an uncoupled equation *linear* in the breakup amplitude, whose kernel involves the previously determined on-shell elastic scattering amplitude. (The uncoupled equation for the reduced elastic amplitude was nonlinear.)

The unitarity relation for the breakup amplitude $\langle \mathbf{k}'\mathbf{K}'|B^{(\pm)}(E)|\mathbf{K};d\rangle$ (for convenience we omit spin labels in this section) may be written

$$\mathrm{Im}\langle \mathbf{k}'\mathbf{K}'|B^{(+)}(E)|\mathbf{K};d\rangle = -\tfrac{1}{2}\pi(2M)^{3/2}(E+\epsilon)^{1/2}\theta(E+\epsilon)\int d\hat{K}''\langle \mathbf{k}'\mathbf{K}'|B^{(-)}(E)|K\hat{K}'';d\rangle\langle \hat{K}''|A^{(+)}(E)|\hat{K}\rangle - \pi\int d\mathbf{K}''$$
$$\times\int d\mathbf{k}''\langle \mathbf{k}'\mathbf{K}'|C^{(-)}(E)|\mathbf{k}''\mathbf{K}''\rangle\delta(E-(2\mu)^{-1}k''^2-(2M)^{-1}K''^2)\langle \mathbf{k}''\mathbf{K}''|B^{(+)}(E)|\mathbf{K};d\rangle, \quad (35)$$

where C is the transition operator for $3\to 3$ collisions. When we try to apply our method of Sec. II for truncating the 5-dimensional integral over 3-body intermediate states in (35), we find a formal difficulty. The operator C contains disconnected parts. The pieces that will cause us trouble are those in which one nucleon interacts with the α while the other nucleon is a spectator. These terms must be included in a correct treatment of the final-state nucleon-α interactions that give rise to the structure at the upper end of, say, the proton energy spectrum. (See Fig. 1.) As we mentioned in Sec. II, these effects are empirically a rather small part of the total differential cross section; we neglect them in order to concentrate on the gross structure. As we shall show in Appendix D,

$$\langle \mathbf{k}'\mathbf{K}'|C^{(\pm)}(E)|\mathbf{k}\mathbf{K}\rangle = \langle \mathbf{k}'\mathbf{K}'|\mathfrak{B}^{(\pm)}(E)|\mathbf{k}\mathbf{K}\rangle + \langle \mathbf{k}'\mathbf{K}'|\mathfrak{B}^{(\pm)}(E)G_0^{(\pm)}(E)t_{np}^{(\pm)}(E)|\mathbf{k}\mathbf{K}\rangle, \quad (36)$$

where $\mathfrak{B}$ is the formal operator whose appropriate matrix elements give the breakup amplitude B, and where $G_0(W)=(W-H_0)^{-1}$. In exactly the way we found Eq. (34), we write

$$\langle \mathbf{k}'\mathbf{K}'|C^{(\pm)}(E)|\mathbf{k}\mathbf{K}\rangle|_{|\mathbf{k}|=0} \simeq \xi^*\langle \mathbf{k}'\mathbf{K}'|B^{(\pm)}(E)|\mathbf{K};d\rangle, \quad (37)$$

where ξ is the same constant factor that appears in Eq. (34). Making a similar approximation for the matrix element $\langle \mathbf{k}''\mathbf{K}''|B^{(+)}(E)|\mathbf{K};d\rangle|_{|\mathbf{k}''|=0}$ that appears in Eq. (35), we find

$$\mathrm{Im}\langle \mathbf{k}'\mathbf{K}'|B^{(+)}(E)|\mathbf{K};d\rangle \simeq -\tfrac{1}{2}\pi(2M)^{3/2}(E+\epsilon)^{1/2}(E+\epsilon)\int d\hat{K}''\langle \mathbf{k}'\mathbf{K}'|B^{(-)}(E)|K\hat{K}'';d\rangle\langle \hat{K}''|A^{(+)}(E)|\hat{K}\rangle$$
$$-2\pi^2(4\mu M)^{3/2}E^2\theta(E)|\xi|^2\int d\hat{K}''\int_0^{\pi/2}d\lambda''\sin^2\lambda''\cos^2\lambda''$$
$$\times\langle \mathbf{k}'\mathbf{K}'|B^{(-)}(E)|\hat{K}''(2ME)^{1/2}\cos\lambda'';d\rangle\langle \hat{K}''(2ME)^{1/2}\cos\lambda''|A^{(+)}(E)|\mathbf{K}\rangle. \quad (38)$$

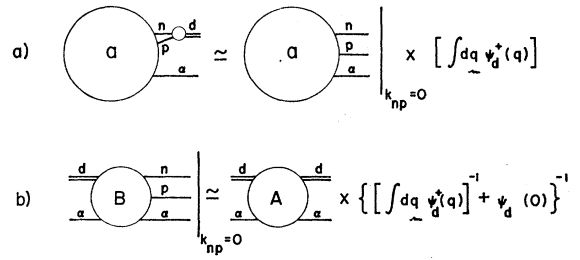

FIG. 4. (a) Schematic representation of the approximate relation given in Eq. (35). (b) Schematic representation of the approximate relation given in Eq. (34). Note that $-\nu(0)\tau_d(0)$ is very nearly $\phi_d(0)$.

