Isospin Invariance and the Pairing-Force Problem*

J. N. GINOCCHIO AND J. WENESER†
Physics Department, Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 11 October 1967)

The isoscalar pairing-force Hamiltonian is studied first in the BCS-like approximation, including neutron-neutron, proton-proton, and neutron-proton interactions. Constraining $\langle\hat{N}\rangle$ and $\langle\hat{T}^2\rangle$, we find a class of BCS-like wave functions which give the *same* minimum energy; they differ in the expectation value of $\hat{T}_z$, $-T\leqslant\langle\hat{T}_z\rangle\leqslant T$. For the limit $\langle\hat{T}_z\rangle=T$, the minimum occurs at zero neutron-proton interaction. The residual interactions, those neglected by the BCS approximation, are treated by the quasiboson approximation. The spurious effects of number and isospin dispersion are identified. A procedure for explicitly displaying the number and isospin dependence of the energy is given, together with one for obtaining excited states with all possible isospins. Finally, as an example, the degenerate case is worked out, and agreement with the exact solution, including ground-state neutron-proton correlations, to the order considered, is demonstrated.

I. INTRODUCTION

THE isoscalar pairing-force problem has been of recent interest.[1–5] In this paper, we study the pure pairing-force problem by starting with a BCS-like first approximation,[6–8] which includes both neutron-neutron, proton-proton, and the neutron-proton interactions. The residual interactions are handled by means of the quasiboson approximation.[9,10] The first of these steps, the BCS approximation, breaks both number and isospin conservation. The terms necessary to restore the conservation are, of course, in the residual interactions. We can examine how and to what extent the number and isospin invariance is recaptured, using the quasiboson approximation to deal with the residual interactions. Here we concentrate on the isospin problem, the number problem having already been treated by such methods.[11]

We begin in Sec. II by linearizing the pairing Hamiltonian, subject to the constraints that the average number of nucleons N and isospin T is fixed. For a separable pairing force we show that there is a class of BCS solutions with the same energy but which differ only in the average value of the isospin projection T_z. These values vary from T to $-T$. The solution with maximum projection $T_z=T$ has no neutron-proton pairing. By using the variational method in Sec. III, we are able to demonstrate that the same conclusions hold also for the most general pairing force. When T_z is constrained rather than T, the BCS solution obtained is the one with no neutron-proton pairing, as pointed out in Sec. IV. The isospin invariance, while not exact, holds within the order of the BCS approximation. To improve on the invariance, one must improve on the approximation. We go on in Sec. V to examine the residual interactions by means of the quasiboson approximation. The method of solving this boson Hamiltonian is outlined in Sec. VI. In particular, the dependence of the boson Hamiltonian on N and T^2 is derived in detail, Eq. (74). Although this dependence is not manifestly isospin invariant, we show that the energy of a general isospin-invariant Hamiltonian will have the same T^2 dependence, Eq. (78). The reason that it does not appear invariant, at first sight, is twofold. First, the expansion is made in a preferred direction in isospin space: namely, the direction in which $T_z=T$, but $T_x=T_y=0$. Second, terms of smaller order have been discarded. With these additions, in Sec. VII, the invariance is made obvious. In Sec. VIII we examine the boson eigenfunctions, showing that the correct correlations are introduced to make them eigenfunctions of $\hat{N}$, $\hat{T}_z$, and $\hat{T}^2$ to the boson order. We also see that not only are the excited states with the isospin T of the ground state given, but also those with isospin $T+1$. Finally, we work out, in detail, the degenerate case, $\epsilon_j=\epsilon$, and compare with the exact solution. We find, in Sec. IX, that the correct ground-state energy is given by the boson approximation, and also the low-lying excited states agree in energy and in angular momentum and isospin structure. Finally, in Sec. X we show that the neutron-proton correlations introduced into the ground state by the boson approximation, agree to leading order with those present in the exact ground-state wave function.

* This work is supported in part through funds provided by the Atomic Energy Commission under Contract AT(30-1)2 098.
† Present address: Physics Department, Brookhaven National Laboratory, Upton, N.Y.
[1] M. K. Pal and M. K. Banerjee, Phys. Letters **13**, 155 (1964).
[2] A. Goswami, Nucl. Phys. **60**, 228 (1964).
[3] P. Camiz, A. Covello, and M. Jean, Nuovo Cimento **42B**, 199 (1966).
[4] J. P. Elliott and D. A. Lea, Phys. Letters **19**, 291 (1965).
[5] E. Salusti and G. Varcaccio, Nuovo Cimento **42B**, 378 (1966); J. Flores and P. A. Mello, Nucl. Phys. **88**, 609 (1966).
[6] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[7] N. N. Bogoliubov, Zh. Eksperim. i Teor. Fiz. **34**, 58 (1958); **34**, 73 (1958) [English transl.: Soviet Phys.—JETP **7**, 41 (1958); **7**, 51 (1958)]; J. G. Valatin, Nuovo Cimento **7**, 843 (1958).
[8] S. T. Beliaev, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **31**, No. 11 (1959).
[9] M. Baranger, Phys. Rev. **120**, 957 (1960); R. Arvieu and M. Veneroni, Compt. Rend. **250**, 922 (1960); **250**, 2155 (1960); T. Marumori, Progr. Theoret. Phys. (Kyoto) **24**, 331 (1960).
[10] S. T. Beliaev and V. G. Zelevinsky, Nucl. Phys. **39**, 582 (1962).
[11] I. Unna and J. Weneser, Phys. Rev. **137**, 1455 (1965).

II. BCS APPROXIMATION FOR A SIMPLE ISOSCALAR PAIRING HAMILTONIAN

The isoscalar pairing Hamiltonian that we study is

$$H=\sum_{jm}\epsilon_j[a_{jm}^\dagger a_{jm}+b_{jm}^\dagger b_{jm}]-\tfrac{1}{2}\sum_{jj'}G_{jj'}\{[\sum_m(-1)^{j-m}a_{jm}^\dagger a_{j-m}^\dagger][\sum_{m'}(-1)^{j'-m'}a_{j'-m'}a_{j'm'}]$$
$$+[\sum_m(-1)^{j-m}b_{jm}^\dagger b_{j-m}^\dagger][\sum_{m'}(-1)^{j'-m'}b_{j'-m'}b_{j'm'}]+2[\sum_m(-1)^{j-m}a_{jm}^\dagger b_{j-m}^\dagger][\sum_{m'}(-1)^{j'-m'}b_{j'-m'}a_{j'm'}]\}. \quad (1)$$

The a's are neutron operators and the b's are proton operators. The Hamiltonian consists of two parts: a single-particle energy term and a pairing interaction between pairs of nucleons coupled to $(J=0,\ T=1)$. The $G_{jj'}=G_{j'j}$ are the coupling constants of the pairing interaction.

The BCS solution, with which we begin, can be obtained by three equivalent procedures: the BCS variational method, the Bogoliubov elimination of dangerous terms, or the self-consistent solution of a linearized form of the Hamiltonian. The last of these methods provides a more intuitive presentation of one of our principal results. Therefore, we begin in this way.

The linearization of this Hamiltonian consists of replacing it by the quadratic form obtained after replacing products of operators by their ground-state expectation values:

$$H_L=\sum_{jm}\epsilon_j(a_{jm}^\dagger a_{jm}+b_{jm}^\dagger b_{jm})-\tfrac{1}{2}\sum_{jj'}G_{jj'}\{\sum_m[a_{jm}^\dagger a_{j-m}^\dagger(-1)^{j-m}\delta_{n,j'}+\delta_{n,j}(-1)^{j'-m'}a_{j'-m'}a_{j'm'}]$$
$$+[b_{jm}^\dagger b_{j-m}^\dagger(-1)^{j-m}\delta_{p,j'}+\delta_{p,j}(-1)^{j'-m'}b_{j'-m'}b_{j'm}]+2[(-1)^{j-m}a_{jm}^\dagger b_{j-m}^\dagger\delta_{np,j'}+\delta_{np,j}(-1)^{j-m'}b_{j'-m}a_{j'm'}]\}$$
$$-2\sum_{jj}G_{jj}[\sum_m a_{jm}^\dagger a_{jm}(\rho_{n,j}+\tfrac{1}{2}\rho_{p,j})+\sum_m b_{jm}^\dagger b_{jm}(\rho_{p,j}+\tfrac{1}{2}\rho_{n,j})+a_{jm}^\dagger b_{jm}(\tfrac{1}{2}\rho_{np,j})+b_{jm}^\dagger a_{jm}(\tfrac{1}{2}\rho_{np,j})], \quad (2)$$

where the δ, ρ are the ground-state expectation values

$$\delta_{n,j}=\langle\sum_m(-1)^{j-m}a_{jm}^\dagger a_{j-m}^\dagger\rangle=\langle\sum_m(-1)^{j-m}a_{j-m}a_{jm}\rangle,$$
$$\delta_{p,j}=\langle\sum_m(-1)^{j-m}b_{jm}^\dagger b_{j-m}^\dagger\rangle=\langle\sum_m(-1)^{j-m}b_{j-m}b_{jm}\rangle, \quad (3a)$$
$$\delta_{np,j}=\langle\sum_m(-1)^{j-m}a_{jm}^\dagger b_{j-m}^\dagger\rangle=\langle\sum_m(-1)^{j-m}b_{j-m}a_{jm}\rangle,$$

$$\rho_{n,j}=\langle a_{jm}^\dagger a_{jm}\rangle,$$
$$\rho_{p,j}=\langle b_{jm}^\dagger b_{jm}\rangle, \quad (3b)$$
$$\rho_{np,j}=\langle a_{jm}^\dagger b_{jm}\rangle=\langle b_{jm}^\dagger a_{jm}\rangle.$$

To carry out the self-consistent program, the δ's and ρ's are finally to take on the numerical values that are obtained by taking the expectation values, (3a) and (3b), in the ground state that follows from the linearized Hamiltonian. In the above, it has already been explicitly assumed that this ground state has spherical spatial symmetry, so that the $\rho_{n,j}$, $\rho_{p,j}$, $\rho_{np,j}$ are independent of m.

There are two kinds of interaction terms in H_L. The terms that involve the δ's correspond to BCS pairing. The ρ-dependent terms are the Hartree-Fock renormalizations of the single-particle energies, ϵ_j. Actually, as we shall see in detail later, the ρ's are an order smaller than the δ's and of the order of other neglected terms. For now, we can note that $\rho_{n,j}$, for example, can be rewritten, to make comparison with the δ's more obvious:

$$\rho_{n,j}=\langle a_{jm}^\dagger a_{jm}\rangle=\frac{1}{2j+1}\langle(\sum_m a_{jm}^\dagger a_{jm})\rangle.$$

Very roughly, the δ's are greater than the ρ's by the average number of single-particle levels $(2j+1)$, which we shall take as large. To simplify the discussion we drop these Hartree-Fock energies, although the same conclusions would be reached were they included.

Dropping the small Hartree-Fock terms from H_L gives us a Hamiltonian we denote as H_L'. This simplified form, H_L', can be usefully simplified by using as additional notation the gap parameters,

$$\Delta_{nj}=\sum_{j'}G_{jj'}\delta_{n,j'},$$
$$\Delta_{pj}=\sum_{j'}G_{jj'}\delta_{p,j'}, \quad (4)$$

and

$$\Delta_{npj}=\sum_{j'}G_{jj'}\delta_{np,j'}.$$

Then H_L' becomes

$$H_L'=\sum_{jm}\epsilon_j(a_{jm}^\dagger a_{jm}+b_{jm}^\dagger b_{jm})$$
$$-\tfrac{1}{2}\sum_{jm}\Delta_{nj}[(-1)^{j-m}a_{jm}^\dagger a_{j-m}^\dagger+(-1)^{j-m}a_{j-m}a_{jm}]$$
$$-\tfrac{1}{2}\sum_{jm}\Delta_{pj}[(-1)^{j-m}b_{jm}^\dagger b_{j-m}^\dagger+(-1)^{j-m}b_{j-m}b_{jm}]$$
$$-\sum_{jm}\Delta_{npj}[(-1)^{j-m}a_{jm}^\dagger b_{j-m}^\dagger+(-1)^{j-m}b_{j-m}a_{jm}]. \quad (5)$$

The diagonalization would be made much easier if the ab-coupling term, the last term in Eq. (5), were eliminated. To do this, we introduce a new set of fermion operators, $\bar{a}$, $\bar{b}$, which are rotated in isospace:

$$\bar{a}_{jm}=\cos\phi_j\ a_{jm}-\sin\phi_j\ b_{jm},$$
$$\bar{b}_{jm}=\sin\phi_j\ a_{jm}+\cos\phi_j\ b_{jm}. \quad (6)$$

Then

$$H_L'=\sum_{jm}\epsilon_j(\bar{a}_{jm}{}^\dagger\bar{a}_{jm}+\bar{b}_{jm}{}^\dagger\bar{b}_{jm})$$
$$-\tfrac{1}{2}\sum_{jm}\Delta_{\bar{a}j}[(-1)^{j-m}\bar{a}_{jm}{}^\dagger\bar{a}_{j-m}{}^\dagger+(-1)^{j-m}\bar{a}_{j-m}\bar{a}_{jm}]$$
$$-\tfrac{1}{2}\sum_{jm}\Delta_{\bar{b}j}[(-1)^{j-m}\bar{b}_{jm}{}^\dagger\bar{b}_{j-m}{}^\dagger+(-1)^{j-m}\bar{b}_{j-m}\bar{b}_{jm}]$$
$$-\sum_{jm}\Delta_{\bar{a}\bar{b},j}[(-1)^{j-m}\bar{a}_{jm}{}^\dagger\bar{b}_{j-m}+(-1)^{j-m}\bar{b}_{j-m}\bar{a}_{jm}], \quad (7)$$

where the new gaps are

$$\Delta_{\bar{a}j}=\cos^2\phi_j\,\Delta_{nj}+\sin^2\phi_j\,\Delta_{pj}-\sin2\phi_j\,\Delta_{npj}, \quad (8a)$$
$$\Delta_{\bar{b}j}=\sin^2\phi_j\,\Delta_{nj}+\cos^2\phi_j\,\Delta_{pj}+\sin2\phi_j\,\Delta_{npj}, \quad (8b)$$
$$\Delta_{\bar{a}\bar{b}j}=\tfrac{1}{2}\sin2\phi_j\,\Delta_{nj}-\tfrac{1}{2}\sin2\phi_j\,\Delta_{pj}+\cos2\phi_j\,\Delta_{npj}. \quad (8c)$$

The coupling is eliminated by choosing the ϕ_j so that $\Delta_{\bar{a}\bar{b}j}$ is zero:

$$\cos2\phi_j/\sin2\phi_j=\tfrac{1}{2}(\Delta_{pj}-\Delta_{nj})/\Delta_{npj}. \quad (9)$$

The discussion of Eq. (9) is especially obvious for the special case of separable coupling constants, $G_{jj'}=g_jg_{j'}$. Then, since it is seen directly from Eq. (4) that the Δ_{nj}/g_j, Δ_{pj}/g_j, and Δ_{npj}/g_j are independent of j, the $\tan2\phi_j$ are independent of j, or

$$\phi_j=\phi+n_j\tfrac{1}{2}\pi.$$

The choice of $n_j\neq0$ actually leads to nothing new, but only to rearrangements of the $\bar{a}_j$, $\bar{b}_j$ or changes of the choice of phases, none of which affect the ground-state solution or the consequent eigenstates of H_L'. Then, we can take

$$\phi_j=\phi. \quad (10)$$

A special, but much discussed, case of the separable coupling constants is that of equal coupling constants, $g_j=\sqrt{G}=g_{j'}$. Actually, this same result of $\phi_j=\phi$ will be shown (Sec. III) to follow in the more general case, but we put that point off to develop the argument for the simpler cases first. We turn to the self-consistent ground-state solution.

