Nuclear Scattering in the Random Phase Approximation

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(Received 9 November 1967)

A simple extension of the random phase approximation (RPA) is described that includes scattering channels of a single nucleon. Coupled equations are derived for the single-nucleon amplitudes that describe the scattering states of the nucleon by an assembly of identical nucleons. These equations reveal that the exclusion principle is taken into account by antisymmetrized interaction matrix elements plus the appearance of the projection operator $1-\rho_0$ (ρ_0 is the single-particle density) into unoccupied single-particle states in the compound nucleus. The use of the RPA also introduces the effects of long-range correlations, the well-known "backward-going graphs" of many-body perturbation theory, in the scattering problem. The system of equations thus obtained contains no parameters beyond those describing the two-body matrix elements that generate the Hartree-Fock field that serves as a zero-order approximation to the excitations considered within the RPA. The scattering solutions of these equations are illustrated with a soluble model. It is shown that the presence of correlations in the wave functions of resonance states can influence their particle decay widths considerably. The important practical problem of angular-momentum decomposition of the coupled-equation system is also discussed.

1. INTRODUCTION

SCATTERING calculations based on an extension of the nuclear shell model into the continuum region of single-particle states have been quite successful in describing many features of nucleon scattering by nuclei.[1] In this model, nuclear states that lie above the nucleon-emission threshold, and are therefore particle unstable, are treated on an equal footing with bound states. One problem that arises immediately in such an approach is the question of properly antisymmetrizing the incident nucleon with the target system.

The general problem of antisymmetry in nuclear reactions is discussed in Ref. 2. One method[2] is to work in the configuration space of all particles and antisymmetrize the full wave function by "brute force." Alternatively, one can introduce creation and destruction operators for particle states as in conventional field theory and develop Chew-Low–type equations for the scattering amplitudes.[3] Both of these methods are quite general. In this article we present a discussion of nucleon scattering by systems of identical nucleons that is more restrictive in that it rests on a simple extension of the method of linearized equations of motion, or the RPA, which has been studied in connection with nuclear and plasma oscillations.[4] However, this approach will allow us to include for the compound system the effects of long-range correlations, i.e., the well-known "backward-going graphs" of many-body perturbation theory and to study their effects on the widths and positions of nuclear states embedded in the single-particle continuum. In this manner, we are able to treat resonant states and bound states on precisely the same footing, and to exactly the same approximation, within the RPA; no special assumptions have to be introduced for the resonating states, such as a weak coupling to the continuum or a "joining radius" dividing configuration space into inside and outside regions to define these states.

Since the Hartree-Fock (HF) approximation for single-particle motion is at the root of the RPA treatment of nuclear excitations, this approach has the added attraction of providing a *self-consistent* treatment of the scattering problem. The average field that scatters the incident nucleon turns out to be the HF potential (this point has already been emphasized by Villars[3]), which in turn is determined by the same two-body interactions that lead to the formation of the resonant (and bound) states in the model. Thus this approach contains no parameters beyond those associated with the form of the two-body interaction employed in determining the HF field.

After introducing the necessary definitions and notation in Sec. 2, we derive the usual RPA equations in Sec. 3, but using the "mixed" representation introduced in Sec. 2. With proper boundary conditions, these equations describe the scattering of a nucleon by a target system of A nucleons. We study this scattering by examining the wave function of the system of $A+1$ nucleons above threshold for particle emission. In fact, since we determine the energy spectrum of the com-

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[1] See, for example, R. H. Lemmer and C. M. Shakin, Ann. Phys. (N. Y.) **27**, 13 (1964); I. Lovas, Nucl. Phys. **81**, 353 (1966); A. Piazza, H. G. Wahsweiler, and W. Greiner, Phys. Letters **25B**, 579 (1967); W. Ebenhöh, W. Glöckle, and J. Hüfner, *ibid.* **24B**, 361 (1967); W. Ebenhöh, W. Glöckle, J. Hüfner, and H. A. Weidenmüller, Z. Physik **202**, 301 (1967); W. P. Beres and W. M. MacDonald, Nucl. Phys. **A91**, 529 (1967).

[2] H. Feshbach, Ann. Phys. (N. Y.) **19**, 287 (1962); A. K. Kerman, in *Lectures in Theoretical Physics*, edited by W. E. Brittin (University of Colorado Press, Boulder, Colo., 1965), Vol. VIII C; W. A. Friedman and H. Feshbach, in Racah Memorial Volume, 1967 (to be published).

[3] F. Villars, in *Fundamentals in Nuclear Theory*, edited by A. de-Shalit and C. Villi (International Atomic Energy Agency, Vienna, 1967), p. 269.

[4] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953); R. Arvieu and M. Vénéroni, Compt. Rend. **250**, 992 (1960); M. Baranger, Phys. Rev. **120**, 957 (1960); G. E. Brown, J. A. Evans, and D. J. Thouless, Nucl. Phys. **24**, 1 (1961).

pound system, our equations below this threshold also describe the bound states of the $A+1$ system. In this last case, they reduce to the usual RPA equations written in the more complicated mixed representation. Section 4 presents an alternative derivation of these equations based on time-dependent HF theory, and Sec. 5 gives an exact solution for a "schematic" scattering problem.