We take partial-wave matrix elements and factor out the threshold dependence in K to obtain

$$\mathrm{Im}B_{l'L';L}{}^{J(+)}(E) = -\pi \sum_{L''} B_{l'L',L''}{}^{J(-)}(E)\rho_{L''}(E)\mathfrak{A}_{L''L}{}^{J(+)}(E), \quad (39)$$

where $\rho_{L''}(E)$ is the (diagonal) matrix element of ρ_J defined by Eq. (25). We may treat $B_{l'L';L}{}^{J(+)}(E)$ as the boundary value of an analytic function $B_{l'L';L}{}^{J}(W)$ that satisfies a dispersion relation of the form

$$B_{l'L',L}{}^{J}(W) = b_{l'L',L}{}^{J}(W) + \sum_{L''} \int_{-\epsilon}^{\infty} ds \frac{B_{l'L',L''}{}^{J}(s-i\eta)\rho_{L''}(s)\mathfrak{A}_{L'',L}{}^{J}(s)}{W-s}, \quad (40)$$

which is known as the Omnés-Mushkhelishvili equation.[4,13] The function $b_{l'L'L}{}^{J}(W)$ contains all of the dynamical singularities of $B_{l'L'L}{}^{J}$. Equation (40) may be solved in the form[13] (we must truncate the matrices by taking $L''=L=J-1$)

$$B_{l'L';L}{}^{J}(W) = N(W)D^{-1}(W),$$

where

$$D(W) = P(W)\exp\left\{-(2\pi i)^{-1} \times \int_{-\epsilon}^{\infty} ds'' \frac{\ln[1+2\pi i\mathfrak{A}_{LL}{}^{J(-)}(s'')\rho_L(s'')]}{W-s''}\right\}, \quad (41)$$

where

$$N(W) = \pi^{-1}\int_{-\infty}^{-\epsilon} \frac{D(E)\,\mathrm{Im}b_{l'L';L}{}^{J}(s)ds}{s-W}, \quad (42)$$

and where $P(W)$ is a real polynomial in W. We choose to let the bound states appear as zeros of $P(W)$, rather than as poles in $N(W)$, for convenience. For the $L=0$, $J^\pi=1^+$ state in Li^6 this fixes

$$P(W) = 1+W/B(\mathrm{Li}^6). \quad (43)$$

Assuming that the integrals in (41) and (42) may be evaluated and that $\mathrm{Im}(b^J)$ is known, the problem is now solved.

Of course, there are practical problems to be overcome in applying this formalism to the calculation of the (kinematically complicated) breakup cross sections. We reserve this calculation for a future paper.

TABLE I. Comparison of s-wave phase shifts calculated from Eq. (28) with experimental results of Ref. 11; for a discussion of the uncertainties in the experimental values see Ref. 11.

E_{Lab} (MeV)	$\delta_0{}^1$ (theory) (deg)	$\delta_0{}^1$ (experimental) (deg)
2.0	118.0	126.4
2.5	112.5	116.5
3.0	108.5	109.2
3.5	105.0	104.4
4.0	102.0	106.0
4.5	99.5	68.0
5.0	98.0	84.6
6.0	94.0	69.9
7.0	91.5	78.4
8.0	89.0	84.8
9.0	87.0	83.6
10.0	85.0	80.5

IV. COMPARISON WITH EXPERIMENT

In Li^6, the lowest $T=0$ channels that are dominated by poles are the $J^\pi=1^+$ and $J^\pi=3^+$ channels. Equations (28) and (29) represent two-parameter fits to the $L'=L=J-1$ amplitudes in these channels. We were able to determine $n_{1;00}$ and $E_0(1^+)$ from the binding energy of the Li^6 ground state and the d-He^4 scattering length as determined from the data of Haeberli and McIntyre.[11] Similarly, we obtained the parameters $n_{3;22}$ and $E_0(3^+)$ by fitting the position and width of the 3^+ resonance. We found $E_0(1^+)=13.0$ MeV and $E_0(3^+)=39.2$ MeV (we do not give the strengths here since they are not simply related to any physical parameter). In Appendix C, which treats the left-hand cut, we discuss the reasonableness of these values of E_0. In Table I, we compare our theoretical values of $\delta_0{}^1(E)$ with the experimental phases[23]; in Table II, we compare our values of $\delta_2{}^3(E)$ with the experimental ones. As a matter of interest we also calculated the 3^+ phase shift using

TABLE II. Comparison of d-wave phase shifts calculated from Eq. (29) for the simple pole representing the left hand at a position equal to 25.0 and 39.2 MeV, with the experimental results of Ref. 11.