The self-consistent solution of the transformed H_L' must be subjected to two sets of constraints. Since the linearized Hamiltonian is not number conserving, the ground-state expectation value of the number operator is constrained to equal the physical number of nucleons. To do this by the usual Lagrange-multiplier method, one adds to the Hamiltonian H_L' the number operator multiplied by a Lagrange multiplier $-\lambda_N\hat{N}$. The number operator $\hat{N}$ in the $\bar{a}$, $\bar{b}$ representation has the uncoupled form

$$\hat{N}=\sum_{jm}(\bar{a}_{jm}{}^\dagger\bar{a}_{jm}+\bar{b}_{jm}{}^\dagger\bar{b}_{jm})=\sum_j(\hat{N}_{\bar{a}j}+\hat{N}_{\bar{b}j}). \quad (11)$$

A second constraint is required to fix the expectation value of the isospin projection, $\hat{T}_z$. The usual treatment[1-3] has done this by adding a term $-\lambda_z\hat{T}_z$. However, $\hat{T}_z$ in the present notation is

$$\hat{T}_z=\tfrac{1}{2}\sum\cos2\phi_j(\bar{a}_{jm}{}^\dagger\bar{a}_{jm}-\bar{b}_{jm}{}^\dagger\bar{b}_{jm})$$
$$+\tfrac{1}{2}\sum\sin2\phi_j(\bar{a}_{jm}{}^\dagger\bar{b}_{jm}+\bar{b}_{jm}{}^\dagger\bar{a}_{jm}), \quad (12a)$$

and so contains a term that couples $\bar{a}$, $\bar{b}$. Instead, it is more expeditious to enforce the further constraint by using instead of T_z the operator $\hat{\tau}$,

$$\hat{\tau}=\tfrac{1}{2}\sum_j(\hat{N}_{\bar{a}j}-\hat{N}_{\bar{b}j})=\tfrac{1}{2}\sum_{jm}(\bar{a}_{jm}{}^\dagger\bar{a}_{jm}-\bar{b}_{jm}{}^\dagger\bar{b}_{jm}), \quad (12b)$$

in the addition to H_L' of $-\lambda_\tau\hat{\tau}$. The significance of this operator $\hat{\tau}$ will become more apparent when we discuss the self-consistent ground-state solution.

The effective linearized Hamiltonian $\mathcal{H}_L$,

$$\mathcal{H}_L=H_L'-\lambda_N\hat{N}-\lambda_\tau\hat{\tau},$$

is

$$\mathcal{H}_L=\sum_{jm}(\epsilon_j-\lambda_N-\tfrac{1}{2}\lambda_\tau)\bar{a}_{jm}{}^\dagger\bar{a}_{jm}+\sum_{jm}(\epsilon_j-\lambda_N+\tfrac{1}{2}\lambda_\tau)\bar{b}_{jm}{}^\dagger\bar{b}_{jm}$$
$$-\tfrac{1}{2}\sum_{jm}\Delta_{\bar{a}j}[(-1)^{j-m}\bar{a}_{jm}{}^\dagger\bar{a}_{j-m}{}^\dagger+(-1)^{j-m}\bar{a}_{j-m}\bar{a}_{jm}]$$
$$-\tfrac{1}{2}\sum_{jm}\Delta_{\bar{b}j}[(-1)^{j-m}\bar{b}_{jm}{}^\dagger\bar{b}_{j-m}{}^\dagger+(-1)^{j-m}\bar{b}_{j-m}\bar{b}_{jm}]$$
$$=\mathcal{H}_{L,\bar{a}}+\mathcal{H}_{L,\bar{b}}. \quad (13)$$

Since there is no coupling between the $\bar{a}$, $\bar{b}$, the ground-state solution is simply the product of the ground-state solutions of $\mathcal{H}_{L,\bar{a}}$ and $\mathcal{H}_{L,\bar{b}}$. The self-consistency requirements on $\Delta_{\bar{a}j}$, $\Delta_{\bar{b}j}$ are easily met since with such a product solution

$$\langle\bar{b}_{jm}{}^\dagger\bar{a}_{j-m}{}^\dagger\rangle=\langle\bar{a}_{j-m}\bar{b}_{jm}\rangle=0,$$

and only terms quadratic in the $\bar{a}$'s or $\bar{b}$'s separately differ from zero. Then, inserting the expression for Δ_{nj}, Δ_{pj}, Δ_{npj} written in terms of the $\bar{a}$, $\bar{b}$ representation into Eqs. (8a) and (8b) for $\Delta_{\bar{a}j}$, $\Delta_{\bar{b}j}$, we obtain

$$\Delta_{\bar{a}j}=\sum_{j'}Gjj'\langle\sum_{m'}(-1)^{j'-m'}\bar{a}_{j'm'}{}^\dagger\bar{a}_{j'-m'}{}^\dagger\rangle,$$
$$\Delta_{\bar{b}j}=\sum_{j'}Gjj'\langle\sum_{m'}(-1)^{j'-m}\bar{b}_{j'm'}{}^\dagger\bar{b}_{j'-m'}{}^\dagger\rangle. \quad (14)$$

The problem is, then, completely reduced to the solution of two separate BCS problems, which are by now very well known. The familiar solution is

$$|\Psi\rangle=\prod_{jm>0}[U_{\bar{a}j}+(-1)^{j-m}V_{\bar{a}j}\bar{a}_{jm}{}^\dagger\bar{a}_{j-m}{}^\dagger]$$
$$\times[U_{\bar{b}j}+(-1)^{j-m}V_{\bar{b}j}\bar{b}_{jm}{}^\dagger\bar{b}_{j-m}{}^\dagger]|0\rangle, \quad (15)$$

where $|0\rangle$ denotes the particle vacuum, and the $U_{\bar{a}j}$,

$V_{\bar{a}j}$ obey the self-consistent equations

$$2(\epsilon_j-\lambda_N-\tfrac{1}{2}\lambda_\tau)U_{\bar{a}j}V_{\bar{a}j}=\Delta_{\bar{a}j}(U_{\bar{a}j}^2-V_{\bar{a}j}^2),$$
$$U_{\bar{a}j}^2+V_{\bar{a}j}^2=1, \tag{16a}$$
$$\Delta_{\bar{a}j}=\sum_{j'}G_{jj'}(2j'+1)U_{\bar{a}j'}V_{\bar{a}j'};$$

and the $U_{\bar{b}j}$, $V_{\bar{b}j}$ the similar equations

$$2(\epsilon_j-\lambda_N+\tfrac{1}{2}\lambda_\tau)U_{\bar{b}j}V_{\bar{b}j}=\Delta_{\bar{b}j}^2(U_{\bar{b}j}^2-V_{\bar{b}j}^2),$$
$$U_{\bar{b}j}^2+V_{\bar{b}j}^2=1, \tag{16b}$$
$$\Delta_{\bar{b}j}=\sum_{j'}G_{jj'}(2j'+1)U_{\bar{b}j'}V_{\bar{b}j'}.$$

The λ_N, λ_τ are determined by the two constraints on the expectation values of the number and the $\hat{\tau}$ operator:

$$\langle\tfrac{1}{2}(\hat{N}+2\hat{\tau})\rangle=\tfrac{1}{2}(N+2\tau)=\sum_j(2j+1)V_{\bar{a}j}^2,$$
$$\langle\tfrac{1}{2}(\hat{N}-2\hat{\tau})\rangle=\tfrac{1}{2}(N-2\tau)=\sum_j(2j+1)V_{\bar{b}j}^2. \tag{16c}$$

It might be remarked that the Hartree-Fock terms in H_L, Eqs. (2), contain a coupling between $\bar{a}$ and $\bar{b}$ which at first sight does not appear to be of the same form as that introduced by the pairing term, Eq. (8). However, with the wave function of the form given in Eq. (15), it can be shown that this coupling also vanishes.

The Eqs. (16) differ slightly from those usually written by the omission of the small Hartree-Fock renormalizations of the single-particle energies, the ρ's of Eq. (3b). These omissions can be shown not to change energies or wave functions, to the order considered in this paper, because they are small terms that cause only small shifts about a stationary minimum point.

We return now to consider the constraint introduced by the $\hat{\tau}$ operator. The ground-state expectation values of $\hat{\tau}$ and $\hat{T}_z$ are simply related by

$$T_z=\langle\Psi|\hat{T}_z|\Psi\rangle=\cos2\phi\langle\Psi|\hat{\tau}|\Psi\rangle=(\cos2\phi)\tau. \tag{17}$$

Further, the expectation value of $\hat{T}^2$ is

$$\langle\Psi|\hat{T}^2|\Psi\rangle=\tau^2+[N-\sum_j(2j+1)(V_{\bar{a}j}^2+V_{\bar{b}j}^2)^2]. \tag{18}$$

Now we shall assume, for order of magnitude purposes that τ is of the order of N, the number of nucleons, and that both these are of the order of the number of single-particle orbits, $\Omega=\sum_j(j+\frac{1}{2})$. The term in square brackets in (18) is, then, an order smaller than the first term. They are of the same relative order as the omitted Hartree-Fock corrections to H_L. As we shall see, the quasiboson treatment of the residual interactions will give corrections of this same order, so that here we need keep only the leading term:

$$\hat{T}^2\cong\tau^2. \tag{19}$$

Equations (17) and (19), provide the physical interpretation of τ as the isospin quantum number T. Henceforth, we shall use T instead of τ.

The expectation value of the energy, $\langle\Psi|H|\Psi\rangle$, becomes

$$\langle\Psi|H|\Psi\rangle=\sum(2j+1)\epsilon_j(V_{\bar{a}j}^2+V_{\bar{b}j}^2)-\sum(2j+1)G_{jj}\times(V_{\bar{a}j}^4+V_{\bar{b}j}^4+V_{\bar{a}j}^2V_{\bar{b}j}^2)-\tfrac{1}{2}\sum G_{jj'}(2j+1)(2j'+1)\times(U_{\bar{a}j}V_{\bar{a}j}U_{\bar{a}j'}V_{\bar{a}j'}+U_{\bar{b}j}V_{\bar{b}j}U_{\bar{b}j'}V_{\bar{b}j'}). \tag{20}$$

It is important to note that this energy is independent of ϕ as are also the equations for the U, V's. The value of T_z can be fixed by choosing ϕ [Eq. (17)] $-T\leq T_z\leq T$. The statement that $\langle\Psi|H|\Psi\rangle$ is independent of ϕ, or of T_z for fixed τ, is the statement of isospin invariance within the BCS approximation. It is, of course, not an exact invariance; Ψ is *not* an eigenfunction of $\hat{T}^2$ exactly.

As a practical procedure, the discussion in terms of the total isospin quantum number and an angle ϕ is unnecessary if all we want is the energy. Namely, it we pick $\phi=0$ we have the solution for $T_z=T$. Then,

$$\Delta_{\bar{a}j}=\Delta_{pj},$$
$$\Delta_{\bar{b}j}=\Delta_{pj},$$

and

$$\Delta_{np,j}=0.$$

The wave function Ψ is reduced to a BCS product form for the neutrons and the protons separately. In this approximation, there is nothing gained from neutron-proton pairing. However, for the same T, but $|T_z|<T$, the neutron-pairing is very important, and, in fact, is just what is required to make the energy T_z-independent. Establishing that there is an energy minimum at $\phi_j=\phi=0$ is very important to the progress of our development because it is about this point that we expand in the quasiboson approximation.

The constraints used above for the isospin are not the only possibilities. Thus the use of $\hat{T}_z$ instead of $\hat{\tau}$ has been noted. One could also employ $\hat{T}^2$. We shall show later that they do not lead to anything new.

III. VARIATIONAL SOLUTION OF THE GENERAL PAIRING HAMILTONIAN

In the previous section, we investigated some pairing Hamiltonians with separable coupling constant by linearizing the Hamiltonian. This linearization procedure is equivalent to a variational method, but we chose the linearization for pedagogical reasons. For the general pairing problem with nonseparable coupling constants it is, however, more convenient to use the variational method directly.

