One-Boson-Exchange Potential and Nuclear Matter*

D. KIANG,‡ M. A. PRESTON, AND P. YIP
Department of Physics, McMaster University, Hamilton, Ontario, Canada
(Received 3 August 1967)

The one-boson-exchange potential model is studied in some detail. The parameters of a velocity-dependent potential based on such a model are fitted to reproduce two-body scattering data. This potential is then used in nuclear-matter calculations. It is shown that the average binding energy of a nucleon in nuclear matter and the saturation density are in reasonable agreement with the experimental values.

1. INTRODUCTION

IT has been apparent for some time that one can fit all the two-nucleon data with any one of several phenomenological potentials, particularly if one allows momentum dependence. It may be expected that nuclear-matter calculations will help choose between such potentials, since these calculations involve matrix elements which do not arise in the two-nucleon system, namely, those between states of different energy. Of course, for this purpose the nuclear-matter calculation must be reliable and of known accuracy; this situation has recently been achieved.

The early calculations by Brueckner and Gammel,[1] Brueckner and Masterson,[2] and Razavy[3] did however suggest that potentials with infinitely hard repulsive cores would give too little binding. Attention has recently shifted to potentials with soft repulsive cores. Also some years ago, Green[4] replaced the infinite hard core by a velocity-dependent phenomenological potential of the type

$$V(r)=V_0(r)+p^2W(r)+W(r)p^2,$$

with $W(r)$ Gaussian. He concluded that such a potential, if $W(r)$ is weak enough to permit use of perturbation theory, does not give saturation at reasonable densities. However, this is not an inherent property of a velocity-dependent potential, but to obtain saturation one must have W so strong that perturbation theory cannot be employed.[5]

With the discovery of vector mesons, capable of producing short-range nucleon-nucleon repulsion, there has been a revival of interest in potentials which are only in part phenomenological, guidance being obtained from observed resonances of the two-nucleon system. Such boson-exchange potentials possess in principle both soft cores and momentum dependence. Impetus for such studies was given by D. Wong's successful calculations of the two-nucleon system based on exchange of observed bosons and using dispersion relations; he and Scotti obtained good agreement with observed cross sections, polarizations, and correlation parameters.[6] Bryan, Dismukes, and Ramsay[7] gave a static-potential model arising from the exchange of three types of mesons: one scalar meson, one vector, and one pseudoscalar meson π. In a later paper, Bryan and Scott[8] improved the fit to two-body scattering data by making use of the exchange of six different mesons.

D. Wong,[9] treating the potential derivation relativistically, arrived at a nonlocal one-boson-exchange potential (OBEP) with explicit velocity dependence as well. By expansion of the nonlocality in powers of relative momentum, D. Wong also expressed his potential in a local but explicitly velocity-dependent form.

Although it was not explicitly related by its author to meson fields, the static soft-core, three-term Yukawa potential used by C. Wong[10] may be regarded as arising from the exchange of three different mesons. With three adjustable parameters, C. Wong reproduced $^1S_0$-state scattering data. Assuming the interaction to occur only in the $^1S_0$ state, he estimated that a soft-core potential would give more binding in nuclear matter than a hard-core potential. Among the various potentials studied by Sprung *et al.*[11] in their recent nuclear-matter calculations is the soft-core Bressel potential. Their calculation also indicated that a soft-core potential provides more binding than hard-core potentials.

The aims of the present investigation are as follows: First, we wish to study the form of the velocity dependence arising from the OBEP model. We shall see that there are rather severe restrictions on the velocity dependence: to order p^2, the radial dependence $W(r)$ is necessarily a Yukawa; the velocity dependence occurs in the tensor and spin-orbit forces as well as in the central forces; and that to order p^2 the Schrödinger equation has a nonphysical singularity. We would like to employ the momentum-dependent potential pro-

* Supported in part by a grant from the National Research Council of Canada.
‡ Now at Dalhousie University, Halifax, Nova Scotia, Canada.
[1] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1953).
[2] K. A. Brueckner and K. S. Masterson, Jr., Phys. Rev. **128**, 2267 (1962).
[3] M. Razavy, Phys. Rev. **130**, 1091 (1963).
[4] A. M. Green, Nucl. Phys. **33**, 218 (1962).
[5] R. K. Bhaduri and M. A. Preston, Can. J. Phys. **42**, 696 (1964).
[6] A. Scotti and D. Y. Wong, Phys. Rev. **138**, B145 (1965).
[7] R. A. Bryan, C. R. Dismukes, and W. Ramsay, Nucl. Phys. **45**, 353 (1963).
[8] R. A. Bryan and B. L. Scott, Phys. Rev. **135**, B434 (1964).
[9] D. Y. Wong, Nucl. Phys. **55**, 212 (1964).
[10] C. W. Wong, Nucl. Phys. **71**, 385 (1965).
[11] D. W. L. Sprung, P. C. Bhargava, and T. K. Dahlblom, Phys. Letters **21**, 538 (1966).

duced by the following bosons, which are the only ones up to nucleon mass which nucleons exchange: π, η, ρ, ω, and an isoscalar scalar. With this potential, one would like to correlate as many observed numbers as possible. One can envisage a quite extensive parameter-fitting problem, as well as the need to solve reliably a good many integrodifferential equations.

Before embarking on such an extensive calculation, we have studied the nuclear-matter properties of a model of this full potential in which the exchange of only three bosons has been considered: π, one "isoscalar" vector, and one isoscalar scalar. The vector meson has been given a different coupling constant in the singlet S state than would arise for an isoscalar exchange; thus the actual existence of both isoscalar and isovector vector mesons is partially mocked up. In Sec. 2, we adjust this OBEP model to fit qualitatively the two-body scattering data. Section 3 is an account of the nuclear-matter calculation using the reference-spectrum method developed by Bethe *et al.*[12] In Sec. 4 we discuss our results and compare them with other works.