E_{Lab} (MeV)	$\delta_2{}^3$ (theory, $E_0=39.2$ MeV) (deg)	$\delta_2{}^3$ (theory, $E_0=25.0$ MeV) (deg)	$\delta_2{}^3$ (experimental) (deg)
2.0	177.2	172.2	171.7
2.5	176.9	171.6	171.5
3.0	176.5	170.7	167.0
3.5	176.1	169.8	169.6
4.0	175.7	168.7	172.4
4.5	175.2	167.7	180.0
5.0	174.7	166.7	164.5
6.0	173.7	164.7	159.7
7.0	172.8	162.8	156.5
8.0	171.9	161.0	154.6
9.0	171.0	159.3	156.8
10.0	170.1	157.6	154.0

[23] These theoretical values are the same as given in I.

$E_0(3^+)=25$ MeV, changing the strength to leave the resonance position unchanged. This gives a width of ~ 50 keV, which is somewhat larger than the experimental width of 21 keV, but gives excellent agreement with the phase shifts. One must recall that a substantial part of the observed lifetime of the 3^+ resonance may be attributed to the Coulomb barrier, which has already been extracted from the experimental data in the usual way[24]: Thus it may be consistent for us to fit a larger width than the experimental one. The reader may notice that our theoretical phases in Table II differ from those of Table II of Ref. 7; in the present work we have used a better treatment of the energy dependence of the 3^+ amplitude near threshold.

As we mention in the Introduction and show in Sec. I A and Appendix B, the bound state can be inserted in the N function or can appear as a zero of the D function. We tested this numerically by calculating the 1^+ amplitude both ways: We insisted that the residue of the bound-state pole be the same in either formulation, which, as we have seen, is equivalent to demanding that $Z=0$. The numerical results agreed within five significant figures.

Our original intention was also to calculate the elastic differential cross section. For this, one needs some model of the amplitude in the other $T=0$ channels (besides $J^\pi=1^+$ and $J^\pi=3^+$). Since the only poles are in the 1^+ and 3^+ channels, it seemed reasonable to approximate the amplitude in the nonresonant channels by Born or impulse approximation.[25] That is, one would write for these channels

$$\langle K'L'1|A_J^{(+)}(E)|KL1\rangle_{\text{Born}} = \langle d; K'L'1|(V_{n\alpha}+V_{p\alpha})_J|d; KL1\rangle$$

and

$$\langle K'L'1|A_J^{(+)}(E)|KL1\rangle_{\text{impulse}} = \langle d; K'L'1|[t_{n\alpha}^{(+)}(E)+t_{p\alpha}^{(+)}(E)]_J|d; KL1\rangle,$$

where $t_{n\alpha}$ and $t_{p\alpha}$ are the n-α and p-α two-body scattering operators in the three-body space, corresponding to $V_{n\alpha}$ and $V_{p\alpha}$, respectively. We found, to our consternation, that the Born approximation exceeded the unitarity limit by about a factor of 5 in the low L channels, and the impulse approximation violated unitarity by a factor of 20. On the other hand, both the Born and impulse approximations give reasonable shapes for the elastic differential cross section, as is typical in nuclear reactions. We shall have more to say on this subject in Sec. V. Because the magnitudes of our predicted cross sections are in such bad agreement with experiment we do not present them here.

V. CONCLUSIONS

We have constructed a simplified dispersion theoretic model of d-He^4 scattering. This system (with the assumption that He^4 is elementary) may of course be treated exactly by solving the Faddeev equation.[26] In practice this represents an extremely difficult numerical problem[27]: The utility of a simple model such as ours is that it gives a direct and intuitively satisfying way to estimate three-body effects. We have no intention of proposing our model as a substitute for the exact equations. Our purpose is rather to qualitatively illustrate how the three-body threshold alters the solution, and to estimate its quantitative effect.

In terms of the structure of its singularities, a reaction amplitude containing three-body thresholds is qualitatively different from one containing only two-body thresholds. The energy dependence at the three-body threshold is $E^2 \ln E$ rather than $E^{1/2}$ as at a two-body threshold.[28] This feature alone, with its implication of an infinite Riemann sheet structure in the complex energy plane, allows a vastly richer array of phenomena involving resonances and bound states. We mention these rather well-known characteristics of three-particle scattering only to indicate the motivation for this work. The fact that no reaction or structure theory currently in vogue (with the exception of a few recent applications of the Faddeev approach[8,22,26]) includes the breakup channels makes it all too easy to forget that heretofore almost no estimate of the magnitude of their effects in a physically realistic model has been made. As a rare example of such a calculation, it is worth noting that Segrè,[29] using the methods of Weinberg,[9] has been able to estimate the effect of the three-body state in a calculation of an upper bound on the neutron-deuteron ^2S scattering length. No detailed knowledge of the nuclear potentials is required in this calculation.