The variational wave function is similar to but not the same as the form (15) discussed above;

$$\Psi=\prod_{jm>0}(U_{\bar{a}j}+(-1)^{j-m}V_{\bar{a}j}\bar{a}_{jm}^\dagger\bar{a}_{j-m}^\dagger)\times(U_{\bar{b}j}+(-1)^{j-m}V_{\bar{b}j}\bar{b}_{jm}^\dagger\bar{b}_{j-m}^\dagger)|0\rangle, \tag{21}$$

and the $\bar{a}$, $\bar{b}$'s are related to the original set by the j-dependent isospace rotation[12] written down in Eq. (6). Here, however, the $U_{\bar{a}j}$, $V_{\bar{a}j}$, $U_{\bar{b}j}$, $V_{\bar{b}j}$, and the ϕ_j's are independent variational parameters, subject to the normalization constraint

$$U_{\bar{a}j}^2+V_{\bar{a}j}^2=U_{\bar{b}j}^2+V_{\bar{b}j}^2=1.$$

The expectation value of the energy is to be varied subject to constraints. The expectation value of the number operator is to be held fixed and equal to the physical number of nucleons N:

$$N=\langle\Psi|\hat{N}|\Psi\rangle=\sum_j(2j+1)(V_{\bar{a}j}^2+V_{\bar{b}j}^2). \tag{22}$$

The isospin constraint consists of the requirement

$$T=\langle\Psi|\hat{\tau}|\Psi\rangle=\tfrac{1}{2}\sum_j(2j+1)(V_{\bar{a}j}^2-V_{\bar{b}j}^2). \tag{23}$$

Employing the Lagrange multiplier formalism, we see that the quantity to be varied is

$$\begin{aligned}\langle\Psi|H-\lambda\hat{N}-\lambda_\tau\hat{\tau}|\Psi\rangle&=\mathcal{E}\\ &=\sum_j\left[(\epsilon_j-\lambda_N-\tfrac{1}{2}\lambda_\tau)(2j+1)V_{\bar{a}j}^2+(\epsilon_j-\lambda_N+\tfrac{1}{2}\lambda_\tau)(2j+1)V_{\bar{b}j}^2\right]-\left[\sum_j(2j+1)G_{jj}(V_{\bar{a}j}^4+V_{\bar{b}j}^4+V_{\bar{a}j}^2V_{\bar{b}j}^2)\right]\\ &\quad-\{\tfrac{1}{4}\sum_{jj'}G_{jj'}(2j+1)(2j'+1)[(U_{\bar{a}j}V_{\bar{a}j}+U_{\bar{b}j}V_{\bar{b}j})(U_{\bar{a}j'}V_{\bar{a}j'}+U_{\bar{b}j'}V_{\bar{b}j'})+\cos2\phi_{jj'}\\ &\quad\times(U_{\bar{a}j}V_{\bar{a}j}-U_{\bar{b}j}V_{\bar{b}j})(U_{\bar{a}j'}V_{\bar{a}j'}-U_{\bar{b}j'}V_{\bar{b}j'})]\},\end{aligned} \tag{24}$$

where $\phi_{jj'}=\phi_j-\phi_{j'}$. It is worth noting in passing that the dependence on the ϕ_j occurs only in differences between these angles; this independence is a statement of the isospin invariance of the effective Hamiltonian.

We first consider the angular variation. If all the $\phi_{jj'}$ were independent, then the condition that $\mathcal{E}$ be at a minimum would be determined by

$$\begin{aligned}0=\partial\mathcal{E}/\partial\phi_{jj'}&=\tfrac{1}{2}G_{jj'}(2j+1)(2j'+1)\sin2\phi_{jj'}\\ &\quad\times(U_{\bar{a}j}V_{\bar{a}j}-U_{\bar{b}j}V_{\bar{b}j})(U_{\bar{a}j'}V_{\bar{a}j'}-U_{\bar{b}j'}V_{\bar{b}j'}).\end{aligned} \tag{25}$$

Putting aside trivial solutions, we deduce $\phi_{jj'}=n_{jj'}\frac{1}{2}\pi$. For the same reasons given in the previous section the choices $n_{jj'}\neq0$ lead to nothing new and we can restrict ourselves to $\phi_{jj'}=0$. Actually the $\phi_{jj'}$ are not all independent of one another and the variation should be restricted by that dependence. However, a restricted variation cannot give a lower minimum than an unrestricted variation. Since the point $\phi_{jj'}=0$ is allowed by both the restricted and unrestricted variations, we see that it is the minimum point in either case. We have, then, the same conclusion as that arrived at for the separable-constants case, $\phi_j=\phi$. Again, the energy is independent of ϕ or T_z, and the special choice $\phi=0$ is available (as it was for the simpler case).

Variation with respect to the other parameters $(V_{\bar{a}j},V_{\bar{b}j})$ leads to the usual BCS equations. Dropping the small Hartree-Fock terms, the second square bracket in (24), we obtain equations of the same form as those given in (16). The energy, $E=\langle\Psi|H|\Psi\rangle$, has the same form as that displayed in Eq. (20). Before going on to study the quasiboson expansion we turn back to consider other prescriptions for the isospin constraint.

IV. PRESCRIPTIONS FOR ISOSPIN CONSTRAINT

An alternative[1–3] isospin constraint is based on requiring that the physical value of the z component of isospin be given by

$$T_z=\langle\Psi|\hat{T}_z|\Psi\rangle=\tfrac{1}{2}\sum_j\cos2\phi_j\,(2j+1)(V_{\bar{a}j}^2-V_{\bar{b}j}^2). \tag{26}$$

The effective Hamiltonian one works with is then

$$H-\lambda_N\hat{N}-\lambda_z\hat{T}_z. \tag{27}$$

One can ask whether this constraint, compared to that based on $\hat{\tau}$, permits a wider variation that will result in a lower energy. We will show that this is not the case.

Variation of the effective Hamiltonian with respect to ϕ_j leads to the condition

$$\begin{aligned}(U_{\bar{a}j}V_{\bar{a}j}-U_{\bar{b}j}V_{\bar{b}j})\sum_{j'}\sin2(\phi_j-\phi_{j'})\,G_{jj'}(2j'+1)\\ \times(U_{\bar{a}j'}V_{\bar{a}j'}-U_{\bar{b}j'}V_{\bar{b}j'})\\ +\lambda_z\sin2\phi_j\,(V_{\bar{a}j}^2-V_{\bar{b}j}^2)=0.\end{aligned} \tag{28}$$

One solution of these equations is obviously $\phi_j=\phi_{j'}=0$. This solution is just that discussed above for the special case $\phi=0$, $T=T_z$. However, there are other solutions to the Eq. (28), and one must ask whether they lead to lower-energy solutions. We answer this question in the negative by a somewhat devious but succinct argument.

Suppose that by constraining $\langle T_z\rangle$ a particular set of $\bar{\phi}_j$, $\bar{V}_{\bar{a}j}$, $\bar{V}_{\bar{b}j}$ is found that gives an energy minimum $\bar{E}(T_z)$. This defines [Eq. (21)] a definite $\bar{\Psi}$. Now ,we can take $\bar{V}_{\bar{a}j}^2\geqslant\bar{V}_{\bar{b}j}^2$ for each j. This follows simply by noting from Eqs. (22), (24), and (26), that if there were a solution with $\bar{V}_{\bar{a}j}^2<\bar{V}_{\bar{b}j}^2$, there exists another solution with the same energy (N and T_z), which has $\bar{V}_{\bar{b}j}'^2=\bar{V}_{\bar{a}j}^2$, $\bar{V}_{\bar{a}j}'^2=\bar{V}_{\bar{b}j}^2$, and $\bar{\phi}_j'=\bar{\phi}_j+\frac{1}{2}\pi$. Given this $\bar{\Psi}$ we define a $\bar{\tau}$:

$$\bar{\tau}=\tfrac{1}{2}\sum(2j+1)(\bar{V}_{\bar{a}j}^2-\bar{V}_{\bar{b}j}^2). \tag{29}$$

We note that $\bar{\tau}\geqslant T_z$. Next we carry out a variational calculation, constraining the expectation value of $\hat{\tau}$ so

[12] C. Bloch and A. Messiah, Nucl. Phys. 39, 95 (1962).

that $\langle\Psi|\hat{\tau}|\Psi\rangle=\bar{\tau}$; the minimum energy so reached we call $E(\bar{\tau})$. Since this variational function Ψ includes $\bar{\Psi}$,

$$E(\bar{\tau})\leq\bar{E}(T_z).$$

We know, however, that for a given $\tau=\bar{\tau}$ we can still choose ϕ so that $\bar{\tau}\cos\phi=T_z$, since $\bar{\tau}\geqslant T_z$. Thus, we see that the variation with the τ and T_z constraints overlap in the region of the minimum and that the T_z constraint will lead to the solution $\phi_j=0$; this is the conclusion of Ref. 3 for the single j-shell case. It is to be noted that this conclusion follows only for an isospin-invariant Hamiltonian.

Having dealt with the T_z constraint, we take up the T^2 constraint and dispose of the question with a similar argument. The effective Hamiltonian is now taken as

$$H-\lambda_N\hat{N}-\lambda_T\hat{T}^2,$$

to include the constraint

$$T^2=\langle\Psi|\hat{T}^2|\Psi\rangle=\tfrac{1}{4}\sum_{jj'}\cos2\phi_{jj'}\,(2j+1)(2j'+1)\times(V_{\bar{a}j}^2-V_{\bar{b}j}^2)(V_{\bar{a}j'}^2-V_{\bar{b}j'}^2)+\{\sum_j(2j+1)\times[(V_{\bar{a}j}^2+V_{\bar{b}j}^2)-\tfrac{1}{2}(V_{\bar{a}j}^2+V_{\bar{b}j}^2)]^2\}. \quad (30)$$

The term in curly brackets in (30) can again be neglected relative to the first or leading term for the reasons we have already discussed in Eq. (18). The variation of $\phi_{jj'}$ leads to a set of conditions which are satisfied by the $\phi_j=\phi$ solution already discussed. This may, however, not be the only set of solutions of these relations. Rather than investigate the existence and extremal properties of all these possibilities, we give a proof that is sufficient to eliminate them for all cases of interest to us here. Suppose there is a solution to these T^2-constraint equations that is not already contained among those obtained via the τ constraint. Say it is a $\bar{\Psi}$ with $\bar{V}_{\bar{a}j}$, $\bar{V}_{\bar{b}j}$, $\bar{\phi}_j$'s, giving an energy minimum at $\bar{E}(T)$. Again, we order $\bar{V}_{\bar{a}j}^2>\bar{V}_{\bar{b}j}^2$, and define a $\bar{\tau}=\langle\bar{\Psi}|\hat{\tau}|\bar{\Psi}\rangle$; then, it can easily be seen that $\bar{\tau}^2>T^2$. Variation constraining the expectation value of $\hat{\tau}$ at $\bar{\tau}$ will give us an $E(\bar{\tau})\leq\bar{E}(T)$, since $\bar{\Psi}$ is contained within the possible variations of Ψ. Now, we add the physical statement that the energy is an increasing function of the isospin—as we shall see in detail later—for the cases of interest here. Then, $E(T)\leq E(\bar{\tau})\leq\bar{E}(T)$, so that no lower-energy solution is obtained by the $\hat{T}^2$-constraint procedure.

V. QUASIBOSON APPROXIMATION

We turn next to the examination of the residual interactions so far omitted in the BCS approximation. We will work only with the simplest of the pairing Hamiltonians ($G_{jj'}=G$) to make a little easier the exposition of this messy problem.

To begin this, let us write the Hamiltonian in terms of the Bogoliubov-Valatin quasiparticle annihilation and creation operators that diagonalize the linearized Hamiltonian:

$$\alpha_{jm}=U_{\bar{a}j}\bar{a}_{jm}-(-1)^{j-m}V_{\bar{a}j}\bar{a}_{j-m}^\dagger,\qquad \beta_{jm}=U_{\bar{b}j}\bar{b}_{jm}-(-1)^{j-m}V_{\bar{a}j}\bar{b}_{j-m}^\dagger. \quad (31)$$

The actual writing out of the Hamiltonian is made somewhat simpler if we make use of the operator combinations

$$\mathcal{A}_{JM}(j,j')=\sum_{m,m'}C(jj'J;m,m'M)\alpha_{j'm'}\alpha_{jm},\qquad \mathcal{A}(j)\equiv\mathcal{A}_{0,0}(j,j),$$
$$\mathcal{B}_{JM}(j,j')=\sum_{m,m'}C(jj'J;mm',M)\beta_{j'm'}\beta_{jm},\qquad \mathcal{B}(j)\equiv\mathcal{B}_{0,0}(j,j), \quad (32a)$$
$$\mathcal{C}_{JM}(j,j')=\sum_{m,m'}C(jj'J;m,m'M)\beta_{j'm'}\alpha_{jm},\qquad \mathcal{C}(j)\equiv\mathcal{C}_{0,0}(j,j)$$

and

$$\mathfrak{N}_\alpha(j)=\sum_m\alpha_{jm}^\dagger\alpha_{jm},\quad \mathfrak{N}_\beta(j)=\sum_m\beta_{jm}^\dagger\beta_{jm},\quad \tau_+(j)=\sum_m\alpha_{jm}^\dagger\beta_{jm},\quad \tau_-(j)=\sum_m\beta_{jm}^\dagger\alpha_{jm}. \quad (32b)$$

We shall also use as notation the gap parameter

$$\Delta_{\bar{a}}=G\sum_j(2j+1)U_{\bar{a}j}V_{\bar{b}j},\ \Delta_{\bar{b}}=G\sum_j(2j+1)U_{\bar{b}j}V_{\bar{b}j} \quad (33a)$$

and the single-quasiparticle energies

$$E_{\bar{a}j}=[(\epsilon_j-\lambda_N-\tfrac{1}{2}\lambda_\tau)^2+\Delta_{\bar{a}}^2]^{1/2},\qquad E_{\bar{b}j}=[(\epsilon_j-\lambda_N+\tfrac{1}{2}\lambda_\tau)^2+\Delta_{\bar{b}}^2]^{1/2}. \quad (33b)$$