2. DEFINITIONS AND BACKGROUND

We introduce a set of creation operators $c_k^\dagger$ corresponding to a complete set of one-particle states $|k\rangle$. In what follows, it will be convenient and useful to identify the wave functions $\langle x|k\rangle=\varphi_k(x)$ with the HF orbitals as our basis of single-particle states. The state obtained by filling the $A+1$ lowest orbitals then corresponds to the HF ground state $|0\rangle$. It acts as a vacuum for the particle and hole operators for which we adopt the notation

$$\begin{aligned} c_k^\dagger &= a^\dagger(\epsilon) \text{ if } k=\epsilon \text{ is unoccupied and unbound,} \\ &= a_m^\dagger \text{ if } k=m \text{ is unoccupied and bound,} \\ &= b_i \text{ if } k=i \text{ is occupied and bound.} \end{aligned} \tag{2.1}$$

In the usual RPA approach one considers the equation of motion for the particle-hole operator $A_{mi}^\dagger = a_m^\dagger b_i^\dagger$, expressed in the occupation number representation of Eq. (2.1), and in a basis consisting only of bound states. However, instead of the operators (2.1), we can also introduce operators in the coordinate space x of the particles

$$\begin{aligned} a^\dagger(x)&=\sum_m \varphi_m^*(x)a_m^\dagger+\int \varphi_\epsilon^*(x)a^\dagger(\epsilon)d\epsilon, \\ b(x)&=\sum_{k,\text{occ}} \varphi_k^*(x)c_k^\dagger=\sum_i \varphi_i^*(x)b_i, \end{aligned} \tag{2.2}$$

so that, for example, the usual field operator for creating a fermion at the point x (with specified spin and isospin) reads

$$\psi^\dagger(x)=\sum_k \varphi_k^*(x)c_k^\dagger=a^\dagger(x)+b(x). \tag{2.3}$$

Of course we have also

$$a(x)|0\rangle=b(x)|0\rangle=0. \tag{2.4}$$

This leads us to consider the particle-hole creation operators $A_\beta^\dagger=a^\dagger(x)b_i^\dagger$ in a mixed representation, where the particle is written in the space representation (the most convenient one for visualizing the scattering process) and the hole in occupation number space (the simplest one for bound states). The commutation relations of the operators A and $A^\dagger$ are

$$[A_{\beta'}^\dagger,A_\beta^\dagger]=[A_{\beta'},A_\beta]=0, \tag{2.5}$$

$$[A_{\beta'},A_\beta^\dagger]=\delta_{ii'}\langle a(x')a^\dagger(x)\rangle-\langle a(x')a^\dagger(x)\rangle b_i^\dagger b_{i'} -\delta_{ii'}a^\dagger(x)a(x'), \tag{2.6}$$

where the angular brackets denote an average with respect to the HF vacuum $|0\rangle$ and where β and β' stand for $\{x,i\}$ and $\{x',i'\}$, respectively.

Acting on the vacuum with $[A_{\beta'},A_\beta^\dagger]$, one gets exactly

$$\begin{aligned} [A_{\beta'},A_\beta^\dagger]|0\rangle&=A_{\beta'}A_\beta^\dagger|0\rangle=\delta_{ii'}\langle a(x')a^\dagger(x)\rangle|0\rangle \\ &=\langle A_{\beta'}A_\beta^\dagger\rangle|0\rangle \end{aligned} \tag{2.7}$$

from Eq. (2.6). The well-known quasiboson approximation is obtained by assuming that the relation

$$[A_{\beta'},A_\beta^\dagger]\simeq\delta_{ii'}\langle a(x')a^\dagger(x)\rangle \tag{2.8}$$

is still valid when acting on states other than the vacuum $|0\rangle$. The quantity $\langle a(x')a^\dagger(x)\rangle$ is directly related to the single-particle density operator $\rho_0(x',x)$ of the compound system. We have (using $[b,b^\dagger]_+$ for the anticommutator of b and $b^\dagger$)

$$\begin{aligned} \rho_0(x',x)&\equiv\langle\psi^\dagger(x)\psi(x')\rangle=\langle b(x)b^\dagger(x')\rangle \\ &=\sum_{i\ \text{occ}}\langle x|i\rangle\langle i|x'\rangle=[b(x),b^\dagger(x')]_+, \end{aligned} \tag{2.9}$$

so that

$$\begin{aligned} \langle a(x')a^\dagger(x)\rangle&=\langle\psi(x')\psi^\dagger(x)\rangle=[a(x'),a^\dagger(x)]_+ \\ &=\delta(x-x')-\rho_0(x',x)=(1-\rho_0)_{x'x}. \end{aligned} \tag{2.10}$$

Equation (2.10) will be important for the developments in Sec. 3. It is clear that $1-\rho_0$ is a projection operator into unoccupied single-particle states of the compound nucleus, since $\rho_0=\rho_0^\dagger$ and $\rho_0^2=\rho_0$.

The total Hamiltonian H of our system is given by a sum of one-body kinetic-energy operators T and two-body interaction energy operators W in the usual fashion,

$$H=\sum T_{kl}c_k^\dagger c_l+\tfrac{1}{2}\sum(kk'|W|ll')c_k^\dagger c_{k'}^\dagger c_{l'}c_l. \tag{2.11}$$

We may "order" H with respect to the HF-vacuum $|0\rangle$ and write instead

$$H=U+\mathfrak{H}+\mathfrak{W}, \tag{2.12}$$

where U is the HF ground-state energy, $\mathfrak{H}$ is the one-particle HF Hamiltonian, and $\mathfrak{W}$ the residual interaction. More precisely,

$$U=\langle H\rangle=\tfrac{1}{2}\sum_i(T_i+E_i), \tag{2.13}$$

$$\mathfrak{H}=\sum_k E_k:c_k^\dagger c_k:=\int dxdx':\psi^\dagger(x)\mathfrak{H}(x,x')\psi(x'):, \tag{2.14}$$

$$\begin{aligned} \mathfrak{W}&=\tfrac{1}{2}\sum(kk'|W|ll'):c_k^\dagger c_{k'}^\dagger c_{l'}c_l: \\ &=\tfrac{1}{2}\int dxdx'dydy' \\ &\quad\times:\psi^\dagger(x)\psi^\dagger(y)(xy|W|x'y')\psi(y')\psi(x'):, \end{aligned} \tag{2.15}$$

where the operators inside the colons $:\cdots:$ are now ordered, and where E_k are the HF single-particle

energies. If k,l refer to the HF representation we have the relation

$$\mathfrak{K}_{kl}=\delta_{kl}E_k=T_{kl}+\sum_{i\ \mathrm{occ}}(ki|\tilde{W}|li)\,, \qquad (2.16)$$

where

$$(ki|\tilde{W}|li)=(ki|W|li)-(ki|W|il)\,,$$

i.e., just the antisymmetrized matrix elements of the two-body interaction W. The last term in Eq. (2.16) is the HF single-particle potential.