We wished to estimate the influence of the breakup channel on the 1^+ scattering amplitude. Keeping the values of $n_{1;00}$ and $E_0(1^+)$ previously determined by our fit to the Li^6 ground-state binding energy and d-He^4 scattering length, we set $|\xi|^2=0$ and recalculated the 1^+D function [Eq. (28)]. This procedure removes the coupling to the breakup channel without changing the pole by which we represent the left-hand cut, which presumably reflects the strength and range of the elementary two-particle interactions. We find that the bound state moves from -3.697 to -3.12 MeV; this represents a change in the binding energy of $\sim 40\%$ as measured from the elastic threshold. To understand this result more clearly, we introduce the familiar equivalent single-channel deuteron-He^4 potential obtained from the three-particle Schrödinger equation by the projection

[24] Reference 13, p. 263.
[25] Reference 13, Chap. 11.
[26] L. D. Faddeev, *Mathematical Aspects of the 3-Body Problem in the Quantum Scattering Theory* (Israel Program for Scientific Translation, Jerusalem, 1965).
[27] R. Aaron and P. Shanley (private communication).
[28] A. J. Dragt and R. Karplus, Nuovo Cimento **26**, 168 (1962). These energies are of course measured from their respective thresholds.
[29] G. Segrè, Nuovo Cimento **38**, 422 (1965).

formalism[30]

$$V_{\text{eff}}=\langle d|\,(V_{n\alpha}+V_{p\alpha})\,|d\rangle+\langle d|\,(V_{n\alpha}+V_{p\alpha})\times(E+i\epsilon-RHR)^{-1}(V_{n\alpha}+V_{p\alpha})\,|d\rangle, \quad (44)$$

where $|d\rangle$ is the deuteron ground-state wave function, R is the projection operator $1-|d\rangle\langle d|$, and H is the three-particle Hamiltonian. Setting $|\xi|^2=0$ (and thereby turning off the coupling to the breakup channel) is equivalent to taking $\langle d|\,(V_{n\alpha}+V_{p\alpha})R\equiv 0$ in Eq. (44). If we assume a simple, local form for the (purely real) effective potential (44), we may calculate the change in its strength needed to produce a given change in the binding energy. In particular, taking the Hulthén form $V_0(e^{\mu r}-1)^{-1}$ gives $\Delta V_0/V_0\approx 6\%$, where we have taken the inverse range μ to correspond to the position of the pole representing the left-hand cut. (For $E_0=13$ MeV, $\mu\simeq 1.8$ fm^{-1}.[31]) We erroneously reported $\Delta V_0/V_0$ to be $\sim 15\%$ in I. One should note that for a weakly bound state, such as the ground state of Li^6, large changes in the binding energy and scattering length may be effected by a small change in the coupling strength.

The energy-dependent second term in Eq. (44) arises from internal excitations (polarization) of the target nucleus (in this case, H^2 since we regard the α as elementary). Below the deuteron breakup threshold, this term is real, negative definite, and monotonically decreasing with increasing energy. Therefore the effective d-He^4 potential given by Eq. (44) will be more attractive at the energy of the 3^+ resonance (which is closer to the breakup threshold) than at the 1^+ bound-state energy. Keeping the parameters $E_0(3^+)$ and $n_{3;22}$ obtained by fitting the position and width of the 3^+ resonance using Eq. (29), we set $|\xi|^2=0$ and tried to locate the new position of the 3^+ resonance. We found that it disappeared entirely, i.e., its position and/or width had increased sufficiently that there was no longer a minimum in $|D_{3;22}|$ up to 2.5 MeV above the breakup threshold. Furthermore, at the 3^+ resonance energy, we found that the term in (29) proportional to $|\xi|^2$ (and thereby arising from the presence of the breakup channel) contributed about 30% of the real part of $[1-D_{3;22}(E)]$.

Let us now discuss some of the implications of our results for nuclear structure and reaction theories. Early shell-model calculations were performed using an infinite single-nucleon potential (for example, an oscillator well), and so the basis of unperturbed single-particle states was entirely discrete. More recent calculations, based on a realistic finite single-particle potential (e.g., Woods-Saxon), have been restricted to a set of states with at most one particle in a continuum state, for reasons of computational simplicity.[32] For bound states in O^{16} lying within 2 MeV of the $O^{15}+n$ threshold, for example, Weidenmuller *et al.*[32] have found corrections on the order of 2–5% to the infinite-harmonic-oscillator results. In view of the results of our calculation (which would be equivalent, in a shell-model picture, to allowing two particles in continuum states), it is not clear that it is possible to set up a hierarchy of successive approximations beginning with one particle in a continuum state and the rest in discrete states.

Once again referring to Eq. (44), we note that our treatment of the scattering of α particles by H^2 is a model of the scattering of an elementary particle by a polarizable nucleus. In other such processes, for example, elastic nucleon-nucleus scattering, the chief unsolved problem is to determine the contribution to the real part of the effective single-channel potential arising from the principal-value integral in the term analogous to the second term of (44). The second-order perturbation theoretic correction to the effective nucleon-nucleus potential is typically of the same order as the first-order result.