The Eqs. (16) are solved by

$$U_{\bar{a}j}=[\tfrac{1}{2}+\tfrac{1}{2}(\epsilon_j-\lambda_N-\tfrac{1}{2}\lambda_\tau)/E_{\bar{a}j}]^{1/2},\quad V_{\bar{a}j}=[\tfrac{1}{2}-\tfrac{1}{2}(\epsilon_j-\lambda_N-\tfrac{1}{2}\lambda_\tau)/E_{\bar{a}j}]^{1/2},$$
$$U_{\bar{b}j}=[\tfrac{1}{2}+\tfrac{1}{2}(\epsilon_j-\lambda_N+\tfrac{1}{2}\lambda_\tau)/E_{\bar{b}j}]^{1/2},\quad V_{\bar{b}j}=[\tfrac{1}{2}-\tfrac{1}{2}(\epsilon_j-\lambda_N+\tfrac{1}{2}\lambda_\tau)/E_{\bar{b}j}]^{1/2}, \quad (34a)$$

together with the requirements

$$1=\tfrac{1}{2}G\sum_j(2j+1)/E_{\bar{a}j},\quad 1=\tfrac{1}{2}G\sum(2j+1)/E_{\bar{b}j},$$
$$\tfrac{1}{2}(N+2T)=\sum_j(2j+1)V_{\bar{a}j}^2, \quad (34b)$$
$$\tfrac{1}{2}(N-2T)=\sum_j(2j+1)V_{\bar{b}j}^2.$$

Then, the effective Hamiltonian

$$\mathcal{H}=H-\lambda_N\hat{N}-\lambda_\tau\hat{\tau}=\mathcal{H}_{\bar{a}}+\mathcal{H}_{\bar{b}}+\mathcal{H}_{\bar{a}\bar{b}} \quad (35a)$$

splits up into three parts. The first part is the effective

Hamiltonian for the $\bar{a}$ fermions, and is

$$\mathcal{H}_{\bar{a}}=[\sum(\epsilon_j-\lambda-\tfrac{1}{2}\lambda_\tau)(2j+1)V_{\bar{a}j}^2-(\Delta_{\bar{a}}^2/2G)-G\sum(2j+1)V_{\bar{a}j}^4]+\sum_j E_{\bar{a}j}\mathcal{N}_{\bar{a}}(j)+\tfrac{1}{2}G\sum_{jj'}[(2j+1)(2j'+1)]^{1/2}$$
$$\times U_{\bar{a}j}^2V_{\bar{a}j'}^2[\mathcal{A}^\dagger(j)\mathcal{A}^\dagger(j')+\mathcal{A}(j')\mathcal{A}(j)]-\tfrac{1}{2}G\sum_{j,j'}[(2j+1)(2j'+1)]^{1/2}(U_{\bar{a}j}^2U_{\bar{a}j'}^2+V_{\bar{a}j}^2V_{\bar{a}j'}^2)\mathcal{A}^\dagger(j)\mathcal{A}(j')$$
$$+G\sum_{j,j'}[(2j+1)]^{1/2}(U_{\bar{a}j}^2-V_{\bar{a}j}^2)U_{\bar{a}j'}V_{\bar{a}j'}[\mathcal{A}^\dagger(j)\mathcal{N}_{\bar{a}}(j')+\mathcal{N}_a(j')\mathcal{A}(j)]-2G\sum_{j'}[(2j+1)]^{1/2}U_{\bar{a}j}V_{\bar{a}j}^3$$
$$\times[\mathcal{A}(j)+\mathcal{A}^\dagger(j)]-2G[\sum_j U_{\bar{a}j}V_{\bar{a}j}\mathcal{N}_{\bar{a}}(j)]^2+2G\sum_j V_{\bar{a}j}^4\mathcal{N}_\alpha(j). \quad (35b)$$

The effective Hamiltonian for the b fermions, $\mathcal{H}_{\bar{b}}$, has the same form as $\mathcal{H}_{\bar{a}}$, except that $\bar{a}\to\bar{b}$, $\alpha\to\beta$, $\mathcal{A}\to\mathcal{B}$, and and $(-\lambda_\tau)\to(+\lambda_\tau)$. The interaction part of the effective Hamiltonian, $\mathcal{H}_{\bar{a}\bar{b}}$, is

$$\mathcal{H}_{\bar{a}\bar{b}}=[-G\sum_j(2j+1)V_{\bar{a}j}^2V_{\bar{b}j}^2]+G\sum_{j,j'}[(2j+1)(2j'+1)]^{1/2}\,U_{\bar{a}j}U_{\bar{b}j}V_{\bar{a}j'}V_{\bar{b}j'}[\mathcal{C}^\dagger(j)\mathcal{C}^\dagger(j')+\mathcal{C}(j')\mathcal{C}(j)]$$
$$-G\sum_{jj'}[(2j+1)(2j'+1)]^{1/2}[U_{\bar{a}j}U_{\bar{b}j}U_{\bar{a}j'}U_{\bar{b}j'}+V_{\bar{a}j}V_{\bar{b}j}V_{\bar{a}j'}V_{\bar{b}j'}]\mathcal{C}^\dagger(j)\mathcal{C}(j')+G\sum_{j,j'}[(2j+1)]^{1/2}U_{\bar{a}j}U_{\bar{b}j}$$
$$\times\{\mathcal{C}^\dagger(j)[U_{\bar{a}j'}V_{\bar{b}j'}\tau_-(j')+V_{\bar{a}j'}U_{\bar{b}j'}\tau_+(j')]+[U_{\bar{a}j'}V_{\bar{b}j'}\tau_+(j')+V_{\bar{a}j'}U_{\bar{b}j'}\tau_-(j')]\mathcal{C}(j)\}-G\sum_{j,j'}[(2j+1)^{1/2}]V_{\bar{a}j}V_{\bar{b}j}$$
$$\times\{\mathcal{C}^\dagger(j)[U_{\bar{a}j'}V_{\bar{b}j'}\tau_+(j')+V_{\bar{a}j'}U_{\bar{b}j'}\tau_-(j')]+[U_{\bar{a}j'}V_{\bar{b}j'}\tau_-(j')+V_{\bar{a}j'}U_{\bar{b}j'}\tau_+(j')]\mathcal{C}(j)\}-G\sum_j(2j+1)^{1/2}V_{\bar{a}j}V_{\bar{b}j}$$
$$\times\{U_{\bar{a}j}V_{\bar{b}j}[\mathcal{A}(j)+\mathcal{A}^\dagger(j)]+V_{\bar{a}j}U_{\bar{b}j}[\mathcal{B}(j)+\mathcal{B}^\dagger(j)]\}-G\sum_{j,j'}[U_{\bar{a}j}V_{\bar{b}j}\tau_+(j)+V_{\bar{a}j}U_{\bar{b}j}\tau_-(j)]$$
$$\times[U_{\bar{a}j'}V_{\bar{b}j'}\tau_-(j')+V_{\bar{a}j'}U_{\bar{b}j'}\tau_+(j')]+G\sum_j V_{\bar{a}j}^2V_{\bar{b}j}^2[\mathcal{N}_\alpha(j)+\mathcal{N}_\beta(j)]. \quad (35c)$$

Some of the terms are readily recognizable. The sum of the constant terms from $\mathcal{H}_{\bar{a}}$, $\mathcal{H}_{\bar{b}}$, $\mathcal{H}_{\bar{a}\bar{b}}$ is just the minimum expectation value of $\mathcal{H}$, defined in Eq. (24). The terms from $\mathcal{H}_{\bar{a}}$, $\mathcal{H}_{\bar{b}}$,

$$\sum_j E_{\bar{a}j}\mathcal{N}_{\bar{a}}(j)+\sum E_{\bar{b}j}\mathcal{N}_{\bar{b}}(j),$$

are the single-quasiparticle Hamiltonians. The remainder is the residual interaction between the quasiparticles. It is this residual interaction that we wish to study in the quasiboson approximation. In this way the rather formidable and opaque residual interaction will be turned into something manageable and understandable.

We develop the quasiboson expansion following the work of Beliaev and Zelevinsky.[10] Two ideas are important in this development. The first is the expansion of the Hamiltonian and various dynamical variables in powers of operators that obey boson commutation rules. The second is that this expansion in powers of boson operators is also an expansion in inverse powers of $\Omega=\sum_j(j+\frac{1}{2})$. This expansion is derived by Beliaev and Zelevinsky on a basis of satisfying commutation rules between the two-quasiparticle fermion operators given in Eq. (32). Their results can also be shown to follow from a diagram summation procedure. The lowest order corresponds to summing bubble diagrams (as shown, for example, in the paper of Baranger[9]). The higher-order terms correspond to various interactions between the lowest-order, bubble diagrams. We do not demonstrate these here since we need only the low orders. To summarize these points: We use only the lowest-order terms explicitly, and take over from Beliaev and Zelevinsky the description of higher orders as higher orders in $1/\Omega$.

The expansion of the two-quasiparticle operators in boson operators to lowest order is

$$\mathcal{A}_{JM}(j,j')\to A_{JM}(j,j')[1+\delta_{j,j'}]^{1/2},\qquad A(j)\equiv A_{0,0}(j,j),$$
$$\mathcal{B}_{JM}(j,j')\to B_{JM}(j,j')[1+\delta_{j,j'}]^{1/2},\qquad B(j)\equiv B_{0,0}(j,j), \quad (36a)$$
$$\mathcal{C}_{JM}(j,j')\to C_{JM}(j,j'),\quad C(j)\equiv C_{0,0}(j,j).$$

Since the operators $\mathcal{A}$ and $\mathcal{B}$ have the symmetry condition

$$\mathcal{A}_{JM}(j,j')=(-1)^{j-j'-J}\mathcal{A}_{JM}(j',j),$$
$$\mathcal{A}_{JM}(j,j')=(-1)^{j-j'-J}\mathcal{B}_{JM}(j',j), \quad (36b)$$

their corresponding boson approximations, $A_{JM}(j,j')$ and $B_{JM}(j,j')$, respectively, also have these symmetry conditions. The boson operators obey the familiar commutation rules

$$[A_{JM}(j,j'),A_{J'M'}^\dagger(k,k')]=[B_{JM}(j,j'),B_{J'M'}^\dagger(k,k')]$$
$$=\delta_{JJ'}\delta_{MM'}[1+\delta_{jj'}]^{-1}[\delta_{jk}\delta_{j'k'}+(-1)^{j-j'+J}\delta_{jk'}\delta_{j'k'}]$$

and

$$[C_{JM}(j',j'),C_{J'M'}^{\dagger}(k,k')]=\delta_{JJ'}\delta_{MM'}\delta_{jk}\delta_{j'k'}. \quad (37)$$

In these commutation rules the C bosons do not have the delta-function term corresponding to exchange, as do the A and B bosons, since the related $\mathcal{C}$'s are bilinear forms of two different kinds of fermions, α's and β's.

The single-quasinucleon operators are bilinear in the bosons and have the closed-form expansions

$$\begin{aligned}
\mathfrak{N}_\alpha(j)&=\sum_{j',J,M}[1+\delta_{jj'}]A_{JM}^{\dagger}(j,j')A_{JM}(j,j')+\sum_{j'JM}C_{JM}^{\dagger}(j,j')C_{JM}(j,j'),\\
\mathfrak{N}_\beta(j)&=\sum_{j',J,M}[1+\delta_{jj'}]B_{JM}^{\dagger}(j,j')B_{JM}(j,j')+\sum_{j'JM}C_{JM}^{\dagger}(j',j)C_{JM}(j',j),\\
\tau_+(j)&=\sum_{j',J,M}[1+\delta_{jj'}]^{1/2}(-1)^{j-j'-J}A_{JM}^{\dagger}(j,j')C_{JM}(j',j)+\sum_{j'JM}[1+\delta_{jj'}]^{1/2}C_{JM}^{\dagger}(j,j')B_{JM}(j,j'),\\
\tau_-(j)&=\sum_{j',J,M}[1+\delta_{jj'}]^{1/2}(-1)^{j-j'-J}C_{JM}^{\dagger}(j',j)A_{JM}(j,j')+\sum_{j',J,M}[1+\delta_{jj'}]^{1/2}B_{JM}^{\dagger}(j,j')C_{JM}(j,j').
\end{aligned} \quad (38)$$

As an illustration, as well as for subsequent use, let us apply this procedure to write out the number operator $\hat{N}$:

$$\hat{N}=\sum_{jm}(a_{jm}^{\dagger}a_{jm}+b_{jm}^{\dagger}b_{jm})=\sum_{jm}(\bar{a}_{jm}^{\dagger}\bar{a}_{jm}+\bar{b}_{jm}^{\dagger}\bar{b}_{jm}).$$

In terms of the quasiparticles it takes the form

$$\hat{N}=\sum_j(2j+1)(V_{\bar{a}j}^2+V_{\bar{b}j}^2)+\sum_j(2j+1)^{1/2}[U_{\bar{a}j}V_{\bar{a}j}(\mathfrak{A}^{\dagger}(j)+\mathfrak{A}(j))+U_{\bar{b}j}V_{\bar{b}j}(\mathfrak{B}^{\dagger}(j)+\mathfrak{B}(j))]$$
$$+\sum_j[(U_{\bar{a}j}^2-V_{\bar{a}j}^2)\mathfrak{N}_\alpha(j)+(U_{\bar{b}j}^2-V_{\bar{b}j}^2)\mathfrak{N}_\beta(j)]. \quad (39)$$

In the quasiboson expansion this becomes

$$\hat{N}_B=[\sum_j(2j+1)(V_{\bar{a}j}^2+V_{\bar{b}j}^2)]+\{\sum_j[2(2j+1)]^{1/2}[U_{aj}V_{aj}(A^{\dagger}(j)+A(j))+U_{bj}V_{bj}(B^{\dagger}(j)+B(j))]\}$$
$$+\{\sum_{j<j',JM}[(U_{\bar{a}j}^2-V_{\bar{a}j}^2+U_{\bar{a}j'}^2-V_{\bar{a}j'}^2)A_{JM}^{\dagger}(j,j')A_{JM}(j,j')+(U_{\bar{b}j}^2-V_{\bar{b}j}^2+U_{\bar{b}j'}^2-V_{\bar{b}j'}^2)B_{JM}^{\dagger}(j,j')B_{JM}(j,j')]$$
$$+\sum_{j,j',JM}(U_{\bar{a}j}^2-V_{\bar{a}j}^2+U_{\bar{b}j'}^2-V_{\bar{b}j'}^2)C_{JM}^{\dagger}(j,j')C_{JM}(j,j')\}. \quad (40a)$$