It is only possible to diagonalize $\mathfrak{K}$ in the occupation number representation since the HF field is nonlocal in coordinate space. However, we will use the fact that the operators $\mathfrak{K}$ and $\mathfrak{W}$ can also usefully be expressed in "mixed representations," which are obtained by replacing some of the fields $\psi^\dagger$ and ψ by their expressions in terms of the $c^\dagger$ and c.

3. LINEARIZED EQUATIONS

Following closely what one does in the usual RPA, let us define an infinite set of operators $O_\lambda^\dagger$:

$$O_\lambda^\dagger=\sum_\alpha(f_\alpha^\lambda A_\alpha^\dagger-g_\alpha^\lambda A_\alpha)$$

$$=\sum_j\int dy[f_j^\lambda(y)a^\dagger(y)b_j^\dagger-g_j^\lambda(y)b_j a(y)]\,, \qquad (3.1)$$

where the eigenvalue index λ still has to be specified.

We now define the approximate ground state $|\psi_0\rangle$, with energy E_0, of our compound system, by the conditions

$$O_\lambda|\psi_0\rangle=0. \qquad (3.2)$$

We also assume that the states

$$|\psi_\lambda\rangle=O_\lambda^\dagger|\psi_0\rangle \qquad (3.3)$$

are approximate eigenstates of our system with eigenvalues E_λ. These approximations are expressed by the relations

$$[H,O_\lambda^\dagger]\simeq\omega_\lambda O_\lambda^\dagger \qquad (3.4)$$

coupled with (3.2), where $\omega_\lambda=E_\lambda-E_0$ is the excitation energy of the state λ. By doing so, one hopes to include in the ground state $|\psi_0\rangle$ the most important part of the long-range correlations, i.e., the ones corresponding to the well-known "backward-going graphs" of perturbation theory.

To the index λ will correspond, in general, a spectrum with a discrete part and a continuum part. The discrete part corresponds to bound states of the $A+1$ system; the continuum part describes the scattering of a nucleon by the A target particles.

We have now to find the equations obeyed by the amplitudes $f_i^\lambda(y)$ and $g_i^\lambda(y)$.

Let us first define two infinite matrices P and Q through the commutator relations

$$[A_\beta,H]=\sum_\gamma(P_{\beta\gamma}A_\gamma+Q_{\beta\gamma}A_\gamma^\dagger)+R,$$

$$[A_\beta^\dagger,H]=\sum_\gamma(-Q_{\beta\gamma}{}^*A_\gamma-P_{\beta\gamma}{}^*A_\gamma^\dagger)-R^\dagger, \qquad (3.5)$$

where the operator R is ordered with respect to the HF vacuum $|0\rangle$ and does not contain any operators A or $A^\dagger$. Linear equations are obtained for A_β and $A_\beta^\dagger$ by neglecting R and $R^\dagger$.

Multiplying the Eqs. (3.5) (with R and $R^\dagger$ suppressed) on the left by the HF vacuum $|0\rangle$, on the right by $A_\alpha^\dagger|0\rangle$, we obtain the relations

$$\langle[A_\beta,H]A_\alpha^\dagger\rangle=\langle A_\beta(H-U)A_\alpha^\dagger\rangle=\sum_\gamma P_{\beta\gamma}\langle A_\gamma A_\alpha^\dagger\rangle, \qquad (3.6)$$

$$\langle A_\alpha[A_\beta,H]\rangle=\langle A_\alpha A_\beta H\rangle=\sum_\gamma Q_{\beta\gamma}\langle A_\alpha A_\gamma^\dagger\rangle. \qquad (3.7)$$

We next consider the expectation values of the linearized commutators (3.5) between approximate eigenstates $|\psi_0\rangle$ and $O_\lambda^\dagger|\psi_0\rangle$ of the system. This procedure yields

$$\langle\psi_0|[A_\beta,H]O_\lambda^\dagger|\psi_0\rangle=\sum_\gamma(P_{\beta\gamma}v_\gamma^\lambda+Q_{\beta\gamma}w_\gamma^\lambda)$$

$$=\omega_\lambda v_\beta^\lambda,$$

$$\langle\psi_0|[A_\beta^\dagger,H]O_\lambda^\dagger|\psi_0\rangle=\sum_\gamma(-Q_{\beta\gamma}{}^*v_\gamma^\lambda-P_{\beta\gamma}{}^*w_\gamma^\lambda)$$

$$=\omega_\lambda w_\beta^\lambda, \qquad (3.8)$$

using $(H-E_\gamma)O_\lambda^\dagger|\psi_0\rangle=(H-E_0)|\psi_0\rangle=0$, after introducing the auxiliary amplitudes

$$v_\beta^\lambda=\langle\psi_0|A_\beta O_\lambda^\dagger|\psi_0\rangle=\sum_\alpha\langle\psi_0|[A_\beta,A_\alpha^\dagger]|\psi_0\rangle f_\alpha^\lambda,$$

$$w_\beta^\lambda=\langle\psi_0|A_\beta^\dagger O_\lambda^\dagger|\psi_0\rangle=\sum_\alpha g_\alpha^\lambda\langle\psi_0|[A_\alpha,A_\beta^\dagger]|\psi_0\rangle. \qquad (3.9)$$