We estimated the second-order correction

$$\langle d|\,(V_{n\alpha}+V_{p\alpha})\frac{P}{E-H_0}(V_{n\alpha}+V_{p\alpha})\,|d\rangle$$

in d-He^4 scattering, using reasonable forms for $V_{n\alpha}$, $V_{p\alpha}$, and $|d\rangle$, and found that it was of the same order as $\langle d|\,(V_n+{}_\alpha V_{p\alpha})\,|d\rangle$. Independent of whether the perturbation series converges,[33] the largeness of the second-order correction, when contrasted with the smallness of the actual correction, as estimated above from the bound-state energy shift in Li^6, indicates that the perturbation series is unlikely to be useful. We feel that the results of our simple model at least render suspect perturbation-theory-based conclusions in the case of nucleon-nucleus scattering. There is indeed evidence for the smallness of polarization corrections to the effective potential for elastic scattering of various projectiles from nuclei: This evidence is found in studies of the systematics of optical-model fits to scattering cross sections.

There have been many attempts to understand the empirically determined optical-model parameters in terms of the basic nucleon-nucleon interaction and some nuclear-structure model, using first-order perturbation theory.[1] The smallness of the polarization corrections to the real part of the effective interaction, as indicated by the above evidence and by our calculations, makes the partial success of these attempts more plausible than heretofore.

Clearly our results also have implications for dispersion-theoretic models of nuclear reactions. Treatment such as those of Shapiro[4] are always restricted to diagrams with only two-body intermediate states (whose contribution diminishes with distance from the physical

[30] J. V. Noble, Phys. Rev. **148**, 1528 (1966).

[31] The value of μ is determined from the relation $\mu^2=8ME_0/\hbar^2$.

[32] W. Ebenhöh, W. Glöckle, J. Hüfner, and H. A. Weidenmüller, Z. Physik **202**, 301 (1967).

[33] The perturbation series almost certainly does not converge, see, e.g., Ref. 20.

energy). Particularly in reactions involving deuterons, our results indicate the danger of *a priori* neglecting contributions from the three-particle intermediate states. These remarks also apply to R-matrix theory,[2] in which many two-body intermediate states are assumed, and three- or more-body intermediate states are neglected.

Finally, it has been suggested that the Li^6 ground state may contain a large admixture of the cluster configuration (He^3+H^3), in addition to the more usually assumed configuration (He^4+H^2).[5,34] Clearly there must be some such admixture since the matrix element of the Hamiltonian between the two types of states is not identically zero. However, in view of the weak binding of the Li^6 ground state with respect to the $\alpha+d$ channel and the $\alpha+n+p$ channel, and its large binding with respect to He^3+H^3, we consider a large admixture of this latter configuration unlikely. Furthermore, the presence of a large component of He^3+H^3 in the Li^6 ground-state wave function would seem to preclude our model, which neglected coupling to this channel, from fitting the $J^\pi=1^+$ $L=0$ phase shift as well as it did, over such a wide range of energy.[35] It may thus be more appropriate to represent the Li^6 ground state as a mixture of $(\alpha+d)$ and $(\alpha+n+p)$ configurations, and in general, in a cluster picture of light nuclei it may be better to include low-lying three-particle configurations than distant two-particle states.

Although we have made little reference to the problems of many-particle intermediate states in high-energy physics, this is clearly a subject of great importance. It is therefore worth noting that pioneering work on this problem from a somewhat different viewpoint has been done by Cook and Lee[36] and by Blankenbecler[37] in the context of the problem of the ρ bootstrap.

ACKNOWLEDGMENTS

We wish to acknowledge the kind hospitality of Professor K. W. McVoy and the University of Wisconsin 1967 Summer Institute in Theoretical Physics. We are also grateful to the University of Wisconsin for the use of its computing facilities. One of us (J.V.N.) would like to acknowledge the hospitality of the Gordon Foundation at the 1967 Gordon Conference in Nuclear Physics for providing the opportunity for many fruitful conversations concerning this work.

APPENDIX A: ANALYTIC BEHAVIOR OF $D_J(E)$

In general, $\mathbf{N}_J(s)$ is a rather complicated function that is analytic for $\mathrm{Re}(s)>-\epsilon$ and has the left-hand cut. This lets us write the integral in (24) as

$$\left[\int_{-\epsilon}^{\infty} ds\varrho_J(s)\mathbf{N}_J(s)(s-z)^{-1}\right]_{L'L}$$

$$=A\int_0^\infty \frac{ds\sqrt{s}}{s-z-\epsilon}\{\beta_{L'}{}^2[(2Ms)^{1/2}]N_{J;L'L}(s-\epsilon)-\beta_{L'}{}^2[(2Mz)^{1/2}]N_{J;L'L}(z)f(z)f^{-1}(s-\epsilon)\}+A\beta_{L'}{}^2[(2Mz)^{1/2}]N_{J;L'L}(z)f(z)$$
$$\times\int_0^\infty \frac{ds\sqrt{s}}{f(s-\epsilon)(s-z-\epsilon)}+B\int_0^\infty \frac{ds}{s-z}\{s^2\gamma_{L'}(s)N_{J;L'L}(s)-z^2\gamma_{L'}(z)N_{J;L'L}(z)f(z)f^{-1}(z)\}$$
$$+Bz^2\gamma_{L'}(z)N_{J;L'L}(z)f(z)\int_0^\infty \frac{ds}{f(s)(s-z)}, \quad \text{(A1)}$$