The first or constant term is just N, the expectation value $\langle\hat{\Psi}|\hat{N}|\Psi\rangle$; it is of order Ω. The second term, which we will denote as $\hat{N}_1$, linear in the boson operators, is of order $\Omega^{1/2}$ assuming that the boson operators themselves have matrix elements of order 1. The third term, quadratic in the boson operators, is of order 1; it will be denoted N_0. We have dropped terms of higher order. In shorthand form,

$$\hat{N}_B=N+\hat{N}_1+\hat{N}_0. \quad (40b)$$

We shall make particular use of $\hat{N}_1$:

$$\hat{N}_1=\sum_j[2(2j+1)]^{1/2}\{U_{\bar{a}j}V_{\bar{a}j}[A^{\dagger}(j)+A(j)]+U_{\bar{b}j}V_{\bar{b}j}[B^{\dagger}(j)+B(j)]\}. \quad (40c)$$

Similarily, the boson expansion of the $\hat{\tau}$ operator is

$$\hat{\tau}_B=[\sum_j\tfrac{1}{2}(2j+1)(V_{\bar{a}j}^2-V_{\bar{b}j}^2)]+\{\sum_j[\tfrac{1}{2}(2j+1)]^{1/2}[U_{\bar{a}j}V_{\bar{a}j}(A^{\dagger}(j)+A(j))-U_{\bar{b}j}V_{\bar{b}j}(B^{\dagger}(j)+B(j))]\}$$
$$+\{\sum_{j\leq j',JM}[\tfrac{1}{2}(U_{\bar{a}j}-V_{\bar{a}j}+U_{\bar{a}j'}-V_{\bar{a}j'})A_{JM}^{\dagger}(j,j')A_{JM}(j,j')-\tfrac{1}{2}(U_{\bar{b}j}^2-V_{\bar{b}j}^2+U_{\bar{b}j'}^2-V_{\bar{b}j'}^2)B_{JM}^{\dagger}(j,j')B_{JM}(j,j')]$$
$$+\sum_{j,j',JM}(U_{\bar{a}j}^2-V_{\bar{a}j}^2-U_{\bar{b}j'}^2+V_{\bar{b}j'}^2)C_{JM}^{\dagger}(j,j')C_{JM}(j,j')\}, \quad (41a)$$

and in shorthand form

$$\hat{\tau}_B=T+\hat{\tau}_1+\hat{\tau}_0, \quad (41b)$$

$$\hat{\tau}_1=\sum_j[\tfrac{1}{2}(2j+1)]^{1/2}[U_{aj}V_{aj}(A^{\dagger}(j)+A(j))-U_{bj}V_{bj}(B^{\dagger}(j)+B(j))]. \quad (41c)$$

The effective Hamiltonian $\mathcal{H}$ in the boson approximation takes on the form

$$
\begin{aligned}
\mathcal{H}_B=\mathcal{E}&+\sum_{j<j';J,M;J\neq0}(E_{\bar{a}j}+E_{\bar{a}j'})A_{JM}^{\dagger}(j,j')A_{JM}(j,j')+\sum_{j<j';J,M;J\neq0}(E_{\bar{b}j}+E_{\bar{b}j'})B_{JM}^{\dagger}(j,j')B_{JM}(j,j')\\
&+\sum_{j,j';J,M;J\neq0}(E_{aj}+E_{bj'})C_{JM}^{\dagger}(j,j')C_{JM}(j,j')+\sum_j 2E_{aj}A^{\dagger}(j)A(j)+\sum_j 2E_{bj}B^{\dagger}(j)B(j)+\sum_j(E_{\bar{a}j}+E_{\bar{b}j})C^{\dagger}(j)C(j)\\
&+G\sum_{jj'}[(2j+1)(2j'+1)]^{1/2}U_{\bar{a}j}^2V_{\bar{a}j'}^2[A^{\dagger}(j)A^{\dagger}(j')+A(j')A(j)]-G\sum_{j,j'}[(2j+1)(2j'+1)]^{1/2}\\
&\times[U_{\bar{a}j}^2U_{\bar{a}j'}^2+V_{\bar{a}j}^2V_{\bar{a}j'}^2]A^{\dagger}(j)A(j')+G\sum_{j,j'}[(2j+1)(2j'+1)^{1/2}U_{\bar{b}j}^2V_{\bar{b}j'}^2[B^{\dagger}(j)B^{\dagger}(j')+B(j')B(j)]\\
&-G\sum_{j,j'}[(2j+1)(2j'+1)]^{1/2}[U_{\bar{b}j}^2U_{\bar{b}j'}^2+V_{\bar{b}j}^2V_{\bar{b}j'}^2]B^{\dagger}(j)B(j)+G\sum_{j,j'}[(2j+1)(2j'+1)]^{1/2}U_{\bar{a}j}U_{\bar{b}j}V_{\bar{a}j'}V_{\bar{b}j'}\\
&\quad\times[C^{\dagger}(j)C^{\dagger}(j')+C(j')C(j)]-G\sum_{j,j'}[(2j+1)(2j'+1)]^{1/2}[U_{\bar{a}j}U_{\bar{b}j}U_{\bar{a}j'}U_{\bar{b}j'}+V_{\bar{a}j}V_{\bar{b}j}V_{\bar{a}j'}V_{\bar{b}j'}]C^{\dagger}(j)C(j'). \quad (42)
\end{aligned}
$$

The constant $\mathcal{E}$ is just the minimum expectation value of $\mathcal{H}$ that we had before Eq. (24). In the expansion, only terms up to and including order Ω have been kept, and higher-order terms are explicitly and implicitly dropped. To keep the order idea clear, it is useful to note that the gap parameters Δ are of order $G\Omega$, as can be seen directly from their defining equations. The ϵ_j are assumed to be of this order ($G\Omega$) because, if they were an order larger the interaction would be handled as weak coupling which is uninteresting, while if taken an order smaller the problem would reduce to the degenerate case. The λ_N, λ_τ are seen directly from Eqs. (16a) and (16b) to also be of this order.

It is worth noting that there are no linear operator terms in $\mathcal{H}_B$. This is an immediate consequence of the fact that we are expanding around an extremal. Turned around, this is a demonstration that we have carried out the variational problem correctly. The absence of linear terms is true for the effective Hamiltonian but not for the boson approximation to the Hamiltonian H_B;

$$H_B=\mathcal{H}_B+\lambda_N\hat{N}_B+\lambda_\tau\hat{\tau}_B, \quad (43)$$

thus emphasizing that it is for $\mathcal{H}$ and not H that the variational problem has been solved.

VI. DIAGONALIZATION OF $\mathcal{H}_B$

The main task is to write the effective Hamiltonian (42) in diagonal form. The terms involving the $J\neq0$ bosons are already in diagonal form. The eigenbosons are $A_{JM}^{\dagger}(j,j')$, $B_{JM}^{\dagger}(j,j')$, and $C_{JM}^{\dagger}(j,j')$ with corresponding energies $(E_{\bar{a}j}+E_{\bar{a}j'})$, $(E_{\bar{b}j}+E_{\bar{b}j'})$, and $(E_{\bar{a}j}+E_{\bar{b}j'})$, respectively. These eigenbosons obey the canonical equations

$$[\mathcal{H}_B,A_{JM}^{\dagger}(j,j')]=(E_{\bar{a}j}+E_{\bar{a}j'})A_{JM}^{\dagger}(j,j'), \quad (44)$$

with similar equations for the others.

The $J=0$ bosons, unlike the $J\neq0$, are coupled by the interaction, and so the diagonalization is appropriately more complicated. It is immediately seen from the form of $\mathcal{H}_B$ [Eq. (42)] that there is no coupling between the A, B, C, $J=0$ bosons so that we can assume the eigenbosons are of the form of the simple linear combinations

$$
\begin{aligned}
\bar{A}_\mu^{\dagger}&=\tfrac{1}{2}\sum_j[a_{\mu j}^{(+)}(A^{\dagger}(j)+A(j))+a_{\mu j}^{(-)}(A^{\dagger}(j)-A(j))],\\
\bar{B}_\mu^{\dagger}&=\tfrac{1}{2}\sum[b_{\mu j}^{(+)}(B^{\dagger}(j)+B(j))+b_{\mu j}^{(-)}(B^{\dagger}(j)-B(j))],\\
\bar{C}_\mu^{\dagger}&=\tfrac{1}{2}\sum[c_{\mu j}^{(+)}(C^{\dagger}(j)+C(j))+c_{\mu j}^{(-)}(C^{\dagger}(j)-C(j))]. \quad (45)
\end{aligned}
$$

The eigenequations (for the $\bar{A}$ case)

$$[\mathcal{H}_B,\bar{A}_\mu^{\dagger}]=\mathcal{E}_{\bar{A}\mu}\bar{A}_\mu^{\dagger}, \quad (46)$$

together with the normalization equation

$$[\bar{A}_\mu,\bar{A}_\mu^{\dagger}]=1, \quad \mathcal{E}_{\bar{A},\mu}\neq0, \quad (47)$$

furnish a set of linear equations sufficient to determine the $a_\mu^{(+)}$, $a_\mu^{(-)}$, and the eigenvalue $\mathcal{E}_{\bar{A}\mu}$. These equations are

$$\mathcal{E}_{\bar{A},\mu}a_{\mu j}^{(-)}=-G(2j+1)^{1/2}\sum_{j'}(2j'+1)^{1/2}a_{\mu j'}^{(+)}+2E_{aj}a_{\mu j}^{(+)}, \quad (48a)$$

$$\mathcal{E}_{\bar{A},\mu}a_{\mu j}^{(+)}=-G(2j+1)^{1/2}(U_{\bar{a}j}^2-V_{\bar{a}j}^2)\sum_{j'}(2j'+1)^{1/2}\times(U_{\bar{a}j'}^2-V_{\bar{a}j'}^2)a_{\mu j'}^{(-)}+2E_{\bar{a}j}a_{\mu j}^{(-)}, \quad (48b)$$

$$\sum_j a_{\mu j}^{(+)}a_{\mu j}^{(-)}=1, \quad \mathcal{E}_{\bar{A}\mu}\neq0. \quad (48c)$$

This is a very familiar procedure and needs no further discussion here except in the special case of a zero-energy eigenvalue.[13]

We examine these and show that there is one and only one zero-energy solution of Eqs. (48a) and (48b). From (48), with $\mathcal{E}_{\bar{A},\mu}=0$, one has

$$a_{\mu j}^{(+)}=G[(2j+1)^{1/2}/2E_{\bar{a}j}]\sum_{j'}(2j'+1)^{1/2}a_{\mu j'}^{(+)}, \quad (49a)$$

$$a_{\mu j}^{(-)}=G[(2j+1)^{1/2}(U_{\bar{a}j}^2-V_{\bar{a}j}^2)/2E_{\bar{a}j}] \times\sum_{j'}(2j'+1)^{1/2}(U_{\bar{a}j'}^2-V_{\bar{a}j'}^2)a_{\mu j'}^{(-)}, \quad (49b)$$

from which it follows that

$$a_{\mu j}^{(+)}=n_+(2j+1)^{1/2}/E_{\bar{a}j}, \quad (50a)$$

$$a_{\mu j}^{(-)}=n_-(2j+1)^{1/2}(U_{\bar{a}j}^2-V_{\bar{a}j}^2)/E_{\bar{a}j}, \quad (50b)$$

where n_+, n_- are constants independent of j;

$$n_+=\tfrac{1}{2}G\sum_{j'}(2j'+1)^{1/2}a_{\mu j'}^{(+)}, \quad (51a)$$

$$n_-=\tfrac{1}{2}G\sum_{j'}(2j'+1)^{1/2}(U_{\bar{a}j'}^2-V_{\bar{a}j'}^2)a_{\mu j'}^{(-)}. \quad (51b)$$

Inserting (50) into (51) leaves us with the relations

$$n_+[1-\tfrac{1}{2}G\sum_{j'}(2j'+1)/E_{\bar{a}j'}]=0, \quad (52a)$$

$$n_-[1-\tfrac{1}{2}G\sum_{j'}(2j'+1)(U_{\bar{a}j'}^2-V_{\bar{a}j'}^2)^2/E_{\bar{a}j'}]=0. \quad (52b)$$

The bracket in (52a) vanishes, as can be seen immediately from the gap equation (34b). On the other hand, the bracket appearing in (52b) can be seen, after using the identity $U_{\bar{a}j}^2+V_{\bar{a}j}^2=1$ and the gap equation, to be

$$2G\sum_{j'}(2j'+1)U_{\bar{a}j'}^2V_{\bar{a}j'}^2/E_{aj'}>0; \quad (52c)$$

then, Eq. (52b) requires that $n_-=0$. The quantity n_+ does not vanish; it is not further determined by the above equations but by a commutation relation between the zero-energy eigenboson and the conjugate boson which will be introduced and discussed later in this section. In summary, the $a_{\mu j}^{(-)}$ vanish and the $a_{\mu j}^{(+)}$ have the unique form given in (50a). There is, then, one and only one zero-energy solution to the A equations. The same conclusion follows for the B equations.

The C bosons have a set of equations similar to (48):

$$\mathcal{E}_{\bar{c},\mu}c_{\mu j}^{(-)}=-G(2j+1)^{1/2}(U_{\bar{a}j}U_{\bar{b}j}+V_{\bar{a}j}V_{\bar{b}j}) \times\sum_{j'}(2j'+1)^{1/2}(U_{\bar{a}j'}U_{\bar{b}j'}+V_{\bar{a}j'}V_{\bar{b}j'})c_{\mu j'}^{(+)} +(E_{\bar{a}j}+E_{\bar{b}j})c_{\mu j}^{(+)}, \quad (53a)$$

$$\mathcal{E}_{\bar{c},\mu}c_{\mu j}^{(+)}=-G(2j+1)^{1/2}(U_{\bar{a}j}U_{\bar{b}j}-V_{\bar{a}j}V_{\bar{b}j}) \times\sum_{j'}(2j'+1)^{1/2}(U_{\bar{a}j'}U_{\bar{b}j'}-V_{\bar{a}j'}V_{\bar{b}j'})c_{\mu j'}^{(-)} +(E_{\bar{a}j}+E_{\bar{b}j})c_{\mu j}^{(-)}, \quad (53b)$$

$$\sum_j c_{\mu j}^{(+)}c_{\mu j}^{(-)}=1, \quad \mathcal{E}_{\bar{c}\mu}\neq 0. \quad (53c)$$

However, there are no analogous zero-energy bosons—that is, at zero energy, independent of the parameters of the Hamiltonian and constraints. A particular one of these C eigenbosons does go to zero energy at the $T=0$ limit, which we will see in detail later.