The quasiboson approximation (2.8) replaces in (3.9) the commutators of A_β and $A_\alpha^\dagger$ by their HF expectation values and provides us with the relations

$$v_\beta^\lambda=v_i^\lambda(x)=\sum_\alpha\langle A_\beta A_\alpha^\dagger\rangle f_\alpha^\lambda$$

$$=f_i^\lambda(x)-\int\rho_0(x,x')f_i^\lambda(x')dx',$$

$$w_\beta^\lambda=w_i^\lambda(x)=\sum_\alpha g_\alpha^\lambda\langle A_\alpha A_\beta^\dagger\rangle$$

$$=g_i^\lambda(x)-\int g_i^\lambda(x')\rho_0(x,x')dx'. \qquad (3.10)$$

If we insert into (3.8) the relation (3.10) and use Eqs. (3.6) and (3.7), we obtain

$$\sum_\alpha [\langle A_\beta(H-U)A_\alpha^\dagger\rangle f_\alpha^\lambda+\langle A_\beta A_\alpha H\rangle g_\alpha^\lambda]=\omega_\lambda v_\beta^\lambda, \quad (3.11)$$

$$\sum_\alpha [-\langle A_\alpha A_\beta H\rangle^* f_\alpha^\lambda - \langle A_\beta(H-U)A_\alpha^\dagger\rangle^* g_\alpha^\lambda]=\omega_\lambda w_\beta^\lambda, \quad (3.12)$$

where $\omega_\lambda=E_\lambda-E_0$ measures the excitation energy of the state λ.

The matrix elements appearing in Eqs. (3.11) and (3.12) refer to expectation values with respect to the HF vacuum. We obtain in the mixed representation $\beta=\{x,i\}$, $\alpha=\{y,j\}$

$$\langle A_\beta(H-U)A_\alpha^\dagger\rangle=\int\int dx'dx''(1-\rho_0)_{xx'} \times[\mathcal{H}(x',x'')\delta_{ij}+(x'j|\tilde{W}|ix'')] \times(1-\rho_0)_{x''y}-\delta_{ij}(1-\rho_0)_{xy}E_i, \quad (3.13)$$

$$\langle A_\beta A_\alpha H\rangle=\int\int dx'\,dx''(1-\rho_0)_{xx'} \times(1-\rho_0)_{yx''}(x'x''|\tilde{W}|ij). \quad (3.14)$$

For local two-body interactions $W(x,x')$, the matrix elements appearing in Eqs. (3.12) and (3.13) are

$$(x'j|\tilde{W}|ix'')=\varphi_j^*(x'')W(x',x'')\varphi_i(x')-\delta(x'-x'') \times\int dy\,\varphi_j^*(y)W(x',y)\varphi_i(y), \quad (3.15)$$

$$(x'x''|\tilde{W}|ij)=W(x',x'') \times[\varphi_i(x')\varphi_j(x'')-\varphi_i(x'')\varphi_j(x')]. \quad (3.16)$$

We now have all the ingredients for Eqs. (3.11) and (3.12), which become

$$(\omega_\lambda+E_i-\mathcal{H})v_i^\lambda(x) =\sum_j\int\int dx'\,dx''(1-\rho_0)_{xx'}[(x'j|\tilde{W}|ix'')v_j^\lambda(x'') +(x'x''|\tilde{W}|ij)w_j^\lambda(x'')], \quad (3.17)$$

$$(\omega_\lambda-E_i+\mathcal{H})w_i^\lambda(x) =\sum_j\int\int dx'\,dx''[-(x'j|\tilde{W}|ix'')w_j^\lambda(x'') -(x'x''|\tilde{W}|ij)v_j^\lambda(x'')](1-\rho_0)_{x'x}, \quad (3.18)$$

where again the amplitudes $v_i^\lambda(x)$ and $w_i^\lambda(x)$ appear only, and understanding the operation $\mathcal{H}v(x)$ to mean $\int dx'\mathcal{H}(x,x')v(x')$, etc.

An important simplification has been introduced into these equations by the fact the $\mathcal{H}$ is the HF Hamiltonian. For then $1-\rho_0$ and $\mathcal{H}$ commute, $[1-\rho_0,\mathcal{H}]=0$, and we have

$$\int dy\,dx'dx''(1-\rho_0)_{xx'}\mathcal{H}(x',x'')(1-\rho_0)_{x''y}f_i^\lambda(y) =\int dx'\,\mathcal{H}(x,x')v_i^\lambda(x') \quad (3.19)$$

since $(1-\rho_0)$ is a projection operator. Thus, no factors appear on the left or right of $\mathcal{H}$ in Eqs. (3.17) and (3.18).

The coupled equations (3.17) and (3.18) are the RPA equations of motion written in a mixed representation. We notice that the amplitudes $v_i^\lambda(x)$ and $w_i^\lambda(x)$ appear naturally in this representation instead of the original amplitudes $f_i^\lambda(x)$ and $g_i^\lambda(x)$ of Eq. (3.1). This is connected with the operation of the Pauli principle, excluding from f and g by means of the projection operator $1-\rho_0$ those states which are already occupied in the ground state of the compound system, but which would appear in the expansions of $f(x)$ and $g(x)$ in a complete set of single-particle states.

Equations (3.17) and (3.18) depend on the one-particle density that is generated by a given two-body interaction W in the HF approximation. One knows, furthermore, that the stability conditions[5] on the HF are closely connected with Eqs. (3.17) and (3.18). One should therefore use the full (nonlocal) HF Hamiltonian $\mathcal{H}$ in a discussion of the scattering problem also. In fact, ρ_0 is precisely the quantity determined in the HF method, so Eqs. (3.17) and (3.18) extend this method quite naturally to scattering problems. We also noted the important simplification, Eq. (3.19), that the HF method introduces.

It is clear from their derivation that Eqs. (3.17) and (3.18) are completely antisymmetrized, except for the violations of the Pauli principle incurred through use of the RPA.[4] The form of the interaction terms in these equations is especially interesting if W is a local two-body interaction. For instance, the right-hand side of Eq. (3.17) then reads:

Right-hand side of Eq. (3.17)=

$$\sum_j\int dx'\,dx''(1-\rho_0)_{xx'}W(x',x'') \times\{\varphi_j^*(x'')[\varphi_i(x')v_j^\lambda(x'')-\varphi_i(x'')v_j^\lambda(x')] +w_j^\lambda(x'')[\varphi_i(x')\varphi_j(x'')-\varphi_i(x'')\varphi_j(x')]\}. \quad (3.20)$$

[5] D. J. Thouless, Nucl. Phys. 21, 225 (1960).