2. ONE-BOSON-EXCHANGE POTENTIAL

A. Potential Model

We shall use D. Wong's expanded, local, but velocity-dependent potential [see his Eq. (12) Ref. 9]. We assume the bosons exchanged to be the pion, one isoscalar scalar meson, and one isoscalar vector meson. Furthermore, for simplicity, we only include direct coupling for the vector meson, and we also neglect the quadratic spin-orbit term. The form of the potential we use is, for a given isospin state,

$$V_{\text{total}}=V+V_TS_{12}+V_{LS}\mathbf{L}\cdot\mathbf{S} +p^2[W+W_TS_{12}+W_{LS}\mathbf{L}\cdot\mathbf{S}] +[W+W_TS_{12}+W_{LS}\mathbf{L}\cdot\mathbf{S}]p^2, \quad (2.1)$$

where

$$V=V_C+V_\sigma\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,$$
$$W=W_C+W_\sigma\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,$$

and S_{12} is the usual tensor operator. We shall also write $V_1=V_C-3V_\sigma$, $V_3=V_C+V_\sigma$ and similarly for W_1 and W_3. We have not put in the $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$ dependence explicitly, but we shall write separate potentials for the $T=0$ and $T=1$ states. The potentials are then, with

$$Y(\alpha)=(1/r)e^{-\mu_\alpha r},$$

$S=0$, $T=1$:

$$V_1=-g_S^2Y(S)(1-\tfrac{1}{4}x_S^2-\tfrac{1}{32}x_S^4)-\tfrac{1}{4}g_\pi^2x_\pi^2Y(\pi) +g_V^2Y(V)(1-\tfrac{1}{32}x_V^4),$$
$$W_1=\tfrac{1}{4}g_S^2Y(S)(1-\tfrac{1}{8}x_S^2-\tfrac{1}{16}x_S^4)+\tfrac{1}{16}g_\pi^2x_\pi^2Y(\pi) +\tfrac{1}{4}g_V^2Y(V)(3-\tfrac{1}{8}x_V^2+\tfrac{1}{16}x_V^4),$$
$$V_T=V_{LS}=W_T=W_{LS}=0; \quad (2.2)$$

$S=0$, $T=0$:

$$V_1=-g_S^2Y(S)(1-\tfrac{1}{4}x_S^2-\tfrac{1}{32}x_S^4)+\tfrac{3}{4}g_\pi^2x_\pi^2Y(\pi) +g_V^2Y(V)(1-\tfrac{1}{32}x_V^4),$$
$$W_1=\tfrac{1}{4}g_S^2Y(S)(1-\tfrac{1}{8}x_S^2-\tfrac{1}{16}x_S^4)-\tfrac{3}{16}g_\pi^2x_\pi^2Y(\pi) +\tfrac{1}{4}g_V^2Y(V)(3-\tfrac{1}{8}x_V^2+\tfrac{1}{16}x_V^4),$$
$$V_T=V_{LS}=W_T=W_{LS}=0; \quad (2.3)$$

$S=1$, $T=0$:

$$V_3=-g_S^2Y(S)(1-\tfrac{1}{4}x_S^2+\tfrac{1}{32}x_S^4)-\tfrac{1}{4}g_\pi^2x_\pi^2Y(\pi) +g_V^2Y(V)(1+\tfrac{2}{3}x_V^2+\tfrac{1}{32}x_V^4),$$
$$V_T=-\tfrac{1}{4}g_\pi^2Y(\pi)[(3/r^2)+(3x_\pi/r)+x_\pi^2] -[g_V^2Y(V)/12][(3/r^2)+(3x_V/r)+x_V^2],$$
$$V_{LS}=-g_S^2Y(S)[(1/r^2)+(x_S/r)](\tfrac{1}{2}-\tfrac{1}{16}x_S^2) -g_V^2Y(V)[(1/r^2)+(x_V/r)](\tfrac{3}{2}+\tfrac{1}{16}x_V^2),$$
$$W_3=\tfrac{1}{4}g_S^2Y(S)[1-(11x_S^2/24)+\tfrac{1}{16}x_S^4]+\tfrac{1}{16}g_\pi^2x_\pi^2Y(\pi) +\tfrac{1}{4}g_V^2Y(V)[3-(11x_V^2/24)-\tfrac{1}{16}x_V^4],$$
$$W_T=[g_S^2Y(S)/96][(3/r^2)+(3x_S/r)+x_S^2] +\tfrac{1}{16}g_\pi^2Y(\pi)[(3/r^2)+(3x_\pi/r)+x_\pi^2] +[g_V^2Y(V)/96][(3/r^2)+(3x_V/r)+x_V^2],$$
$$W_{LS}=\tfrac{1}{16}g_S^2Y(S)[(1/r^2)+(x_S/r)](3-\tfrac{1}{2}x_S^2) +\tfrac{1}{16}g_V^2Y(V)[(1/r^2)+(x_V/r)](5+\tfrac{1}{2}x_V^2); \quad (2.4)$$

$S=1$, $T=1$:

$$V_3=-g_S^2Y(S)(1-\tfrac{1}{4}x_S^2+\tfrac{1}{32}x_S^4)+[g_\pi^2x_\pi^2Y(\pi)/12] +g_V^2Y(V)(1+\tfrac{2}{3}x_V^2+\tfrac{1}{32}x_V^4),$$
$$V_T=[g_\pi^2Y(\pi)/12][(3/r^2)+(3x_\pi/r)+x_\pi^2] -[g_V^2Y(V)/12][(3/r^2)+(3x_V/r)+x_V^2],$$
$$V_{LS}=-g_S^2Y(S)[(1/r^2)+(x_S/r)](\tfrac{1}{2}-\tfrac{1}{16}x_S^2) -g_V^2Y(V)[(1/r^2)+(x_V/r)](\tfrac{3}{2}+\tfrac{1}{16}x_V^2),$$
$$W_3=\tfrac{1}{4}g_S^2Y(S)[1-(11x_S^2/24)+\tfrac{1}{16}x_S^4] -(g_\pi^2x_\pi^2/48)Y(\pi) +\tfrac{1}{4}g_V^2Y(V)[3-(11x_V^2/24)-\tfrac{1}{16}x_V^4],$$
$$W_T=[g_S^2Y(S)/96][(3/r^2)+(3x_S/r)+x_S^2] -[g_\pi^2Y(\pi)/48][(3/r^2)+(3x_\pi/r)+x_\pi^2] +[g_V^2Y(V)/96][(3/r^2)+(3x_V/r)+x_V^2],$$
$$W_{LS}=\tfrac{1}{16}g_S^2Y(S)[(1/r^2)+(x_S/r)](3-\tfrac{1}{2}x_S^2) +\tfrac{1}{16}g_V^2Y(V)[(1/r^2)+(x_V/r)](5+\tfrac{1}{2}x_V^2). \quad (2.5)$$