The two zero-energy bosons are well known in the literature, where they are frequently designated the "spurious solutions" related to number dispersion. In fact, the bosons can be seen from Eqs. (40c) and (41c) to be proportional to the linear terms $\hat{N}_{\bar{a}1}=\tfrac{1}{2}(\hat{N}_1+2\hat{\tau}_1)$, $\hat{N}_{\bar{b}1}=\tfrac{1}{2}(N_1-2\hat{\tau}_1)$ by recognizing the identities

$$2U_{\bar{a}j}V_{\bar{a}j}=\Delta_{\bar{a}}/E_{\bar{a}j}, \quad 2U_{\bar{b}j}V_{\bar{b}j}=\Delta_{\bar{b}}/E_{\bar{b}j}.$$

One of the C eigenbosons will be seen to be the boson approximation of the isospin operator $\hat{T}_-$; it is this eigenboson whose corresponding eigenenergy goes to zero at $T=0$. The existence of these three eigenbosons can be inferred also from the commutation rules related to certain exact conservation laws, as will be shown below. However, for a system with p different j orbits, there will be $3(p-1)$ other eigenbosons with $J=0$, which can only be obtained by solving the linear equations (48) and (53). From now on we shall designate these eigenbosons as

$$\bar{A}_\mu,\bar{B}_\mu,\bar{C}_\mu; \quad \mu=1\cdots(p-1).$$

The employment of conservation laws to get some of the eigenbosons is very instructive, although it cannot,

[13] In order for the eigenbosons to be stable, their eigenenergies must be real. Writing the right-hand side of Eq. (48a) as $\sum_{j'}M_{jj'}^{(+)}a_{\mu j'}^{(+)}$ and of Eq. (48b) as $\sum_{j'}M_{jj'}^{(-)}a_{\mu j'}^{(-)}$, these energies ξ_μ will be real if both $M^{(+)}$ and $M^{(-)}$ are positive definite. (See Ref. 14.) Since both of these matrices are separable except for a diagonal part, the eigenvalues of each satisfy a dispersion relation. For $M^{(+)}$ this relation is

$$f^{(+)}(\lambda)\equiv G\sum_j\frac{(2j+1)}{2E_{\bar{a}j}-\lambda}=1,$$

while for $M^{(-)}$ it is

$$f^{(-)}(\lambda)\equiv G\sum_j(2j+1)\frac{(U_{\bar{a}j}^2-V_{\bar{a}j}^2)^2}{2E_{\bar{a}j}-\lambda}=1.$$

Since the quasiparticle energies $E_{\bar{a}j}$ are positive, the functions $f^{(+)}$, $f^{(-)}$ must decrease monotonically as λ decreases from zero; that is,

$$f^{(\pm)}(0)\geqslant f^{(\pm)}(\lambda)\geqslant 0,$$

for

$$0\geqslant\lambda\geqslant-\infty.$$

Using the gap equation [Eq. (34b)] we can evaluate $f^{(+)}(0)$:

$$f^{(+)}(0)=G\sum_j\frac{(2j+1)}{2E_{\bar{a}j}}=1;$$

this shows that $\lambda=0$ is a solution, but, according to the above inequality, there cannot be any negative λ solutions to this dispersion relation. Likewise, for $f^{(-)}(0)$,

$$f^{(-)}(0)=G\sum\frac{(2j+1)}{2E_{\bar{a}j}}(1-4U_{aj}^2V_{aj}^2)\leqslant G\sum_j\frac{(2j+1)}{2E_{\bar{a}j}}=1,$$

and so there cannot be any negative λ solutions to this relation. This proof also follows for the more general separable pairing force, $G_{jj'}=g_jg_{j'}$. Thus the A eigenbosons are stable, and by similar arguments, the B and C eigenbosons are also seen to be stable.

of course, give us anything beyond that derived from the above eigenequations. We want to discuss from this viewpoint the eigenbosons related to the number and isospin operators, and to find out how the Hamiltonian depends on them. To make this as simple and physical as possible we take the $(\bar{a},\bar{b})$ representation to coincide with the (a,b), the usual (neutron, proton) description, by choosing the isospin rotation angle ϕ equal to zero. Then, for example, $\hat{\tau}=\hat{T}_z$. We shall use this special choice throughout the remainder of the paper although it could all be done more generally.

The exact conservation laws that we want to use are number and isospin conservation. These can be stated as the commutator relations:

$$[H,\hat{N}]=[H,\hat{T}_z]=[H,\hat{T}_\pm]=0 \tag{54a}$$

or, for the effective Hamiltonian,

$$[\mathcal{H},\hat{N}]=0;\quad [\mathcal{H},\hat{T}_z]=0,\quad [\mathcal{H},\hat{T}_\pm]=\mp\lambda_\tau\hat{T}_\pm. \tag{54b}$$

How are these to be stated in the boson approximation?

Let us begin with the first of these—number conservation. The number operator has been written as a sum of terms [Eq. (40)]. The first term is a constant N, the second, the linear term $\hat{N}_1$, is of order $\Omega^{1/2}$, and the third, the quadratic term $\hat{N}_0$, is of order 1; higher terms are dropped. The effective Hamiltonian $\mathcal{H}$ is similarly a sum of terms; a constant, the quadratic terms of order Ω, and neglected higher-order terms. Schematically we write these as

$$\mathcal{H}=(\text{const}=\mathcal{E})+\hat{\mathcal{H}}_2+\cdots, \tag{55}$$

$$\hat{N}=(\text{const}=N)+\hat{N}_1+\hat{N}_0+\cdots. \tag{56}$$

Then to order $\Omega^{3/2}$ the commutator relation is

$$[\hat{\mathcal{H}}_2,\hat{N}_1]=0. \tag{57}$$

Since $\mathcal{H}_2$ differs from $\mathcal{H}_B$ by a constant, which of course commutes with $\hat{N}_1$, we have

$$[\mathcal{H}_B,\hat{N}_1]=0. \tag{58}$$

Thus, N_1 is an eigenboson corresponding to *zero* eigenenergy.

Very similarly if we write

$$\hat{T}_z=(\text{const}=T)+\hat{T}_{z1}+\hat{T}_{z0}+\cdots, \tag{59}$$

we can deduce a second eigenboson of zero energy since

$$[\mathcal{H}_B,\hat{T}_{z1}]=0. \tag{60}$$

These two eigenbosons commute with each other, $[\hat{N}_1,\hat{T}_{z1}]=0$.

To use the third commutator relation we write out $\hat{T}_+$ in its boson expansion. Thus it is easily seen that $\hat{T}_+$,

$$\hat{T}_+=\sum_{jm} a_{jm}{}^\dagger b_{jm}=\sum_j(2j+1)^{1/2}\times[U_{nj}V_{pj}\mathcal{C}^\dagger(j)+V_{nj}U_{pj}\mathcal{C}(j)]+\sum_j[U_{nj}U_{pj}\tau_+(j)-V_{nj}V_{pj}\tau_-(j)], \tag{61}$$

becomes, to order $\Omega^{1/2}$ (which is all that we shall require),

$$\hat{T}_{B+}=\sum_j(2j+1)^{1/2}[U_{nj}V_{pj}C^\dagger(j)+V_{nj}U_{pj}C(j)]. \tag{62}$$

Terms of order 1 have been dropped. Then from Eq. (45), used in order $\Omega^{3/2}$, and recalling that λ_τ is of order Ω,

$$[\mathcal{H}_B,\hat{T}_{B+}]=-\lambda_\tau\hat{T}_{B+}. \tag{63}$$

In analogous fashion we can also obtain the Hermitian conjugate relation

$$[\mathcal{H}_B,\hat{T}_{B-}]=+\lambda_\tau\hat{T}_{B-}. \tag{64}$$

We immediately see, then, that $\hat{T}_{B-}$ is proportional to an eigenboson of $\mathcal{H}_B$ with eigenenergy λ_τ. The commutator $[\hat{T}_{B+},\hat{T}_{B-}]$ can be evaluated from similar arguments. Starting from the exact relation $[\hat{T}_+,\hat{T}_-]=2\hat{T}_z$, the order Ω gives us

$$[\hat{T}_{B+},\hat{T}_{B-}]=2(\Omega\text{ terms of }\hat{T}_z)=2T. \tag{65}$$

For the special case $T\to 0$, since $N_n\to N_p$, $(\lambda_N-\frac{1}{2}\lambda_\tau)\to(\lambda_N+\frac{1}{2}\lambda_\tau)$, and so $\lambda_\tau\to 0$. We see then from (64) that the eigenenergy approaches zero. In fact, $\hat{T}_{B-}$ is the C boson that takes on zero energy at $T=0$—as was noted above.

Having deduced these eigenbosons, we must next explicitly display how $\mathcal{H}_B$ depends on them. The $\hat{T}_{B+}$, $\hat{T}_{B-}$ dependence is straightforward. The $\hat{T}_{B+}/(2T)^{1/2}$, $\hat{T}_{B-}/(2T)^{1/2}$ are normalized conjugate eigenbosons, corresponding to the nonzero eigenvalue λ_τ; they appear in $\mathcal{H}_B$ just as do the $J\neq 0$ $A_{JM}{}^\dagger$ discussed above. The dependence is then

$$+(\lambda_\tau/2T)\hat{T}_{B-}\hat{T}_{B+}. \tag{66}$$

The display of the $\hat{N}_1$, $\hat{T}_{z1}$ dependence is more complicated because they are zero-energy eigenbosons, and a more circumspect approach is necessary.[14] The form of the dependence can, however, be easily understood on general grounds. Since $\mathcal{H}_B$ must be quadratic in the boson operators and Hermitian, and since $\hat{N}_1$ and $\hat{T}_{z1}$ are themselves Hermitian (and the only zero-energy eigenbosons), the part of $\mathcal{H}_B$ containing these eigenbosons can only have the form

$$\omega_N\hat{N}_1{}^2+\omega_{NT}\hat{T}_{z1}\hat{N}_1+\omega_T\hat{T}_{z1}{}^2. \tag{67}$$

We can go a little further on general grounds by noting that in the effective Hamiltonian [Eq. (42)] there are no terms coupling A and B, A and C, or B and C bosons, so that the A, B, and C parts are separately diagonalizable. There can, then be no A, B cross terms. This turns out to require that $\omega_T=4\omega_N$, and (67) can be rewritten in a more transparent form, separated into a neutron

[14] D. J. Thouless, Nucl. Phys. **22**, 78 (1961); D. J. Thouless and J. G. Valatin, *ibid.* **31**, 211 (1962).

term and a proton term:

$$\omega_n \hat{N}_{n1}{}^2 + \omega_p \hat{N}_{p1}{}^2, \qquad (68)$$

where

$$\omega_n = (\omega_T + \omega_{NT})/2, \quad \omega_p = (\omega_T - \omega_{NT})/2, \qquad (69a)$$

$$\hat{N}_{n1} = \tfrac{1}{2}(\hat{N}_1 + 2\hat{T}_{z1}), \quad \hat{N}_{p1} = \tfrac{1}{2}(\hat{N}_1 - 2\hat{T}_{z1}). \qquad (69b)$$

To determine the ω's it is convenient to introduce the variables canonically conjugate to N_{n1} and N_{p1}. These variables, $\hat{X}_n$ and $\hat{X}_p$, are defined by the following conditions:

$$[\hat{N}_{n1}, \hat{X}_n] = 1 = [\hat{N}_{p1}, \hat{X}_p], \qquad (70a)$$

$$[\hat{N}_{p1}, \hat{X}_n] = 0 = [\hat{N}_{n1}, \hat{X}_p], \qquad (70b)$$

$$[\mathcal{H}_B, \hat{X}_n] = 2\omega_n \hat{N}_{n1}, \qquad (70c)$$

$$[\mathcal{H}_B, \hat{X}_p] = 2\omega_p \hat{N}_{p1}, \qquad (70d)$$

together with the requirement that all the other commutators vanish, including those with nonzero-energy eigenbosons. To fulfill these conditions explicitly, we write $\hat{X}_n, \hat{X}_p$ as a linear superposition of the $J=0$ A and B bosons, respectively:

$$\hat{X}_n = \sum_j x_{nj}(A^\dagger(j) - A(j)),$$
$$\hat{X}_p = \sum_j x_{pj}(B^\dagger(j) - B(j)). \qquad (71)$$

Only the combinations $(A(j) - A^\dagger(j))$, $(B(j) - B^\dagger(j))$ appear since the $(A(j) + A^\dagger(j))$, $(B(j) + B^\dagger(j))$ commute with $\hat{N}_{n1}$ and $\hat{N}_{p1}$. When (71) is inserted into the commutator equations (70), a set of simultaneous equations for the x's and the ω's is obtained. The results of this calculation, which is outlined in Appendix A, are

$$x_{nj} = [2(2j+1)]^{1/2}(\omega_n/E_{nj}) \times [(U_{nj}{}^2 - V_{nj}{}^2)\Gamma_n + U_{nj}V_{nj}], \qquad (72a)$$

where

$$\Gamma_n = \frac{\sum_{j'}(2j'+1)(U_{nj'}{}^2 - V_{nj'}{}^2)U_{nj'}{}^2 V_{nj'}{}^2}{4\sum_j (2j+1)U_{nj}{}^3 V_{nj}{}^3},$$
$$\omega_n = \frac{\Delta_n}{8[\sum_j(2j+1)U_{nj}{}^3V_{nj}{}^3][4\Gamma_n{}^2+1]}, \qquad (72b)$$

together with a similar set for x_{pj}, ω_p obtained by $n \to p$. If instead of working with $(\hat{N}_{n1}, \hat{N}_{p1})$ we want the alternative linear combinations $(\hat{N}_1, \hat{T}_{z1})$, the corresponding conjugate bosons are

$$\hat{X}_N = \tfrac{1}{2}(\hat{X}_n + \hat{X}_p), \quad \hat{X}_z = (\hat{X}_n - \hat{X}_p). \qquad (73)$$

We have thus determined explicitly that the part of $\mathcal{H}_B$ that depends on the number and isospin bosons $\hat{N}$, $\hat{T}_{z1}$, $\hat{T}_{B+}$, and $\hat{T}_{B-}$ is

$$(\omega_n + \omega_p)(\tfrac{1}{4}\hat{N}_1{}^2 + \hat{T}_{z1}{}^2) + (\omega_n - \omega_p)\hat{N}_1\hat{T}_{z1} + (\lambda_T/2T)\hat{T}_{B-}\hat{T}_{B+}. \qquad (74)$$

The remaining parts of $\mathcal{H}_B$, involving the other eigenbosons, commute with $\hat{N}_1$, $\hat{T}_{z1}$, $\hat{T}_{B+}$, and $\hat{T}_{B-}$. The number and isospin dependence of $\mathcal{H}_B$ is then displayed completely in (74). This result is not in the least transparent. Especially obscured are the simple isospin properties of the original Hamiltonian. We show in the next section that we do have just the form resulting from an isoscalar Hamiltonian.