We comment on the normalization of the states $|\psi_\lambda\rangle = O_\lambda^\dagger|\psi_0\rangle$. The scalar product

$$\langle\psi_{\lambda'}|\psi_\lambda\rangle = \sum_{\alpha\alpha'}\left(f_{\alpha'}^{\lambda'*}\langle\psi_0|[A_{\alpha'},A_\alpha^\dagger]|\psi_0\rangle f_\alpha^\lambda - g_{\alpha'}^{\lambda'*}\langle\psi_0|[A_\alpha,A_{\alpha'}^\dagger]|\psi_0\rangle g_\alpha^\lambda\right)$$

$$= \sum_j \int dx dx' [f_j^{\lambda'*}(x')(1-\rho_0)_{x'x} f_j^\lambda(x) - g_j^{\lambda'*}(x')(1-\rho_0)_{xx'} g_j^\lambda(x)]$$

$$= \sum_j \int dy [v_j^{\lambda'*}(y) v_j^\lambda(y) - w_j^{\lambda'*}(y) w_j^\lambda(y)] \quad (3.21)$$

is obtained using the quasiboson approximation again and the fact that $1-\rho_0$ is a projector. Notice that this scalar product can also be expressed in terms of the amplitudes v and w. The normalization condition is then obtained by prescribing (3.21) to be either $\delta_{\lambda\lambda'}$ or $\delta(\lambda-\lambda')$, depending on whether λ labels a bound or a scattering state.

Finally, if we suppress the amplitudes $w(x)$ in Eq. (3.17) we recover the one-particle–one-hole Tamm-Dancoff approximation (TDA). The equations for the amplitude $v_i^\lambda(x)$ are

$$(\omega_\lambda + E_i - \mathcal{H}) v_i^\lambda(x) = \sum_j \int dx' dx'' (1-\rho_0)_{xx'} W(x',x'') \varphi_j^*(x'') \times [\varphi_i(x') v_j^\lambda(x'') - \varphi_i(x'') v_j^\lambda(x')] \quad (3.22)$$

if we use Eq. (3.20). Equations (3.22) are in the subspace of one-hole–one-particle states since the ground state $|\psi_0\rangle$ is reduced in this case to the HF vacuum $|0\rangle$. In structure, this set of equations is similar in form to the coupled equations studied by Melkanoff, Raynal, and Sawada,[6] and by Buck and Hill[6] (without, however, identifying $\mathcal{H}$ with a self-consistent Hamiltonian). The effect of the exclusion principle is seen to be rather complicated. If one ignores antisymmetry, as is usually done for high-energy scattering, only a term of the form $-\sum_j F_{ij}(x) v_j^\lambda(x)$ survives on the right-hand side of Eq. (3.22). This term exhibits the familiar form factor

$$F_{ij}(x) = \int dx'\, \varphi_j^*(x') W(x,x') \varphi_i(x') \quad (3.23)$$

that appears in all discussions of high-energy particle scattering. To this term the exclusion principle adds an "exchange" term and also a term depending on the density operator ρ_0. We have already discussed the origin of the density operator in the problem. The appearance of the density operator in scattering problems has been noted previously.[2] However, our results differ in detail from those given in Ref. 2 in that the density operator of the compound nucleus and not the target nucleus appears in our case.

The discussion so far has not distinguished between bound states and scattering states of Eqs. (3.17) and (3.18). To distinguish between these, we recall that the separation energy of a particle in a HF state $\varphi_i(x)$ is $S_i = -|E_i|$ (Koopman's theorem), leaving the residual nucleus with a hole in state i. Thus, when $\omega_\lambda \geq -E_i = S_i$, Eq. (3.17) defines a scattering problem for all such channels i. Notice in passing that the thresholds for inelastic scattering to excited states of the target are determined by the hole energies E_i in Eq. (3.22). These energies are determined in principle by the HF field $\mathcal{H}$ and cannot therefore be adjusted to fit experiment as in the calculations under Ref. 1. Disagreement between the calculated and experimental positions of such threshold energies in the present treatment of the scattering problem will stem from the failure of the HF field to provide an adequate description of the target nucleus. At large distances, the amplitudes $v_i^\lambda(x)$ will have outgoing waves in all the open channels, in addition to incoming waves in channels specified by the scattering process one wants to study. By contrast, the amplitudes $w_i^\lambda(x)$ do not give rise to any particle flux at infinite separation since we always find $\omega_\lambda + |E_i|$ positive in Eq. (3.18). However, the $w_i^\lambda(x)$ do affect the amplitudes $v_i^\lambda(x)$ through their mutual coupling in Eqs. (3.17) and (3.18).

4. TIME-DEPENDENT DERIVATION

Before discussing some general features of the solution of Eqs. (3.17) and (3.18), it is instructive to present a direct derivation of these equations based on an extension of the time-dependent HF approximation to scattering problems. This derivation makes clear that the excited states $O_\lambda^\dagger|\psi_0\rangle$ of Sec. 3 are associated with density oscillations about the equilibrium density ρ_0 of the ground state. Changes in the density with time obey the equation of motion[7]

$$i(\partial/\partial t)\rho(t) = [\mathcal{H}[\rho(t)],\rho(t)], \quad (4.1)$$

where $\mathcal{H}$ is the time-dependent version of the HF Hamiltonian introduced in Eq. (2.12). Its dependence on the density $\rho(t)$ appears through the HF single-particle potential $\mathcal{U}_{kl}$, where

$$\mathcal{H} = \sum (T_{kl} + \mathcal{U}_{kl}) : c_k^\dagger c_l :,$$
$$\mathcal{U}_{kl} = \sum (kk'|\bar{W}|ll')\rho_{l'k'}(t), \quad (4.2)$$

and $\rho_{lk}(t) = \langle\psi_{\rm HF}(t)|c_k^\dagger c_l|\psi_{\rm HF}(t)\rangle$ is explicitly time-dependent through the HF ground state $|\psi_{\rm HF}(t)\rangle$. We achieve a linear form of Eq. (4.1) by writing $\rho(t)$

[6] J. Raynal, M. A. Melkanoff, and T. Sawada, Nucl. Phys. A101, 369 (1967); B. Buck and A. D. Hill, *ibid.* A95, 271 (1967); C. Bloch and V. Gillet, Phys. Letters 18, 58 (1965); C. A. Caine and R. H. Lemmer, Laboratory for Nuclear Science Technical Report No. 78, 1964 (unpublished).