Here

$$Y(\alpha)=e^{-\mu_\alpha r}/r,$$
$$x_\alpha=\mu_\alpha/M,$$

and the g^2's are the coupling constants. We adopt a unit system in which $\hbar=c=1$, and the nucleon mass M is set to be unity also. It should be added that the potential functions $W_i(r)$ of the velocity-dependent part as listed above are D. Wong's results divided

[12] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. 129, 225 (1963).

by two, owing to the Hermitization process

$$p^2W_i \rightarrow \tfrac{1}{2}(p^2W_i+W_ip^2).$$

B. Schrödinger Equation

Let us consider the triplet case, as the singlet case can also be readily obtained from it. We expand the total wave function

$$\Psi(\mathbf{r})=\sum_L \frac{u_{JL}(r)}{r}\mathcal{Y}_{JLS=1}{}^M.$$

The two-body Schrödinger equation is then (in units of $\hbar=M=c=1$), for

$L=J-1$:

$$u_{J,J-1}''\left[-(1+2W_3)+\frac{4(J-1)}{(2J+1)}W_T-2(J-1)W_{LS}\right]+u_{J,J-1}'\left[-2W_3'+\frac{4(J-1)}{(2J+1)}W_T'-2(J-1)W_{LS}'\right]$$
$$+u_{J,J-1}\left[\frac{J(J-1)(1+2W_3)}{r^2}+V_3-\frac{2(J-1)}{(2J+1)}V_T+(J-1)V_{LS}-W_3''-\frac{4(J-1)^2J}{(2J+1)}\frac{W_T}{r^2}+2J(J-1)^2\frac{W_{LS}}{r^2}\right.$$
$$\left.+\frac{2(J-1)}{(2J+1)}W_T''-(J-1)W_{LS}''\right]+u_{J,J+1}''\left[-\frac{12[J(J+1)]^{1/2}}{(2J+1)}W_T\right]+u_{J,J+1}'\left[-\frac{12[J(J+1)]^{1/2}}{2J+1}W_T'\right]$$
$$+u_{J,J+1}\left[\frac{6[J(J+1)]^{1/2}}{(2J+1)}V_T-\frac{6[J(J+1)]^{1/2}}{(2J+1)}W_T''+\frac{12(J^2+J+1)[J(J+1)]^{1/2}}{(2J+1)}\frac{W_T}{r^2}\right]=k^2u_{J,J-1}; \quad (2.6)$$

$L=J+1$:

$$u_{J,J+1}''\left[-(1+2W_3)+\frac{4(J+2)}{(2J+1)}W_T+2(J+2)W_{LS}\right]+u_{J,J+1}'\left[-2W_3'+\frac{4(J+2)}{(2J+1)}W_T'+2(J+2)W_{LS}'\right]$$
$$+u_{J,J+1}\left[\frac{(J+1)(J+2)(1+2W_3)}{r^2}+V_3-\frac{2(J+2)}{(2J+1)}V_T-(J+2)V_{LS}-W_3''-\frac{4(J+1)(J+2)^2W_T}{(2J+1)r^2}\right.$$
$$\left.-\frac{2(J+1)(J+2)^2}{r^2}W_{LS}+\frac{2(J+2)}{(2J+1)}W_T''+(J+2)W_{LS}''\right]+u_{J,J-1}''\left[-\frac{12[J(J+1)]^{1/2}}{(2J+1)}W_T\right]$$
$$+u_{J,J-1}'\left[-\frac{12[J(J+1)]^{1/2}}{(2J+1)}W_T'\right]+u_{J,J-1}\left[\frac{6[J(J+1)]^{1/2}}{(2J+1)}V_T-\frac{6[J(J+1)]^{1/2}}{(2J+1)}W_T''\right.$$
$$\left.+\frac{12(J^2+J+1)[J(J+1)]^{1/2}}{(2J+1)}\frac{W_T}{r^2}\right]=k^2u_{J,J+1}. \quad (2.7)$$

Here $k^2=E$. We note that Eqs. (2.6) and (2.7) are coupled because of the tensor force. For the $L=J$ state, we have an uncoupled equation:

$L=J$:

$$u_{J,J}''[-(1+2W_3)-4W_T+2W_{LS}]+u_{J,J}'[-2W_3'-4W_T'+2W_{LS}']+u_{J,J}\left[\frac{J(J+1)(1+2W_3)}{r^2}+V_3+2V_T-V_{LS}\right.$$
$$\left.-W_3''-2W_T''+4J(J+1)\frac{W_T}{r^2}+W_{LS}''-\frac{2J(J+1)}{r^2}W_{LS}\right]=k^2u_{J,J}. \quad (2.8)$$

The equation for the singlet state is simply

$$(1+2W_1)u_l''+2W_1'u_l'-\left[\frac{(1+2W_1)l(l+1)}{r^2}+V_1-W_1''\right]u_l=k^2u_l. \quad (2.9)$$

C. Results of Phase-Shift Analysis

We find that the parameters used by Bryan, Dismukes, and Ramsay[7] give us surprisingly good results. We are forced, however, to make a few *ad hoc* state-dependent adjustments. The vector meson coupling