VII. ISOSPIN PROPERTIES

To understand the isospin and number dependence obtained in Eq. (74), let us go back and analyze the expansion of a general isospin-invariant Hamiltonian H. To make the number and isospin dependence clear, we make explicit the dependence on these variables

$$H = H(\hat{N}, \hat{T}^2).$$

The expansion of H around the mean values of $\hat{N}$ and $\hat{T}^2$ is given by the Taylor series

$$H(\hat{N},\hat{T}^2) = H(N,T^2) + \left[\frac{\partial}{\partial N}H(\hat{N},T^2)\right](\hat{N}-N) + \left[\frac{\partial}{\partial(T^2)}H(N,T^2)\right](\hat{T}^2 - T^2) + \frac{1}{2}\left[\frac{\partial^2}{\partial N^2}H(N,T^2)\right](\hat{N}-N)^2$$
$$+ \frac{1}{2}\left[\frac{\partial^2}{\partial(T^2)^2}H(N,T^2)\right](\hat{T}^2 - T^2)^2 + \left[\frac{\partial}{\partial N}\frac{\partial}{\partial(T^2)}H(N,T^2)\right](\hat{N}-N)(\hat{T}^2 - T^2) + \cdots. \qquad (75)$$

Recall that we have already seen the expansion of $\hat{N}$ in decreasing order of $\Omega^{1/2}$;

$$\hat{N} - N = \hat{N}_1 + \hat{N}_0 + \cdots.$$

We can also see that on inserting the boson expansion in

$$\hat{T}^2 = \hat{T}_z{}^2 + \tfrac{1}{2}(\hat{T}_+\hat{T}_- + \hat{T}_-\hat{T}_+),$$

we have

$$\hat{T}^2 = (T + \hat{T}_{z1} + \hat{T}_{z0} +)^2 + \tfrac{1}{2}(\hat{T}_{B+} + \cdots)(\hat{T}_{B-} + \cdots) + \tfrac{1}{2}(\hat{T}_{B-} + \cdots)(\hat{T}_{B+} + \cdots).$$

Thus, collecting terms according to orders of $\Omega^{1/2}$,

$$\hat{T}^2 - T^2 = [2T\hat{T}_{z1}] + [2T\hat{T}_{z0} + \hat{T}_{z1}{}^2 + \tfrac{1}{2}(\hat{T}_{B+}\hat{T}_{B-} + \hat{T}_{B-}T_{B+})] + \cdots, \qquad (76)$$

the first term on the right side being of order $\Omega^{3/2}$, the second of order Ω. The constant term $H(N,T^2)$ can also be expanded in powers of $\Omega^{1/2}$:

$$H(N,T^2)=H_4+H_2+\cdots. \tag{77}$$

H_4 is of order Ω^2, H_2 of order Ω.

Inserting these expansions and collecting according to orders of $\Omega^{1/2}$, we have, up to order Ω,

$$H(\hat{N},\hat{T}^2)=H_4+\left\{\left(\frac{\partial H_4}{\partial N}\right)\hat{N}_1+\left[\frac{\partial H_4}{\partial(T^2)}\right]2T\hat{T}_{z1}\right\}+\left\{H_2+\left(\frac{\partial H_4}{\partial N}\right)\hat{N}_0+\left[\frac{\partial H_4}{\partial(T^2)}\right][2T\hat{T}_{z0}+\hat{T}_{z1}{}^2+\tfrac{1}{2}(\hat{T}_{B+}\hat{T}_{B-}+\hat{T}_{B-}\hat{T}_{B+})]\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial^2 H_4}{\partial N^2}\right)\hat{N}_1{}^2+\left[\frac{\partial^2 H_4}{\partial(T^2)^2}\right]2T^2\hat{T}_{z1}{}^2+\frac{\partial^2 H_4}{\partial N\partial(T^2)}2T\hat{T}_{z1}\hat{N}_1\right\}, \tag{78a}$$

which can be rearranged into the form

$$H(\hat{N},\hat{T}^2)=H_4+H_2+T\frac{\partial H_4}{\partial(T^2)}+\frac{\partial H_4}{\partial N}(\hat{N}_1+\hat{N}_0)+\left(2T\frac{\partial H_4}{\partial T^2}\right)(\hat{T}_{z1}+\hat{T}_{z0})+\left(\frac{1}{2}\frac{\partial^2 H_4}{\partial N^2}\right)\hat{N}_1{}^2$$
$$+\left[\frac{\partial H_4}{\partial(T^2)}+2T^2\frac{\partial^2 H_4}{\partial(T^2)^2}\right]\hat{T}_{z1}{}^2+2T\frac{\partial^2 H_4}{\partial N\partial(T^2)}\hat{N}_1\hat{T}_{z1}+\left[\frac{\partial H_4}{\partial(T^2)}\right]\hat{T}_{B-}\hat{T}_{B+}. \tag{78b}$$

The variational energy E is defined [Eq. (20)] as the expectation value of H in the ground-state variational function, which is the boson-vacuum state. We have, then,

$$E_4+E_2=H_4+H_2+T\frac{\partial H_4}{\partial(T^2)}+\frac{\partial H_4}{\partial N}\langle\hat{N}_1+N_0\rangle+2T\frac{\partial H_4}{\partial T^2}\langle\hat{T}_{z1}+\hat{T}_{z0}\rangle+\frac{1}{2}\frac{\partial^2 H_4}{\partial N^2}\langle\hat{N}_1{}^2\rangle$$
$$+\left[\frac{\partial H_4}{\partial(T^2)}+2T^2\frac{\partial^2 H_4}{\partial(T^2)^2}\right]\langle\hat{T}_{z1}{}^2\rangle+2T\frac{\partial^2 H_4}{\partial N\partial(T^2)}\langle\hat{N}_1\hat{T}_{z1}\rangle+\frac{\partial H_4}{\partial(T^2)}\langle\hat{T}_{B-}\hat{T}_{B+}\rangle. \tag{78c}$$

The various expectation values in the boson vacuum, appearing in Eq. (78c), are not zero because the operator products are not in normal form; they contribute terms of order Ω. Then $E_4=H_4$, but H_2 differs from E_2. In fact, these order-Ω correction terms are part of the zero-point energy corrections.

This general result can now be applied to our pairing problem. In any such variational problem, the first derivative of the energy with respect to a constrained variable is equal to the corresponding Lagrange multiplier. Then, for general reasons, or by detailed calculations, we have

$$\frac{\partial E_4}{\partial N}=\lambda_N,\quad 2T\frac{\partial E_4}{\partial(T^2)}=\lambda_\tau. \tag{79a}$$

By detailed calculations, outlined in Appendix A, we see that

$$\frac{\partial^2 E_4}{\partial N^2}=T^2\frac{\partial^2 E_4}{\partial(T^2)^2}+\frac{1}{2}\frac{\partial E_4}{\partial(T^2)}=\tfrac{1}{2}(\omega_n+\omega_p),$$
$$T\frac{\partial^2 E_4}{\partial N\partial(T^2)}=\tfrac{1}{2}(\omega_n-\omega_p). \tag{79b}$$

Then, to order Ω,

$$H(\hat{N},\hat{T}^2)=(E_4+E_2)+\lambda_N(\hat{N}_1+\hat{N}_0)+\lambda_\tau(\hat{T}_{z1}+\hat{T}_{z0})+(\omega_n+\omega_p)(\tfrac{1}{4}\hat{N}_1{}^2+\hat{T}_{z1}{}^2)+(\omega_n-\omega_p)\hat{N}_1\hat{T}_{z1}+(\lambda_\tau/4T)(\hat{T}_{B+}\hat{T}_{B-}+\hat{T}_{B-}\hat{T}_{B+}).$$

Finally, to order Ω,

$$\mathcal{H}(\hat{N},\hat{T}^2)=H(\hat{N},T^2)-\lambda_N\hat{N}-\lambda_\tau\hat{T}_z=(E_4+E_2-\lambda_N N-\lambda_\tau T+\tfrac{1}{2}\lambda_\tau)+(\omega_n+\omega_p)\times(\tfrac{1}{4}\hat{N}_1{}^2+\hat{T}_{z1}{}^2)+(\omega_n-\omega_p)\hat{N}_1\hat{T}_{z1}+(\lambda_\tau/2T)\hat{T}_{B-}\hat{T}_{B+}. \tag{80}$$

This is in agreement with the operator equation (74). In this way we have demonstrated how the isospin invariance looks when expanded in orders $\Omega^{1/2}$.

Having started with an isospin-invariant Hamiltonian and kept the isospin invariance in each order we have ended up with an expression (78b) that is not manifestly an isospin invariant. The reason for this, as we have seen in detail, can be traced to the asymmetry in the expansions of $\hat{T}_z$ as against $\hat{T}_+$, $\hat{T}_-$. To this can also be traced the fact that $\hat{T}_{B+}$, $\hat{T}_{B-}$ appear as conjugate nonzero-energy bosons, while $\hat{T}_{z1}$ enters as a zero-energy boson. Actually they must all be treated together. Discarding a piece, say $\hat{T}_{z1}$ as a "spurious state," would ruin the invariance in the order considered.

We have considered the $\hat{T}_+$, $\hat{T}_-$, $\hat{T}_z$, and $\hat{N}$ dependences in the boson approximation and related these dependences to derivatives of the variational energy $E(N,T^2)$. Nothing further is really learned than is already known from E, given as a function of N and T^2. The first derivatives are known on general grounds, to be given by the Lagrange multipliers which are determined simultaneously with the determination of E. The second derivatives can, for a pairing Hamiltonian, be written as functions of the already determined U, V's.

VIII. WAVE FUNCTION IN THE QUASIBOSON APPROXIMATION

We come now to consider the eigenstates of the Hamiltonian rather than the effective Hamiltonian we have worked with before. In the quasiboson approximation the Hamiltonian H_B is found by adding to $\mathcal{H}_B$ the additional terms

$$\lambda_N\hat{N}_B+\lambda_\tau\hat{T}_{zB}=\lambda_N(N+\hat{N}_1+\hat{N}_0)+\lambda_\tau(T+\hat{T}_{z1}+\hat{T}_{z0}).$$

To know the energy to order Ω we need to know the effect of $\hat{N}_0$, $\hat{T}_{z0}$.

Rather than handle this directly, it is convenient to transform these $\lambda_N N_0$, $\lambda_\tau T_{z0}$ terms away and work with the transformed Hamiltonian. That is, following Unna,[11] we produce a transformation e^{iS} such that

$$\begin{aligned} e^{-iS}(N+\hat{N}_1+\hat{N}_0)e^{iS}&=N+\hat{N}_1,\\ e^{-iS}(T+\hat{T}_{z1}+\hat{T}_{z0})e^{iS}&=T+\hat{T}_{z1}. \end{aligned} \tag{81}$$

The Hermitian operator S need be specified only to its leading term, order $\Omega^{-1/2}$:

$$\begin{aligned} S=i\sum_j(U_{nj}^2-V_{nj}^2)/4[2(2j+1)]^{1/2}U_{nj}V_{nj}[\tfrac{2}{3}(A^\dagger(j)-A(j))^3+(A^\dagger(j)-A(j))\mathfrak{N}_\alpha(j)+\mathfrak{N}_\alpha(j)(A^\dagger(j)-A(j))]\\ +i\sum(U_{pj}^2-V_{pj}^2)/4[2(2j+1)]^{1/2}U_{pj}V_{pj}[\tfrac{2}{3}(B^\dagger(j)-B(j))^3+(B^\dagger(j)-B(j))\mathfrak{N}_\beta(j)+\mathfrak{N}_\beta(j)(B^\dagger(j)-B(j))]. \end{aligned} \tag{82}$$

Since the transformed effective Hamiltonian $\mathcal{H}_B'$ is to order Ω, the same as $\mathcal{H}_B$, Eq. (42), the transformed Hamiltonian H_B'

$$H_B'=e^{-iS}H_Be^{iS}$$

is, to order Ω,

$$H_B'=\lambda_N N+\lambda_\tau T+\mathcal{H}_B+\lambda_N\hat{N}_1+\lambda_\tau\hat{T}_{z1}. \tag{83}$$

This transformed boson Hamiltonian can now be discussed simply. It is diagonalized by the same eigenbosons that diagonalized $\mathcal{H}_B$ since they differ only by the linear terms involving the boson operators $\hat{N}_1$ and $\hat{T}_{z1}$, which are themselves among the eigenbosons. Further, the leading corrections H_C' to the transformed Hamiltonian

$$H'=e^{-iS}He^{+iS}=H_B'+H_C'$$

can be shown by using the commutation rules (54a) to commute with $\hat{N}_1$ and $\hat{T}_{z1}$. Therefore, the eigenfunctions of H_B' are sufficiently accurate to give the energies to order Ω, even though there are operator terms in H_B' of order $\Omega^{3/2}$. Henceforth we will work with the transformed problem.