where $A=\frac{1}{2}(2M)^{3/2}$, $B=2\pi(4\mu M)^{3/2}|\xi|^2$, and $f(s)$ is an entire function of s with no zeros to the right of $-\epsilon$ that increases at least as fast as s. The functions $\beta_L{}^2[(2Mz)^{1/2}]$ must be analytic in z for $\mathrm{Re}(z)\geq-\epsilon$, and clearly the left-hand side of (A1) must be a convergent integral or our whole analysis would be meaningless. We have written the right-hand side of (A1) in this way in order to isolate the singularities arising from the elastic and breakup thresholds. For simplicity we choose $f(s)=s+s_0$, where $s_0>\epsilon$. We find

$$\int_0^\infty ds\sqrt{s}(s+s_0-\epsilon)^{-1}(s-z-\epsilon)^{-1}=\frac{\pi[(s_0-\epsilon)^{1/2}+i(z+\epsilon)^{1/2}]}{s_0+z}$$
$$=\pi[(s_0-\epsilon)^{1/2}+i(z+\epsilon)^{1/2}]f^{-1}(z)$$

and

$$\int_0^\infty ds(s+s_0)^{-1}(s-z)^{-1}=f^{-1}(z)[\ln s_0-\ln(-z)]$$
$$=f^{-1}(z)[\ln s_0-\ln|z|+i\pi].$$

Inserting these results back into the right-hand side of (A1) and noting that the remaining integrals are functions analytic on the positive real z axis, we see that we may write

$$\lim_{\eta\to0}[\mathbf{D}_J(E+i\eta)]_{L'L}$$
$$=[\mathbf{R}_J(E)]_{L'L}+i\pi[\varrho_J{}^{(E)}\cdot\mathbf{N}_J(E)]_{L'L}$$
$$-2\pi(4\mu M)^{3/2}|\xi|^2E^2\gamma_{L'}(E)\ln|E|N_{J;L'L}(E). \quad \text{(A2)}$$

[34] F. C. Young, P. D. Forsyth, and J. B. Marion, Nucl. Phys. **A91**, 209 (1967).
[35] See Table I.
[36] L. G. Cook, Jr., and B. W. Lee, Phys. Rev. **127**, 297 (1962).
[37] R. Blankenbecler, Phys. Rev. **125**, 155 (1962).

APPENDIX B: RENORMALIZATION OF Li^6

In exact analogy with Sec. I A, we may identify a zero of $\det[\mathbf{D}_J(E)]$ with a bound state of a purely composite system. For convenience, we examine this result in the specific case of the (weakly bound) $T=0$, $J^\pi=1^+$ Li^6 ground state, assuming it to be pure $L=0$. Again we write the full Hamiltonian

$$H=H_0+V,$$

where

$$V=V_{np}+V_{n\alpha}+V_{p\alpha}.$$

If there is an elementary component to the Li^6 ground state, it must satisfy

$$(H_0+V_{np})|\psi_0\rangle=-B_0|\psi_0\rangle. \tag{B1}$$

Conversely, the actual ground state satisfies

$$H|\psi_B\rangle=-B|\psi_B\rangle. \tag{B2}$$

The completeness relation for the eigenstates of H_0+V_{np} is

$$\mathbf{1}=|\psi_0\rangle\langle\psi_0|+\int d^3K|\mathbf{K},d\rangle\langle\mathbf{K},d| + \int d\mathbf{K}\int d\mathbf{k}|\mathbf{K},\mathbf{k}^{(+)}\rangle\langle\mathbf{K},\mathbf{k}^{(+)}|, \tag{B3}$$

where

$$\langle\mathbf{R}_{d,\alpha},\mathbf{r}_{np}|\mathbf{K},d\rangle=(2\pi)^{-3/2}\exp(i\mathbf{K}\cdot\mathbf{R}_{d,\alpha})\psi_d(\mathbf{r}_{np}),$$
$$\langle\mathbf{R}_{d,\alpha},\mathbf{r}_{np}|\mathbf{K},\mathbf{k}^{(+)}\rangle=(2\pi)^{-3/2}\exp(i\mathbf{K}\cdot\mathbf{R}_{d\alpha})\psi_{\mathbf{k}}^{(+)}(\mathbf{r}_{np}),$$

and $|\mathbf{R}_{d,\alpha}|$ is the distance between the n-p center of mass and that of the α. Defining $Z=|\langle\psi_0|\psi_B\rangle|^2$, we have, as before,

$$1-Z=\int d^3K\frac{|\langle\mathbf{K},d|(V_{n\alpha}+V_{p\alpha})|\psi_B\rangle|^2}{[(2M)^{-1}K^2-\epsilon+B]^2} + \int d^3K\int d^3k\frac{|\langle\mathbf{K},\mathbf{k}^{(+)}|(V_{n\alpha}+V_{p\alpha})|\psi_B\rangle|^2}{[(2M)^{-1}K^2+(2\mu)^{-1}k^2+B]^2}. \tag{B4}$$