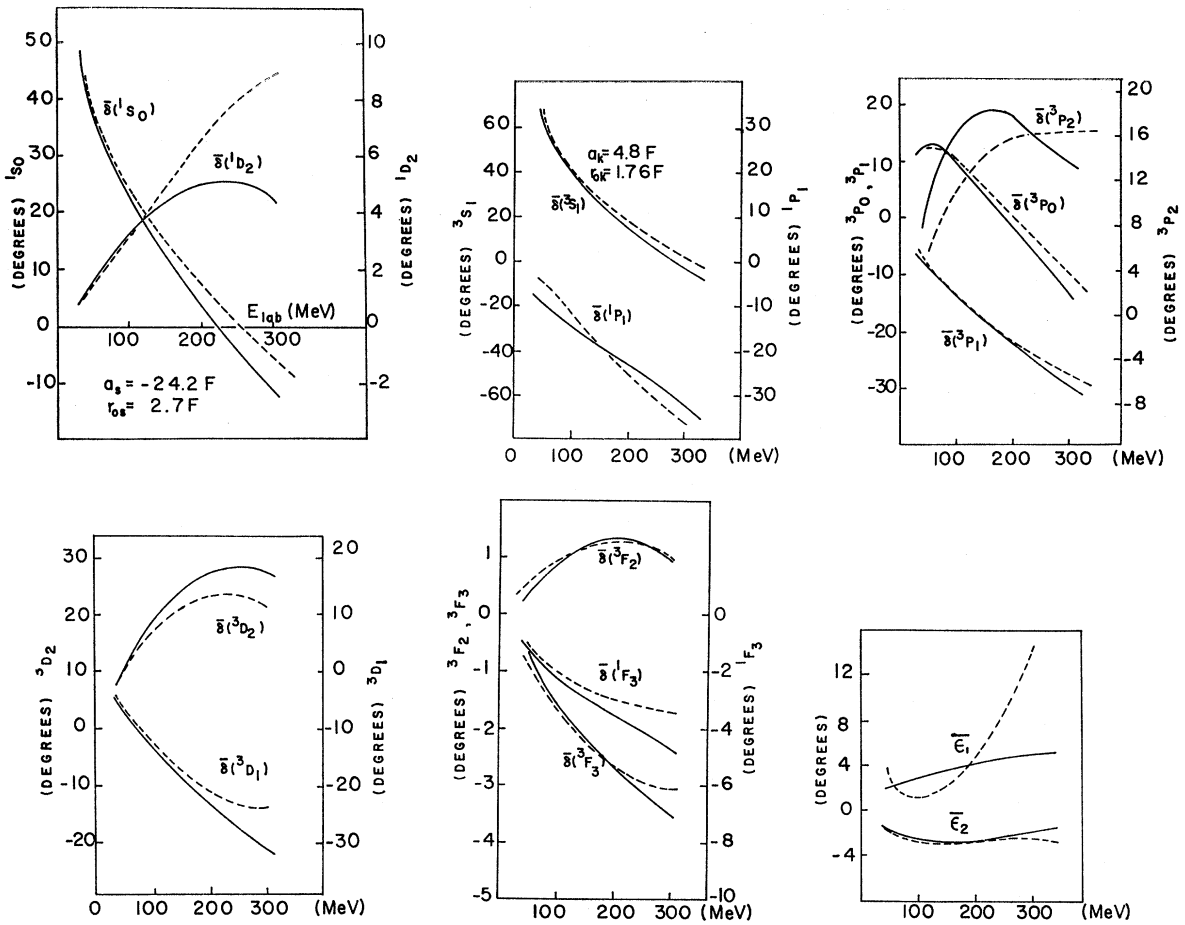

FIG. 1. Phase-shift characteristics. The solid curves show the results of calculation with potential M1. Dotted curves display the values of Table V of the analysis of R. A. Arndt and M. H. McGregor [Phys. Rev. 141, 873 (1966)].

constant has to be greatly increased in order to get any reasonable result for the singlet-odd states. We also need to modify the parameters for the singlet-even state slightly. The potential parameters are listed in Table I. All states have a "mathematical" hard core of $r_c=0.074$ F, i.e., we set our wave function to be zero inside this region and start our numerical integration of our differential equations at r_c.

The phase shifts calculated from the potential are compared in Fig. 1 with "experimental" phase shifts. The agreement may be described as adequate. We have not attempted to obtain a very precise fit, since our potential is only a model of the six-boson OBEP and our concern was to determine if such a potential would give encouraging results for nuclear matter.

In order to compare with C. Wong's work,[10] we also fix $g_\pi^2=14.0$, $m_\pi=0.147$ nucleon mass, $m_V=1.051$ nucleon mass, and $r_c=0.001$ F, all very close to C. Wong's values. We then adjust the remaining three potential parameters to reproduce his scattering data for the $^1S_0$ state. Table II shows the parameters for this model which is assumed to act in the $^1S_0$ state only. We designate this model by M2 and the full-potential model of Table I by M1.

TABLE I. Parameters for potential model M1. $r_c=0.074$, μ in units of reciprocal nucleon Compton wavelength.

	Singlet even	Singlet odd	Triplet even	Triplet odd
μ_π	0.147	0.147	0.147	0.147
g_π^2	14.0	14.0	14.0	14.0
μ_V	0.8085	0.8085	0.8085	0.8085
g_V^2	34.0	74.0	34.0	34.0
μ_S	0.598	0.588	0.588	0.588
g_S^2	15.3	15.4	15.4	15.4

TABLE II. Parameters for potential model M2. $r_c=0.0001$ F, M in units of nucleon mass.