[7] J. Goldstone and K. Gottfried, Nuovo Cimento 13, 849 (1959).

$=\rho_0+\rho^{(1)}(t)$ and only keeping terms linear in $\rho^{(1)}(t)$. One finds that

$$i(\partial/\partial t)\rho_{kl}{}^{(1)}=[\mathcal{H},\rho^{(1)}]_{kl}+[\mathcal{U}^{(1)},\rho_0]_{kl}, \quad (4.3)$$

where now $\mathcal{H}=\mathcal{H}[\rho_0]$ is the usual time-independent HF Hamiltonian and the time-dependent part of the HF potential is given by

$$\mathcal{U}_{kl}{}^{(1)}=\sum (kk'|\tilde{W}|ll')\rho_{l'k'}{}^{(1)}(t). \quad (4.4)$$

In the approximation (4.3), the amplitudes $\rho_{kl}{}^{(1)}$ can only involve the creation or destruction of particle-hole pairs, i.e., if l is a particle state, k must be a hole state and vice versa. As in Sec. 2, let us reserve the index i or j for holes in the occupation number representation. Then there are two cases to consider in Eq. (4.3): (i) $k=x$, a particle state; $l=i$, a hole state; and (ii) $k=i$, a hole state; $l=x$, a particle state. The commutators are

$$[\mathcal{H},\rho^{(1)}]=\int dx'\,\mathcal{H}(x,x')\rho_{x'i}{}^{(1)}-E_i\rho_{xi}{}^{(1)}$$

$$=(\mathcal{H}-E_i)\rho_{xi}{}^{(1)} \quad (4.5)$$

and

$$[\mathcal{U}^{(1)},\rho_0]_{xi}=\sum_k \mathcal{U}_{xk}{}^{(1)}(\rho_0)_{ki}-\int dx'\,\rho_0(x,x')\mathcal{U}_{x'i}{}^{(1)}$$

$$=\sum_j \int dx'\,dx''(1-\rho_0)_{xx'}$$

$$\times[(x'j|\tilde{W}|ix'')\rho_{x''j}{}^{(1)}$$

$$+(x'x''|\tilde{W}|ij)\rho_{jx''}{}^{(1)}] \quad (4.6)$$

if we choose intermediate states in either the HF occupation number representation $i,j,\cdots$, or the coordinate representation x, and note that $(\rho_0)_{ki}=\delta_{ki}$, $\mathcal{H}_{ki}=E_i\delta_{ki}$ in the former representation. Similar expressions are obtained for $[\mathcal{H},\rho^{(1)}]_{ix}$ and $[\mathcal{U}^{(1)},\rho_0]_{ix}$. We insert these expressions into Eq. (4.3) and look for solutions of the type $\rho^{(1)}(t)=\rho^{(1)\lambda}e^{-i\omega_\lambda t}+$c.c. of frequency ω_λ. Then one has

$$(\omega_\lambda+E_i-\mathcal{H})\rho_{xi}{}^{(1)\lambda}$$

$$=\sum_j \int dx'\,dx''(1-\rho_0)_{xx'}[(x'j|\tilde{W}|ix'')\rho_{x''j}{}^{(1)\lambda}$$

$$+(x'x''|\tilde{W}|ij)\rho_{jx''}{}^{(1)\lambda}] \quad (4.7)$$

and

$$(\omega_\lambda-E_i+\mathcal{H})\rho_{ix}{}^{(1)\lambda}$$

$$=\sum_j \int dx'\,dx''[-(x'j|\tilde{W}|ix'')\rho_{jx''}{}^{(1)\lambda}$$

$$-(x'x''|\tilde{W}|ij)\rho_{x''j}{}^{(1)\lambda}](1-\rho_0)_{x'x}, \quad (4.8)$$

which are precisely equivalent to Eqs. (3.17) and (3.18) if we identify $\rho_{xi}{}^{(1)\lambda}$ as $v_i{}^\lambda(x)$, $\rho_{ix}{}^{(1)\lambda}$ as $w_i{}^\lambda(x)$. We emphasize again that these equations are only consistent if $\rho_{xi}{}^{(1)}$ refers to states with particles above the Fermi surface and holes below. Thus $\rho_{xi}{}^{(1)}$ absorbs the projection operator $1-\rho_0$ onto unoccupied particle states in its definition.

The complexity introduced by the exclusion principle is doubled by the use of the RPA in our coupled equations. However, it may be worthwhile to keep such complexities if long-range correlations in the system are important. One knows that such correlations play an important role for γ-ray widths of particle-hole excitations.[8] Similar effects can enter for particle-emission widths of eigenstates $|\psi_\lambda\rangle$ lying in the single-particle continuum.

5. SCHEMATIC MODEL

Generally, we have to resort to numerical methods to study the solutions of Eqs. (4.7) and (4.8). However, the so-called schematic model introduced by Brown *et al.*[8] can be solved exactly for its scattering amplitude within the framework of the set of coupled equations for $v_i(x)$ and $w_i(x)$. We introduce the approximation of the schematic model, i.e., separable matrix elements

$$(xj|\tilde{W}|ix')=(xx'|\tilde{W}|ij)\approx\pm\lambda D_i(x)D_j(x') \quad (5.1)$$

where, in our representation, $D_i(x)$ is proportional to the dipole operator integrand $x\varphi_i(x)$ for γ-ray absorption. The connection with the usual amplitudes D_{mi} defined, for example, in Ref. 8 is $D_{mi}=\int dx\,\varphi_m{}^*(x)D_i(x)$, where m is a particle state above the Fermi surface. The coupled equations (4.7) and (4.8) can be solved without further approximation. Dropping the index λ on v and w for clarity, we have