We wish to compare the leading terms (in B/E_0) of the residue of the 1^+ amplitude as computed from Eq. (28), that is,

$$-[C_1\pi/2(B-\epsilon)^{1/2}+\tfrac{3}{2}C_2E_0]^{-1},$$

with the residue

$$\lim_{K\to i(8mB/3)^{1/2}}|\langle\mathbf{K},d|(V_{n\alpha}+V_{p\alpha})|\psi_B\rangle|^2$$

of the T matrix, as obtained from (B4) with $Z=0$. We remove the matrix element from the integral in the first term of (B4) as in (15); unfortunately we cannot use this procedure to remove the matrix element from the second integral, as the remaining integral diverges. To get around this, we put in a convergence factor

$$\left(\frac{\frac{3}{2}E_0}{\frac{3}{2}E_0+(2M)^{-1}K^2+(2\mu)^{-1}k^2}\right)^2,$$

chosen to make the coefficient $\frac{3}{2}$ appear in front of the C_2 term and to simulate the large-energy behavior of the matrix element. We then apply the approximation detailed in Sec. II to obtain

$$1-Z\simeq 4\pi|\langle\hat{K}i[2M(B-\epsilon)]^{1/2},d|(V_{n\alpha}+V_{p\alpha})|\psi_B\rangle|^2 \times[C_1\pi/2(B-\epsilon)^{1/2}+\tfrac{3}{2}C_2E_0].$$

When $Z=0$, the residue of the $L=0$ amplitude then agrees with that computed from Eq. (28), to order $(B-\epsilon)/(E_0-\epsilon)$.

APPENDIX C: NATURE OF THE LEFT-HAND CUT

When we simulate the left-hand cut by a simple pole, two questions arise: (1) Does a simple pole give reasonable asymptotic dependence? (2) Is the position of the pole in the particular channels $J=1^+$ and 3^+, $T=0$ reasonable? If d-α potential were a pure Yukawa form, the partial-wave on-shell Born term would be proportional to

$$E^{-1}Q_l(1+2E_0/E),$$

which is asymptotically dominated by $E^{-1}\ln E$, independent of l. On the other hand, when we take the nucleon-α interaction to be a superposition of Yukawa potentials that we smear over the deuteron (i.e., $V_{d-\alpha}=\langle d|[V_{n\alpha}+V_{p\alpha}]|d\rangle$), we obtain a more rapid fall-off with E; in fact, an E^{-1} asymptotic dependence. Therefore, the pole that represents the left-hand cut should be simple.

As mentioned in Sec. IV, we found the position of the left-hand cut pole in the 1^+ channel to be -13.0 MeV, while the position of this pole in the 3^+ channel was approximately 2–3 times this amount. We believe these results are reasonable. To see why this is so, we examine the discontinuity across the left-hand cut. The lowest-order contribution to the left-hand cut is the Born approximation to the elastic scattering amplitude, which has the form

$$B(|\mathbf{K}'-\mathbf{K}|)=2\int\psi_d^*(\mathbf{k}+\tfrac{1}{2}(\mathbf{K}'-\mathbf{K})) \times V(|\mathbf{K}-\mathbf{K}'|)\psi_d(\mathbf{k})d\mathbf{k}, \tag{C1}$$

where V is a local potential supposed to represent the nucleon-α interaction and ψ_d is the momentum-space deuteron wave function. Assuming for convenience that in coordinate space[38]

$$\psi_d(\mathbf{r})=\frac{Ne^{-\kappa r}}{(4\pi)^{1/2}}\frac{(1-e^{-\alpha r})}{r},$$

we find for this integral

$$B(Q)=(2V(Q)/Q)\{\tan^{-1}(Q/4\kappa)-2\tan^{-1}(Q/2(\alpha+2\kappa)) +\tan^{-1}(Q/4(\alpha+\kappa))\}\{(\kappa)^{-1}+(\alpha+2\kappa)^{-1} +(4(\alpha+\kappa))^{-1}\}^{-1}. \tag{C2}$$

[38] The essential feature of ψ_d that determines the left-hand branch point is the asymptotic behavior $e^{-\kappa r}/r$, common to any deuteron wave function.

The partial-wave on-shell projection of $B(Q)$ is

$$\frac{1}{2}\int_{-1}^{1} dt P_l(t) B[(2K^2(1-t))]^{1/2}. \quad \text{(C3)}$$

Assuming that the leading singularity in V is a pole $(Q^2+\mu^2)^{-1}$ it is easily seen that for large positive K^2, $B_l(K)\sim O(K^{-2})=O(E^{-1})$. $B_l(K)$ has several branch points along the negative real K^2 axis. The nearest one comes when $K/2\kappa=\pm i$ or $K^2=-4\kappa^2$. Numerically, this occurs for $E=-5.6$ MeV, which is to be compared with the 1^+ bound state at -3.697 MeV. Thus we see that it is reasonable to approximate the effect of the cut for $L=0$ by a pole at $E=-13.0$ MeV. The discontinuity resulting from the nearest singularity in (C2) above is a weak one and quickly diminishes with increasing l. Therefore, it is not surprising that we find, when we simulate the higher partial-wave channels by a pole, that the pole recedes to the left.