Predetermined		Adjusted	
g_π^2	14.0	g_S^2	2.42
M_π	0.147	g_V^2	42.24
M_V	1.0508	M_S	0.40

3. NUCLEAR-MATTER CALCULATION

Using potential model M1 we then calculate nuclear-matter properties by means of the reference-spectrum method developed by Bethe *et al.*[12] For the first-order calculation we follow quite closely Razavy's outline.[3] We differ from him in that we calculate the hole potential energy for only an average collision momentum k_0 and that we do not attempt separation in any angular momentum state. The basic quantity involved is G^R, the reaction matrix in the reference spectrum, given by

$$(k_0|G_L{}^R|k_0)=\frac{(\gamma^2+k_0^2)}{m^*}\int\phi_L(k_0r)\chi_L dr\,, \qquad (3.1)$$

where the subscript L denotes the contribution from the Lth partial wave ϕ_L is the unperturbed wave function, χ_L is the wave defect, and

$$\gamma^2=2\Delta k_F{}^2-k_0{}^2 \qquad (3.2)$$

for holes, and

$$\gamma^2=3k_0{}^2+(3\Delta-0.6)k_F{}^2$$

for particles. The gap Δ and the effective mass m^* are two parameters which should be generated self-consistently if higher-order terms are to be small.

The total wave defect

$$\zeta^R=\frac{1}{k_0r}\sum_L(2L+1)i^L\chi_L(k_0r)P_L(\cos\theta)$$

satisfies the equation

$$(\gamma^2-\Delta^2)\zeta^R=m^*V\psi^R. \qquad (3.3)$$

This equation is solved by Ridley's method.[13,14] Before solving this differential equation, we must first decide on an approximate expression for γ^2. The expressions (3.2) used by Bethe *et al.* are meant to take the off-shell propagation effect into account, and are tailored for a hard-core potential. In Fig. 2 we show a plot of the $^1S_0$-state wave defect χ versus r for our nonstatic, soft-core potential, and in Fig. 3 we plot the square of the Fourier transform of the wave defect χ. We see that our important intermediate states are in a momentum region comparable to that for a hard-core potential, and therefore we conclude we may use the same expression as Bethe *et al.*[12] for γ^2.

Once the G^R matrix is obtained, the mean single-particle potential energy for the hole state m is given by

$$\bar{U}_m\approx(2k_F{}^3/3\pi^2)(k_0=(\sqrt{0.3})k_F|G^R|k_0)\times 41.5\text{ MeV}. \qquad (3.4)$$

Here G^R denotes the sum of all the partial waves, including their appropriate statistical weights. We then

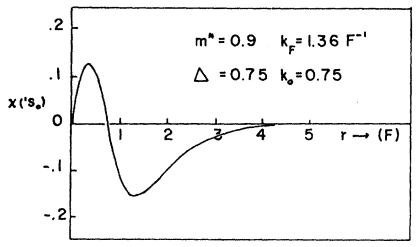

FIG. 2. Wave defect for the singlet S state.

calculate the G matrix for the particle states. Let

$$W^{\text{particle}}(k_0)\equiv 41.5\frac{2k_F{}^3}{3\pi^2}\sum_{\text{even }L}(k_0|G_L{}^R|k_0). \qquad (3.5)$$

We compute W^{particle} for two k_0's; namely, for $k_0=1.5k_F$ and $k_0=2k_F$, and we fit

$$W^{\text{particle}}(k_0)=A'+B'k_0{}^2 \qquad (3.6)$$

by a parabola. Then the single-particle potential energy for particle state is, as shown in Razavy,[3]

$$U(k_b)=(A'+\tfrac{1}{4}0.6k_F{}^2B')+(\tfrac{1}{4}B')k_b{}^2\,, \qquad (3.7)$$

and

$$m_f{}^*=[1+(B'M/2\hbar^2)]^{-1}, \qquad (3.8)$$

$$\Delta_f=\frac{m_f{}^*}{41.5}\frac{1}{k_F{}^2}\{U[\sqrt{(0.6)}k_F]-\bar{U}\}. \qquad (3.9)$$

For a given k_F, we choose a guessed Δ_i and $m_i{}^*$ in Eq. (3.3) and see if the outputs Δ_f and $m_f{}^*$ are consistent with the input. If not, we repeat the cycle. The average energy per particle is given by

$$\bar{E}=\bar{T}+\tfrac{1}{2}\bar{U}_m=\tfrac{3}{10}(\hbar^2/M)k_F{}^2+\tfrac{1}{2}\bar{U}_m. \qquad (3.10)$$

The above procedure forms a major cycle in our computation. We execute major cycles for $k_F=1.25$, 1.36, and 1.50 F^{-1}. In Table III we show the first-order contribution to the G matrix from various partial waves.

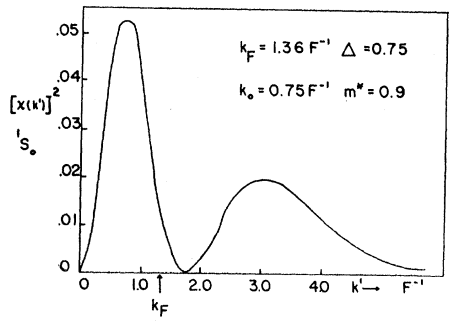

FIG. 3. Square of the Fourier transform of the wave defect for $^1S_0$ state.

[13] E. C. Ridley, Proc. Cambridge Phil. Soc. **53**, 442 (1957).
[14] M. Razavy and D. W. L. Sprung, Phys. Rev. **133**, B300 (1964).

TABLE III. Partial-wave contribution to the G matrix for $k_F=1.36$ F^{-1}.