$$v_j=u_j{}^{(+)}\delta_{ij}$$

$$+C_i(\omega)\frac{1}{\omega+E_j-\mathcal{H}+i\eta}\lambda(1-\rho_0)D_j, \quad \eta\to 0^+ \quad (5.2)$$

and

$$w_j=-C_i(\omega)\frac{1}{\omega-E_j+\mathcal{H}}\lambda D_j(1-\rho_0), \quad (5.3)$$

since v_j must satisfy outgoing-wave boundary conditions at infinity. We reserve the index i for the initial state of the target (described as a single-hole state in our model). Then v_j only has incoming waves in channel $j=i$. These are contained in the potential scattering wave function $u_i{}^{(+)}$ of the HF field

$$(\omega+E_i-\mathcal{H})u_i{}^{(+)}=0 \quad (5.4)$$

at energy $\omega+E_i$. The function $C_i(\omega)$ in Eqs. (5.2)

[8] G. E. Brown, L. Castillejo, and J. A. Evans, Nucl. Phys. **22**, 1 (1961).

and (5.3) is given by

$$C_i(\omega)=\sum_j \int dx' \,[v_j(x')+w_j(x')]D_j(x')$$

$$=D_{ki}\Big\{1-\lambda\sum_j\Big[D_j\frac{1}{\omega+E_j-\mathcal{H}+i\eta}(1-\rho_0)D_j - D_j\frac{1}{\omega-E_j+\mathcal{H}}D_j(1-\rho_0)\Big]\Big\}^{-1}, \quad (5.5)$$

where $D_{ki}=\int u_k^{(+)}(x)D_i(x)dx$ is proportional to the amplitude for the incident particle of momentum $\mathbf{k}$, making a transition into the hole state i. The full-transition amplitude for particle scattering follows immediately from the asymptotic behavior of v_j in Eq. (5.2):

$$T_{ji}=T_{\mathrm{HF}}\delta_{ji}+\lambda C_i(\omega)\langle u_j^{(-)},(1-\rho_0)D_j\rangle, \quad (5.6)$$

where T_{HF} is the potential scattering amplitude of the HF potential, and $u_j^{(-)}$ an incoming-wave solution of $\mathcal{H}$ at energy $\omega+E_j$. Eq. (5.6) is exact; we study the resonance behavior of the scattering cross section by looking for poles of T_{ji} in the energy variable ω. Apart from possible single-particle resonances in the HF field amplitudes T_{HF} and $\langle u_j^{(-)},(1-\rho_0)D_j\rangle$ we see that the poles of T_{ji} are contained in $C_i(\omega)$. These poles satisfy the relation

$$1=\lambda\sum_j\Big[D_j\frac{1}{\omega+E_j-\mathcal{H}+i\eta}(1-\rho_0)D_j - D_j\frac{1}{\omega-E_j+\mathcal{H}}D_j(1-\rho_0)\Big], \quad (5.7)$$

which is independent of the initial conditions contained in the label i on $C_i(\omega)$. Equation (5.7) is simply the dispersion relation for the eigenmodes of Eqs. (4.7) and (4.8) that endows these modes with an escape width if the energy of the mode happens to lie in the single-particle continuum, $\omega>-E_j$. We make Eq. (5.7) more explicit by introducing the complete set $\varphi_k(x)$ of states of $\mathcal{H}$. The projection operator $1-\rho_0$ only allows unoccupied states to appear (we call these n) so that

$$1=\lambda\sum_{nj}\Big(\frac{D_{nj}^2}{\omega+E_j-E_n+i\eta}-\frac{D_{nj}^2}{\omega-E_j+E_n}\Big), \quad (5.8)$$

which is precisely the RPA dispersion for the schematic model, but where the sum on unoccupied states n includes an integration over the single-particle continuum of the HF field. This second feature introduces the decay of the eigenmodes. For example (recall that E_j and E_n are negative numbers for bound states),

$$\sum_{nj}\frac{D_{nj}^2}{\omega+E_j-E_n+i\eta}=\sum_{nj}{}'\frac{D_{nj}^2}{\omega-|E_j|+|E_n|} +\sum_j P\int_0^\infty d\epsilon\frac{D_{\epsilon j}^2}{\omega-|E_j|-\epsilon} -i\pi\sum_j\int_0^\infty d\epsilon\, D_{\epsilon j}^2\delta(\omega-|E_j|-\epsilon), \quad (5.9)$$

where the prime in the first sum restricts n to bound states; ϵ labels continuum states of $\mathcal{H}$ and P denotes a principal value integral. An imaginary part appears in the pole whenever one or more open channels are present. Notice that there is no contribution to the imaginary part from the second term in Eq. (5.8) which reflects the effects of correlations. The widths of resonances in the schematic model are still changed, however, by the introduction of correlations, since the energy eigenvalue assumes a different form [see Eq. (5.12)]. The dispersion relation (5.8) completely unifies the determination of the resonating and bound states in the RPA and their mutual effects. Roots of Eq. (5.8) that lie below $|E_i|$ (the lowest-hole state) are automatically divested of an imaginary part, since

$$i\pi\sum_j\int_0^\infty d\epsilon\,\lambda D_{\epsilon j}^2\delta(\omega-|E_j|-\epsilon) =i\pi\sum_j\lambda D_{\omega-|E_j|,j}^2 \quad (5.10)$$

(sum on states $\omega-|E_j|>0$ only—i.e., the sum includes only open channels). The imaginary part then grows according to the sum on the right-hand side of Eq. (5.10) as more channels open up.