APPENDIX D: RELATIONS BETWEEN SCATTERING OPERATORS

We now state several identities satisfied by the formal scattering operators of our model of d-He^4 scattering. The system of n, p, and α is described by the Hamiltonian

$$H=H_0+V_{np}+V_{n\alpha}+V_{p\alpha}. \quad \text{(D1)}$$

The formal elastic scattering operator describing He^4-(d,d)He^4 is given by

$$\mathfrak{A}(W)=(V_{n\alpha}+V_{p\alpha})+(V_{n\alpha}+V_{p\alpha})\times(W-H)^{-1}(V_{n\alpha}+V_{p\alpha}). \quad \text{(D2)}$$

In terms of $\mathfrak{A}(W)$ and the deuteron internal wave function, the elastic scattering matrix may be written

$$\langle \mathbf{K}'|A^{(+)}(E)|\mathbf{K}\rangle=\lim_{\eta\to 0^+}\int d\mathbf{k}'\int d\mathbf{k}\phi_d^*(\mathbf{k}')\times\langle \mathbf{K}'\mathbf{k}'|\mathfrak{A}(E+i\eta)|\mathbf{K}\mathbf{k}\rangle\phi_d(\mathbf{k}), \quad \text{(D3)}$$

where spin sums are left implicit. Similarly, the amplitude for the breakup reaction He$^4(d,np)$He^4 is given by

$$\langle \mathbf{K}'\mathbf{k}'|B^{(+)}(E)|\mathbf{K};d\rangle=\lim_{\eta\to 0^+}\int d\mathbf{k}\langle \mathbf{K}'\mathbf{k}'|\mathfrak{B}(E+i\eta)|\mathbf{K},\mathbf{k}\rangle\phi_d(\mathbf{k}), \quad \text{(D4)}$$

where the formal breakup operator $\mathfrak{B}$ is given by either of the forms

$$\mathfrak{B}(W)=(V_{np}+V_{n\alpha}+V_{p\alpha})\times(1+(W-H)^{-1}(V_{n\alpha}+V_{p\alpha})) \quad \text{(D5a)}$$

or

$$\mathfrak{B}(W)=[1+(V_{np}+V_{n\alpha}+V_{p\alpha})\times(W-H)^{-1}](V_{n\alpha}+V_{p\alpha}). \quad \text{(D5b)}$$

Since both these forms yield identical on-shell breakup amplitudes, and since we deal only with on-shell quantities in our dispersion relations, we have not bothered to distinguish between the two forms.

Comparing Eq. (D5b) with (D2), we see that

$$\mathfrak{B}(W)=\mathfrak{A}(W)+V_{np}(W-H)^{-1}(V_{n\alpha}+V_{p\alpha}). \quad \text{(D6)}$$

However, using the ubiquitous identity

$$(W-H)^{-1}=[(W-H)^{-1}(V_{n\alpha}+V_{p\alpha})+1]\times(W-H_0-V_{np})^{-1} \quad \text{(D7)}$$

together with Eq. (D2), we find

$$(W-H)^{-1}(V_{n\alpha}+V_{p\alpha})=(W-H_0-V_{np})^{-1}\mathfrak{A}(W). \quad \text{(D8)}$$

Finally, we require one other well-known identity, namely,

$$V_{np}(W-H_0-V_{np})^{-1}=t_{np}(W)(W-H_0)^{-1}, \quad \text{(D9)}$$

where

$$t_{np}(W)=V_{np}+V_{np}(W-H_0)^{-1}t_{np}(W) \quad \text{(D10)}$$

defines the n-p scattering matrix in the 3-particle space. Combining (D6), (D8), and (D9), we obtain Eq. (30), which may be formally written

$$\mathfrak{B}(W)=\mathfrak{A}(W)+t_{np}(W)G_0(W)\mathfrak{A}(W). \quad \text{(D11)}$$

The scattering operator for He$^4+n+p\to$ He^4+n+p is written

$$\mathfrak{C}(W)=V+V(W-H)^{-1}V, \quad \text{(D12)}$$

where $V\equiv V_{np}+V_{n\alpha}+V_{p\alpha}$. Comparing (D12) with (D5a), we find

$$\mathfrak{C}(W)=\mathfrak{B}(W)+V(W-H)^{-1}V_{np}. \quad \text{(D13)}$$

Using the transpose of Eq. (D7) and (D5a) we find

$$V(W-H)^{-1}=\mathfrak{B}(W)(W-H_0-V_{np})^{-1}, \quad \text{(D14)}$$

which together with the transpose of (D9) immediately yields

$$\mathfrak{C}(W)=\mathfrak{B}(W)+\mathfrak{B}(W)(W-H_0)^{-1}t_{np}(W). \quad \text{(D15)}$$

Equation (D15) is just Eq. (36).