States	W_l (MeV)
$^1S_0$	−32.30
$^1P_1$	+ 8.50
$^1D_2$	− 5.29
$^1F_3$	+ 1.79
$^1G_4$	− 0.64
$^3P_0$	− 8.97
$^3P_1$	+21.43
$^3D_2$	− 5.67
$^3F_3$	+ 0.84
$^3G_4$	− 0.19
$^3S_1$	−43.41
$^3D_1$	+ 2.98
$^3P_2$	− 9.43
$^3F_2$	− 1.18
$^3D_3$	− 0.08
$^3G_3$	+ 0.28
$^3F_4$	− 0.62
$^3H_4$	− 0.06
Total	−72.02

Having obtained the first-order result G^R, we then proceed to calculate the second-order term in

$$G^N\approx G^R+G^R[(1/e^R)-(Q/e^N)]G^R, \quad (3.11)$$

where e^R is the reference spectrum, e^N the actual spectrum, and Q is the Pauli operator which ensures that all the intermediate states are projected outside the Fermi sea. As in Bethe *et al.*[12] e^R and e^N are given by

$$e^{R,N}(\bar{k}_0,k')=2[T(k')+U^{R,N}(k')-T(\bar{k}_0)-U^N(\bar{k}_0)], \quad (3.12)$$

where T and U are kinetic and potential energies' respectively, and $\bar{k}_0$ is the average relative momentum of hole states. The calculations follow closely that of Bethe *et al.*,[12] where the total second-order correction is given by

$$\int_0^\infty \epsilon(k')\mathfrak{F}_{S,T}{}^M(k')dk', \quad (3.13)$$

where

$$\epsilon(k')=e^R \quad \text{for} \quad k'<k_F$$
$$=(e^N-e^R)e^R/e^N \quad \text{for} \quad k'\geqslant k_F$$

and $\mathfrak{F}_{S,T}{}^M(k')$ is the weighted sum of squares of Fourier transforms from the wave-function defects in different states. In our case, only contributions from singlet S state and triplet S-coupled state are calculated, contributions from other states being relatively small. Therefore, we have

$$\mathfrak{F}_{S,T}{}^M(k')=3F_0{}^2(k')+3[F_{0,1}{}^2(k')+F_{2,1}{}^2(k')], \quad (3.14)$$

where F is the Fourier transform of the appropriate wave-function defect, and the notation follows that of Bethe *et al.*[12]

$U^R(k')$, the reference-particle potential energy is as given in (3.7). The nuclear spectrum $U^N(k')$ for the particle states is calculated explicitly at four different values of k', and interpolation is used to obtain U^N at the values of k' needed for the second-order corrections. Results of this calculation are included in Table IV.

The above consideration of the second-order correction does not include the three-body cluster effect except for the fact that Rajaraman's prescription[15] is used in calculating the particle potential energy. Bethe,[16] however, suggested that this effect be incorporated in the calculation by evaluating

$$(k_0|G^Rf(r)|k_0)_L$$

for all states, where $f(r)$ is the so-called "suppression factor," given by

$$f(r_{23})=F(r_{23})/F_1(r_{23}), \quad (3.15)$$

where

$$F_1(r_{23})=\int d\tau_1\eta_{12}(\eta_{12}+\eta_{13}),$$

$$F(r_{23})=\int d\tau_1\eta_{12}[\Phi-\Psi^{(1)}(r_{12},r_{13},r_{23})],$$

$$\Phi-\Psi^{(1)}=\frac{\eta_{12}u_{13}+\eta_{13}u_{12}-\eta_{23}(u_{12}+u_{13}-2u_{12}u_{13})}{u_{12}u_{13}+u_{12}u_{23}+u_{31}u_{32}-2u_{12}u_{23}u_{31}},$$

$$u_{12}=1-\zeta_{12},$$

as in Bethe.[16] η and ζ are the normalized wave-function distortions for hole-hole and particle-hole interactions, respectively. These are calculated by the modified Moszkowski-Scott separation method[12] in which both the wave-function distortion and its derivative are defined to be zero at the separation distance. We solve this second-order differential equation by a variation of Ridley's method.[13] The suppression factor $f(r)$ is then calculated at three particle momenta (3, 4, and 5 F^{-1}) representative of the region of wave number where the wave-defect Fourier transform is largest. We find that $f(r)$ varies only slightly in this region, and the result at 4 F^{-1} is used for our calculations. In obtaining this suppression factor, we use only the ζ and η for the $^1S_0$ state; these functions represent the angular averages of the total ζ and η. The procedure follows closely that adopted by Bhargava and Sprung.[17]

TABLE IV. Nuclear-matter calculation results.

k_F	M_i^*	Δ_i	M_f^*	Δ_f	$\bar{E}/A$ (1st order) (MeV)	2nd order without 3-body effect (MeV)	2nd order with 3-body effect (MeV)
1.25 F^{-1}	0.80	0.70	0.83	0.50	−10.0	−4.60	−1.37
1.36 F^{-1}	0.90	0.70	0.83	0.64	−13.25	−2.72	+0.82
1.50 F^{-1}	0.80	0.65	0.77	0.53	− 8.75	−5.66	−1.30

[15] R. Rajaraman, Phys. Rev. 129, 265 (1963).
[16] H. A. Bethe, Phys. Rev. 138, B804 (1965).
[17] P. C. Bhargava and D. W. L. Sprung, Ann. Phys. (N. Y.) 42, 222 (1967).

The over-all effect of $f(r)$ is to lower the particle energy spectrum, thus making a recalculation of the second-order correction necessary. The results are tabulated in Table IV. The final binding energy of the nuclear matter calculation is shown in Fig. 4. Also, in Table V, we compare our results with those of Sprung *et al.*,[11] using the Reid soft-core and Bressel-Kerman soft-core potentials. They are in reasonable agreement.

It is also interesting to employ the simple potential model M2 to estimate nuclear-matter properties by means of the Moszkowski-Scott separation method.[18] This will allow us to compare the binding effect of a velocity-dependent potential with C. Wong's static potential.[10] We follow C. Wong's procedure and use the same notations. It is expected that two potentials yielding comparable two-body scattering data will also produce comparable first-order Born term

$$V_l=\frac{4\pi}{k_0^2}\int_d^\infty \sin^2(k_0r)V_l(r)dr \tag{3.16}$$

in nuclear matter. Here the V_l in the integral denotes the long-range part of the potential, and d the separation distance. The two potentials may have very different short-range correlation, thus producing different values for the dispersion term. Difference in binding energy is then mainly attributed to the difference in the dispersion term. An estimate of this dispersion term E_D is given by

$$\begin{aligned} E_D &\simeq \tfrac{3}{16}(2\Delta U)\frac{4\pi\rho}{k_0^2}\int_0^d \chi_{k_0}{}^2dr \\ &\equiv \tfrac{3}{16}(2\Delta U)D. \end{aligned} \tag{3.17}$$