If we make the still more special assumption that the bound particle-hole energies $|E_n-E_j|=E_0$ are degenerate and suppress the contribution from the principal value integral in Eq. (5.9) we regain the degenerate schematic-model results: For N bound particle-hole excitations, $N-1$ stay degenerate at E_0, and one moves to $\omega=E_0+\Sigma_c$ (we write $\Sigma_c=\sum_{nj}{}'\lambda D_{nj}^2$) if we suppress ground-state correlations (the TDA result), or to $\omega=(E_0^2+2E_0\Sigma_c)^{1/2}$ if we include them (the RPA result), provided ω so obtained lies below the first threshold at $|E_i|$ for particle emission. For $\omega>E_i$ the eigenvalues are complex: We replace ω by $\omega-i\gamma$ and find ($\Sigma_0=\sum_{j\ \mathrm{open}}\lambda D_{\omega-|E_j|,j}^2$)

$$\omega=E_0+\Sigma_c(1+\pi^2\Sigma_0^2)^{-1}$$

and

$$(\gamma/\pi)=\Sigma_0\Sigma_c(1+\pi^2\Sigma_0^2)^{-1}, \quad (5.11)$$

in TDA, and

$$\omega=\{E_0^2-\gamma^2+2E_0\Sigma_c(1+\pi^2\Sigma_0^2)^{-1}\}^{1/2},$$
$$(\gamma/\pi)=(E_0/\omega_{\mathrm{TDA}})\Sigma_0\Sigma_c(1+\pi^2\Sigma_0^2)^{-1}. \quad (5.12)$$

We have neglected contributions coming from principal-value integrals like that in Eq. (5.9) in RPA. The coupling to the continuum via the imaginary terms in Eq. (5.8) is seen to "renormalize" the particle-hole matrix elements between bound pairs by a factor $(1+\pi^2\Sigma_0^2)^{-1}<1$ and effectively makes these matrix elements weaker. The energy shifts away from E_0 are correspondingly smaller. In the RPA we also obtain the usual factor $E_0/\omega_{\mathrm{TDA}}$ multiplying the particle width γ of the TDA, as in the case of dipole transition rates. This factor decreases the particle width if the level ω is pushed up, increases it if ω is pushed down. We also see from the second term in Eq. (5.6) that the strength of the transition into the resonance states through the factor $\langle u_j^{(-)}(1-\rho_0)D_j\rangle$ is cut down by the exclusion principle, in that fewer particle-hole transitions are allowed by the projection factor $(1-\rho_0)$.

6. DISCUSSION

We have seen that the extension of the usual RPA equations to cover scattering problems is straightforward and leads to coupled equations similar in structure to those obtained for nucleon-nucleus scattering when antisymmetry is ignored. The exclusion principle introduces important differences in detail that are partly carried by the projection operator $1-\rho_0$ in Eqs. (3.17) and (3.18). The procedure gives a unified treatment of bound-state and scattering-state problems within the RPA method. Starting from a Hartree-Bogoliubov basis instead of a HF basis, one can extend the method to treat two quasiparticle excitations within the RPA framework instead of particle-hole excitations.

In actual calculations we would have to introduce states $|\psi_\lambda\rangle$ carrying a good angular momentum. This is easily accomplished by writing

$$O_\lambda{}^{\dagger JM}=\sum_{i,lj}\int_0^\infty dr\, r^2\{f_{i,lj}{}^{\lambda J}(r)[a_{lj}{}^\dagger(r)b_i{}^\dagger]_M^J - g_{i,lj}{}^{\lambda J}(r)[b_i a_{lj}(r)]_{-M}^J(-)^{J+M}\}, \quad (6.1)$$

where lj are the partial waves which couple with the hole states $i=\{n_il_ij_i\}$ to a total angular momentum JM. The operators $a_{ljm}{}^\dagger(r)$ and $b_{im_i}{}^\dagger$ are defined by (m is the z component of $\mathbf{j}$)

$$a_{ljm}{}^\dagger(r)=\sum_n R_{nlj}(r)a_{nljm}{}^\dagger+\int d\epsilon R_{\epsilon lj}(r)a_{\epsilon ljm}{}^\dagger,$$

where R_{nlj} and $R_{\epsilon lj}$ are radial wave functions for unoccupied bound states (nlj) and continuum states (ϵlj), respectively. The hole operator $b_{im_i}{}^\dagger$ is

$$b_{im_i}{}^\dagger=(-)^{j_i+m_i}c_{n_il_ij_i-m_i}. \quad (6.2)$$

The meaning of the operators $a_{ljm}{}^\dagger(r)$ and $a_{ljm}(r)$ is clear: they are simply the creation and destruction operators of a nucleon at radial position $r=|\mathbf{r}|$, with angular momentum (ljm). Their anticommutator is

$$[a_{ljm}(r),a_{l'j'm'}{}^\dagger(r')]_+ = \delta_{ll'}\delta_{jj'}\delta_{mm'}[\delta(r-r')/r^2-\rho_{lj}(r,r')], \quad (6.3)$$

where

$$\rho_{lj}(r,r')=\sum_{n,\mathrm{occ}} R_{nlj}(r)R_{nlj}(r') \quad (6.4)$$

is the radial density matrix for nucleons with a given l and j.

The expression (6.3) replaces Eq. (2.10). Otherwise the derivation of coupled equations of fixed JM proceeds as before and gives *radial* equations for $f^J(r)$ and $g^J(r)$ [or, rather, their counterparts $v^J(r)$ and $w^J(r)$]. From their construction, these equations should not contain any spurious *resonance* excitations, contrary to the case of the TDA equations (3.22), which will contain spurious center-of-mass excitations. It is not clear whether the center-of-mass motion in the scattering problem is completely accounted for by using the RPA equations, however, since in addition to resonance scattering, these equations also give rise to a background of potential scattering from a HF potential fixed in space and additional corrections may be necessary. This point is under investigation.

HF potentials are now available for several nuclei, including O^{16}. Work is in progress to obtain numerical solutions of Eqs. (3.17) and (3.18) for O^{16}, based on an extension of the numerical methods described in Ref. 9.

ACKNOWLEDGMENTS

One of us (R. H. L.) would like to thank Prof. M. Jean and members of the Institut de Physique Nucléaire at Orsay for their kind hospitality.

[9] D. Vautherin and M. Véneroni, Phys. Letters **25B**, 175 (1967).