Here ΔU is the difference in the single-particle potential between an average excited state and an average state inside the Fermi sea, k_0 is the average value of the relative momentum k, ρ is the nuclear-matter density, and χ is the wave defect. Thus the difference in the binding energy per particle in a nuclear-matter calculation using two different potentials which produce the same phase shifts is

$$\begin{aligned} \Delta E \approx \Delta E_D &\approx \tfrac{3}{16}(2\Delta U)\Delta D \\ &= 26.25\Delta D \end{aligned} \tag{3.18}$$

if ΔU is taken to be 70 MeV. Comparison of our results with C. Wong's is shown in Table VI. We observe that, relative to C. Wong, we have an extra binding energy of $|\Delta E_D| \approx 1.2$ MeV. We must note, however, that comparison in the $^1S_0$ state alone may not be very conclusive because of the dependence on r_0 and thus the added arbitrariness. Also, all we have seen is that perhaps we shall get more binding, but nothing is said of the saturation problem. Moreover, we have not considered the interference terms in the separation method.

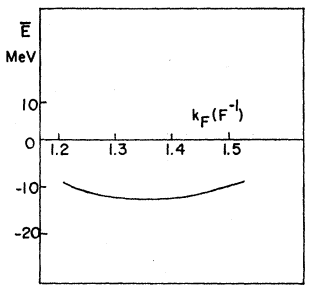

FIG. 4. Binding-energy curve.

TABLE V. Comparison of binding-energy and saturation results.

Potential	E/A at saturation (MeV)	Saturation (k_F) (F^{-1})
Reid[a]	−18.6	1.40
Bressel-Kerman[a]	−15.1	1.50
M1	−12.8	1.33

[a] These results include a third-order correction term and are quoted from Bhargava (see Ref. 22).

4. DISCUSSION

The Schrödinger equations (2.6)–(2.8) for the triplet states in general have a singularity. (The singlet case presents no problem.) In particular, let us examine the uncoupled triplet equation (2.8), which has the structure

$$a(r)u''+b(r)u'+c(r)u=0. \tag{4.1}$$

Suppose $a(r)$ vanishes at a certain point $r=r_0$, the precise location of which depends on the potential parameter used. It can be seen by usual series solution method that r_0 is a logarithmic singularity. With the potential parameters we use, this point $r=r_0$ occurs at about 0.19 F for the $^3P_1$ and $^3F_1$ states, whereas for the $^3D_2$ and $^4G_3$ states this occurs at $r_0=0.11$ F.

The singularity is unphysical. It arises because we have expanded the potential to the p^2 term, resulting in a particular type of velocity dependence. Had we expanded our potential to the p^4 term, we would have a fourth-order differential equation which would have a singularity presumably at a smaller r_0. If we use the

TABLE VI. Comparison of static and velocity-dependent three-meson potential models. Fermi momentum $k_F=1.36$ F^{-1}; relative momentum $k_0=0.75$ F^{-1}.

	C. Wong's Y_{npl}	Model M2
a	− 23.75 F	− 23.58 F
	2.67 F	2.78 F
δ (250 MeV)	− 0.000 rad	0.019 rad
d	1.06 F	1.04 F
V_l	−460.1 MeV F^3	−462.8 MeV F^3
D	0.0622	0.0179

[18] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) 11, 65 (1960).

originally nonlocal potential, we have actually an integrodifferential equation. Even in the singlet $^1S_0$ state, however, the solution of this integrodifferential equation would be lengthy.

In our numerical computation, we ignore this regrettable singularity. Fortunately, the singularity does not seem to be too serious, since the results are not sensitive to the mesh size Δ (variation around $\Delta \simeq 0.01$ F, say). We cannot, of course, reduce the mesh indefinitely. One reason why the singularity is not serious is perhaps because the logarithmic divergence is relatively weak, and therefore the singular admixture in our solution remains finite and small, unless we get really close to the singular point. Another reason may be seen if we note that the singular point r_0 occurs at small distance (never farther out than 0.49 F). In this region, the wave function is still very small, and certainly much smaller than that in the asymptotic region. Thus a little change in the wave function in this innermost region due to a change in the mesh around the singular point will be covered up in the asymptotic region, thus producing no difference in the phase shifts. For nuclear-matter calculations, we are confronted with a bound-state problem: The magnitude of the wave function at large distance is not greater than that in the inside region. Actual computation indicates again, however, that even here the difference in the wave function due to a change in the mesh is tolerable, and the situation is further helped by noting that the interesting quantity, the G matrix, is

$$G \sim \int (\phi - \psi)\phi dr,$$

where ψ is the actual wave function, and ϕ is the unperturbed wave function. For the S state, ϕ is just a sine wave. For most collision momentum, sin kr is quite small for small r and thus, essentially, $(\phi-\psi)$ is reduced greatly and the integrand, in this region where the singularity occurs, has only a small contribution to the G matrix.

In fitting the phase-shift data we try to achieve only a qualitative agreement. We choose $g_\pi^2 = 14.0$, $m_\pi = 0.147$ nucleon mass, and $m_V^2 = 0.8085$ nucleon mass, as predetermined in potential model M1. That we have to make *ad hoc* adjustment for the singlet-odd states is perhaps because we have not included enough mesons and thus we do not have enough adjustable parameters. It could also be due to our ignoring the quadratic spin-orbit term. Our fit to the $^1D_2$ and $^3P_2$ state is especially poor. Bryan, in his investigation of OBEP and velocity dependence, seems to have this difficulty also.

The values of the meson-nucleon coupling constants given in Table I do not, at first sight, appear to be in agreement with values determined from nucleon form factors, particularly for the vector mesons. They also appear to differ from values of the ω, ϕ, and ρ coupling constants found to fit nucleon-nucleon scattering in certain recent analyses.[19] While Table I gives $g_V^2 = 34.0$ (or 74.0 in the singlet-odd state), typical values are $g_\omega^2 + g_\phi^2 \sim 4$, $g_\rho^2 \sim 1$. However, we must recognize that the ρ meson has derivative as well as direct coupling, whereas we use an "effective" vector meson with only direct coupling. If the ratio of derivative to direct coupling for the ρ is about 4, the strength of the leading terms of the nucleon-nucleon potential generated by the ρ corresponds to the exchange of a meson with only direct coupling with a coupling constant greater than the actual direct ρ-coupling constant by a factor of at least 20, but depending on the spin state of the nucleon system. In other words, roughly speaking, if $g_\rho^2 \sim 1$ and $f_\rho/g_\rho \sim 4$, an effective equivalent direct coupling constant for ρ has a value dependent on the state but ~ 20. Moreover, the ρ is an isovector particle, and, apart from the use of a much stronger coupling in the singlet-odd state, we have attempted to fit the data with a single isoscalar vector meson. For this single particle representing both ρ and $\omega - \phi$, it is then reasonable that g^2 should be around 30.

In the Bryan-Dismukes-Ramsay static potential,[7] a zero cutoff is used, i.e., $V = 0$ for $r \leqslant 0.54$ F. Our velocity-dependent potential extends to almost $r = 0$ and yet we find their parameters also give us reasonable results. It seems we should actually use a smaller g_V^2 because we have the static repulsive vector-meson term contributing to $r < 0.54$ F as well. (Of course the scalar meson and the pion also contribute at small r, but the vector meson dominates here.) The fact is that our velocity dependence actually has an attractive effect.[20] This is also evident in the comparison of our M2 model with C. Wong's static potential. The long-range part of the potential determines mainly the phase shifts, whereas the short-range part is important in determining the dispersion term in the separation method. At short range, our potential model M2 is softer than C. Wong's potential, producing a smaller wave defect and thus more binding energy, as actually shown by our calculation.

Qualitatively at least, we see the one-boson-exchange potential model can lead to reasonable results in nuclear-matter calculation, so far as the binding energy and saturation density are concerned. We also estimated the nuclear compressibility according to the results of this calculation. By definition,[21] the nuclear compressibility is $R^2 \partial^2 \bar{E}/\partial R^2$ evaluated at the equilibrium radius, where $\bar{E}$ is the total binding energy per nucleon and R the radius of the nucleus.

Extending this definition to an infinite, noncharged

[19] R. A. Arndt, R. A. Bryan, and M. H. McGregor, Phys. Letters **21**, 314 (1966); G. Köpp and P. Söding, *ibid.* **23**, 494 (1966).

[20] D. Kiang, Phys. Letters **24B**, 132 (1967).

[21] W. J. Swiatecki, Proc. Phys. Soc. (London) **A63**, 1208 (1950). This definition is by no means the only one.

medium, we get the compressibility of nuclear matter as

$$k_F^2(\partial^2\bar{E}/\partial k_F^2)$$

evaluated at the saturation Fermi momentum.

Our graphical estimate, from the binding-energy curve, gives a value of about 400 MeV. This is in disagreement with values generally quoted. However, until a value of nuclear compressibility is better determined, one can really say very little about this rather rough estimate.

As we observe in Table IV, the self-consistency in our reference-spectrum parameters is only modest, but sufficient to reduce second-order terms. We can claim that our results are good up to second order and accurate to within 1 or 2 MeV. Bhargava and Sprung[17] have found in their calculations that if a third-order correction is included (estimated in our case to be 1–2 MeV in magnitude), different spectrum parameters yield almost the same binding and saturation. Thus we argue that a modest amount of self-consistency should be sufficient in this qualitative consideration.

In dealing with the three-body effect, we have followed Bethe's prescription.[18] Subsequently, both Day,[22] and Kirson[23] made improvements on Bethe's solution of the Fadeev equation. However, we have not recalculated $f(r)$ with Day's more accurate prescription, trusting that it will not change our qualitative conclusion, since Day's solution coincides with Bethe's solution within the repulsive core in which the suppression due to three-body cluster is most important.

More recently, Bethe[24] pointed out a discrepancy in the solution of the three-body problem, which would result in less suppression of the repulsive core than previously calculated. Although this effect is significant with a core of infinite height, it may well be of minor importance with a soft core like M1. However, since we do not wish to make a precise quantitative conclusion on our nuclear-matter calculations, we will only say that the binding energy curve shown in Fig. 4 will be pushed up somewhat, if the three-body problem is treated properly. This also applies to the results for the Reid potential and the Bressel-Kerman potential that we have quoted from Bhargava.[25]

[22] B. Day, Phys. Rev. **151**, 826 (1966).
[23] M. Kirson, Ph.D. thesis, Cornell University (unpublished).
[24] R. Rajaraman and H. A. Bethe, Rev. Mod. Phys. **39**, 745 (1967). We are grateful to Dr. D. W. L. Sprung for discussion of this point.

5. SUMMARY AND CONCLUSION

We have studied a nucleon-nucleon potential based on the one-boson-exchange model. This potential has a specific type of velocity dependence, which occurs not only in the central-force part, but also in the tensor and spin-orbit parts. The radial dependence of the velocity-dependent part is also Yukawa.

We have fitted qualitatively the potential parameters to the two-body scattering data. In view of the comparatively few parameters we have, our fit is fairly satisfactory. The fit may be improved if we use, for example, a six-pole model.

Our nuclear-matter calculation indicates that such a potential gives saturation at reasonable density. The potential we use nevertheless possesses a very strong soft core, and, from the curve for the Fourier transform of the defect, Fig. 2, we see that it is in some states almost equivalent to a hard core of 0.3 F.

ACKNOWLEDGMENTS

We are indebted to Dr. R. K. Bhaduri, Dr. G. E. Brown, Dr. Y. Nogami, and Dr. D. W. L. Sprung for many interesting and helpful discussions, and to Dr. Sprung and Dr. Bhargava for sharing some of their computer programs with us.

[25] P. C. Bhargava, Ph.D. thesis, McMaster University, 1966 (unpublished).