The ground-state eigenfunction is especially interesting. It is defined by the requirement that the conjugate of the eigenbosons operative on this function, Ψ_B, make it vanish.[15] Thus, in our example,

$J\neq0$:

$$A_{JM}(j,j')\Psi_B=B_{JM}(j,j')\Psi_B=C_{JM}(j,j')\Psi_B=0, \tag{84a}$$

$J=0$:

$$\bar{A}_\mu\Psi_B=\bar{B}_\mu\Psi_B=\bar{C}_\mu\Psi_B=0;\ \mu=1\cdots(p-1), \tag{84b}$$

$$\hat{N}_1\psi_B=\hat{T}_{z1}\psi_B=\hat{T}_{B+}\psi_B=0. \tag{84c}$$

[15] R. A. Sorensen, Nucl. Phys. **25**, 674 (1961); E. A. Sanderson, Phys. Letters **19**, 141 (1965); J. da Providencia, *ibid.* **22**, 478 (1966).

The condition (84a) is simply met by taking as Ψ_B factors independent of $A_{JM}^\dagger$, $B_{JM}^\dagger$, and $C_{JM}^\dagger$ $(J\neq0)$ times the boson vacuum $|0_B\rangle$. The condition (84b) and (84c) requires the form

$$\begin{aligned} \Psi_B=&\exp[-\tfrac{1}{2}\sum_{j,j'}g_{jj'}{}^A A^\dagger(j)A^\dagger(j')]\\ &\times\exp[-\tfrac{1}{2}\sum_{j,j'}g_{jj'}{}^B B^\dagger(j)B^\dagger(j')]\\ &\times\exp[-\tfrac{1}{2}\sum_{j,j'}g_{jj'}{}^C C^\dagger(j)C^\dagger(j')]|0_B\rangle. \end{aligned} \tag{85}$$

Inserting the forms of the $\bar{A}$, Eq. (45) into (84b) and (84c), commuting the A's through and matching the coefficients of the $A^\dagger$ leads to the equations for the g^A:

$$\sum_{j'}(a_{\mu j'}{}^{(+)}+a_{\mu j'}{}^{(-)})g_{jj'}{}^A=(a_{\mu j}{}^{(+)}-a_{\mu j}{}^{(-)}); \tag{86}$$

similar equations follow for the g^B, g^C by replacing the $(a^{(+)},a^{(-)})$ by $(b^{(+)},b^{(-)})$, and $(c^{(+)},c^{(-)})$. Some of these requirements have special physical significance. The conditions $N_1\psi_B=0$, $\hat{T}_{z1}\psi_B=0$ would make ψ_B an eigenfunction of $\hat{N}$ and $\hat{T}_z$ up to the order considered here. With $T_{B+}\psi_B=0$, these would also make ψ_B an eigenfunction of $\hat{T}^2$ to this order.

The solution of these equations is straightforward. Even though there are zero-energy eigenbosons, the Eqs. (86) are still solvable, and the $g_{jj'}{}^A$ are well behaved. However, it can be shown that the resulting function ψ_B has an undefined norm; the series formed for the norm by expanding each of the Ψ_B in powers of the exponent can be shown to diverge as does

$$\sum_n 1/\sqrt{n}.$$

The difficulty comes from the fact that, for a zero-energy solution, there is no bound on the corresponding "coordinate" or conjugate boson.

To avoid this divergence problem for the $\hat{N}_1$, $\hat{T}_{z1}$ zero-energy bosons that we have to deal with, we use instead of Ψ_B,

$$\bar{\Psi}_B = f_N(\hat{X}_N) f_z(\hat{X}_z) \Psi_B\,, \tag{87}$$

where the f_N, f_z are taken so that $\bar{\Psi}_B$ is a bound function. $\bar{\Psi}_B$ is, then, not an eigenfunction of the $\hat{N}_1$ and $\hat{T}_{z1}$, but rather a packet that can be chosen to give the desired expectation values to the operators that enter. To the order of interest here, these are $\hat{N}_1$, $\hat{N}_1^2$, $\hat{T}_{z1}$, and $\hat{T}_{z1}^2$. Once having chosen the expectation values, it is assured that excited states formed by operating on $\bar{\Psi}_B$ with nonzero-energy boson operators will have the same expectation values as does the ground state, since these eigenbosons commute with $\hat{X}_N$, $\hat{X}_z$. These conditions, together with the relation $\hat{T}_{B+}\bar{\Psi}_B = 0$, fix the expectation value of $\hat{T}^2$;

$$\langle \bar{\Psi}_B | \hat{T}^2 | \bar{\Psi}_B \rangle = \langle \bar{\Psi}_B | (\hat{T}_z^2 + \hat{T}_z) | \bar{\Psi}_B \rangle = T(T+1)\,. \tag{88a}$$

States formed by operating on $\bar{\Psi}_B$ with any of the nonzero-energy bosons except $\hat{T}_{B-}$ will have the same expectation value of $\hat{T}^2$. The boson $\hat{T}_{B-}$ changes T by one unit; that is, to the order considered in this paper,

$$\begin{aligned}\langle [(\hat{T}_{B-}\bar{\Psi}_{B-})/\sqrt{(2T)}] | \hat{T}^2 | [\hat{T}_{B-}\bar{\Psi}_{B-}/\sqrt{(2T)}] \rangle &= (1/2T) \langle \bar{\Psi}_B | \hat{T}_{B+}\hat{T}^2\hat{T}_{B-} | \bar{\Psi}_B \rangle \\ &= \langle \bar{\Psi}_B | \hat{T}^2 | \bar{\Psi}_B \rangle + 2T\,,\end{aligned} \tag{88b}$$

as can be seen by using $[\hat{T}_{B+}, \hat{T}^2] = 2T\hat{T}_{B+}$ together with $\hat{T}_{B+}\,\bar{\Psi}_B = 0$ and Eq. (65).

In practical work, we can see that applying these procedures to the $H(\hat{N}, \hat{T}^2)$ of (78b), replacing the operator dependences by their expectation values, is equivalent to dropping the $\hat{N}_1$, $\hat{T}_{z1}$ dependence and keeping only the constant term and that dependent on $\hat{T}_{B-}$, $\hat{T}_{B+}$:

$$H_4 + H_2 + T\frac{\partial H_4}{\partial (T^2)} + \frac{\partial H_4}{\partial (T^2)} \hat{T}_{B-}\hat{T}_{B+}\,. \tag{89}$$

Applied to the actual Hamiltonian H_B', Eq. (83) with the full boson dependence explicit, these prescriptions can be simply stated: drop the linear terms $\lambda_N \hat{N}_1 + \lambda_\tau \hat{T}_{z1}$ and the terms quadratic in $\hat{N}_1$ and $\hat{T}_{z1}$. We will give a detailed illustration when we discuss the degenerate case in the next section.

We have so far not made explicit whether we are talking about even-even or odd-odd nuclei or both. This simple piece of information is, of course, contained in the wave function, and we will now proceed to erect a cumbersome machine to extract it. We begin by noting a symmetry of the problem. Under the operation $R(a)$ defined by

$$a_{jm} \to -a_{jm}\,, \quad a_{jm}^\dagger \to -a_{jm}^\dagger \tag{90a}$$

or, equivalently,

$$\alpha_{jm} \to -\alpha_{jm}\,, \quad \alpha_{jm}^\dagger \to -\alpha_{jm}^\dagger\,, \tag{90b}$$

the Hamiltonian remains invariant. It also remains invariant under the similarly defined $R(b)$. Then, any eigenfunction of H must be simultaneously an eigenfunction of $R(a)$ and of $R(b)$;

$$R(a)\Psi = \pm\Psi\,, \quad R(b)\Psi = \pm\Psi\,. \tag{91}$$

The eveness or oddness under $R(a)$ corresponds to the eveness or oddness of the number of a particles; similarly for $R(b)$. The BCS wave function, Eq. (21), is even under both $R(a)$ and $R(b)$.

From the fact that the operator combinations $\mathcal{A}^\dagger$, $\mathcal{A}$, $\mathcal{B}^\dagger$, $\mathcal{B}$, (32a) are even under each of these transformations and $\mathcal{C}^\dagger$, $\mathcal{C}$ are odd, we see that the boson operators $A^\dagger$, A, $B^\dagger$, B are even while $C^\dagger$, C are odd. The correlated ground-state wave function Ψ_B (85), as well as $\bar{\Psi}_B$ (87), is then clearly even under $R(a)$ and under $R(b)$, and thus corresponds to an even-even system. The excited-state wave functions

$$\begin{gathered} A_{JM}^\dagger(j,j')\bar{\Psi}_B\,,\ B_{JM}^\dagger(j,j')\bar{\Psi}_B;\ j \leqslant j',\ J \neq 0, \\ \bar{A}_\mu^\dagger\bar{\Psi}_B,\ \bar{B}_\mu^\dagger\bar{\Psi}_B;\ \mu = 1\cdots(p-1)\,, \end{gathered} \tag{92}$$

are similarly even and thus correspond to excited states of the even-even system. Contrastingly,

$$C_{JM}^\dagger\bar{\Psi}_B\,,\ \bar{C}_\mu^\dagger\bar{\Psi}_B\,,\ \hat{T}_{B-}\bar{\Psi}_B \tag{93}$$

are odd under the transformations, and thus do not belong in the even-even system to which we confine our attention. However, the states

$$\begin{gathered} C_{JM}^\dagger(j,j')\hat{T}_{B-}\bar{\Psi}_B;\ \ J \neq 0, \\ C_\mu^\dagger\hat{T}_{B-}\bar{\Psi}_B;\ \ \mu = 1\cdots(p-1),\ J = 0\,, \end{gathered} \tag{94}$$

are even-even excited states. We can immediately see that the states referred to in (92) have the same isospin T as the ground state, while those in (94) have isospin $T+1$.

In summary, we have solved for the states of an even-even system, including those with isospin greater than that of the ground-state value, $T = T_z$. Since the Hamiltonian is an isoscalar we have also solved for the spectrum of odd-odd nuclei with the same isospins. We omit the discussion of the wave functions of the neighboring odd-odd systems which can be obtained by suitable operations with the isospin operators on the even-even wave functions.

IX. DEGENERATE CASE: ENERGY SPECTRUM

As an illustration of these methods we consider the special case of degenerate single-particle orbitals, $\epsilon_j = 0$.

The BCS equations (16) are easily solved; the (U,V)'s are independent of the orbit index j since all the ϵ_j's are equal, and the gap parameter, for the constant-G case considered here, is also j-independent.

The constraints immediately give us

$$V_n^2=(N+2T)/4\Omega,\quad U_n^2=(4\Omega-N-2T)/4\Omega,$$
$$V_p^2=(N-2T)/4\Omega,\quad U_p^2=(4\Omega-N+2T)/4\Omega,\qquad (95a)$$
$$\Omega=\sum_j(j+\tfrac{1}{2}),$$

from which we calculate

$$E_n=E_p=G\Omega,\ \lambda_N=G(\tfrac{1}{2}N-\Omega),\ \lambda_\tau=2GT,$$
$$\Delta_n^2=\tfrac{1}{4}G^2(N+2T)(4\Omega-N-2T),\qquad (95b)$$
$$\Delta_p^2=\tfrac{1}{4}G^2(N-2T)(4\Omega-N+2T).$$

The corresponding energy is

$$E=-\tfrac{1}{4}G[N(4\Omega-N)-4T^2]-(G/2\Omega)(\tfrac{1}{2}N^2+T^2).\qquad (95c)$$

It should be noted that in calculating the (UV)'s we have used the equations (16) which omit the Hartree-Fock renormalizations of the single-particle energies. However, their inclusion will not produce a change in (95c), either in the leading term which is of order Ω^2 or the second term which is of order Ω. The exact energy for an even-even nucleus with $T_z=T$ is well known[16]:

$$-\tfrac{1}{4}G[N(4\Omega-N)-4T^2]-G(\tfrac{3}{2}N-T).\qquad (96)$$

The two expressions (95c) and (96) agree, as expected, in the order-Ω^2 but not in the order-Ω term.

The Hamiltonian in the boson approximation H_B' [Eq. (83)] is

$$H_B'=E+\lambda_N\hat{N}_1+\lambda_\tau\hat{T}_{z1}+2G\Omega\sum_{J,M,j\leq j',J\neq0}[A_{JM}^\dagger(j,j')A_{JM}(j,j')+B_{JM}^\dagger(j,j')B_{JM}(j,j')]+2G\Omega\sum_{JM,j,j',J\neq0}C_{JM}^\dagger(j,j')$$
$$\times C_{JM}(j,j')+\{2G\Omega\sum_j A^\dagger(j)A(j)+GU_n^2V_n^2\sum_{j,j'}(2j+1)^{1/2}(2j'+1)^{1/2}[A^\dagger(j)A^\dagger(j')+A(j')A(j)]-G(U_n^4+V_n^4)$$
$$\times\sum_{j,j'}(2j+1)^{1/2}(2j'+1)^{1/2}A^\dagger(j)A(j')\}+\{2G\Omega\sum_j B^\dagger(j)B(j)+GU_p^2V_p^2\sum_{j,j'}(2j+1)^{1/2}(2j'+1)^{1/2}$$
$$\times[B^\dagger(j)B^\dagger(j')+B(j')B(j)]-G(U_p^4+V_p^4)\sum_{j,j'}(2j+1)^{1/2}(2j'+1)^{1/2}B^\dagger(j)B(j')\}$$
$$+\{GU_nV_nU_pV_p\sum_{j,j'}[C^\dagger(j)C^\dagger(j')+C(j')C(j)]-G(U_n^2U_p^2+V_n^2V_p^2)\sum_{j,j'}C^\dagger(j)C(j')\};\qquad (97)$$

and

$$\hat{N}_1=\hat{N}_{n1}+\hat{N}_{p1}=U_nV_n\sum_j[2(2j+1)]^{1/2}(A^\dagger(j)+A(j))+U_pV_p\sum_j[2(2j+1)]^{1/2}(B^\dagger(j)+B(j))$$
$$\hat{T}_{z1}=\tfrac{1}{2}(\hat{N}_{n1}-\hat{N}_{p1})=\tfrac{1}{2}U_nV_n\sum_j[2(2j+1)]^{1/2}(A^\dagger(j)+A(j))-\tfrac{1}{2}U_pV_p\sum_j[2(2j+1)]^{1/2}(B^\dagger(j)+B(j)).\qquad (98)$$

The eigenbosons are easily obtained. As pointed out before, the Hamiltonian is separable into parts dependent on A alone, B alone, and C alone. The $J\neq0$ eigenbosons are trivially obtained as in Eq. (44). There are as many $J=0$ $\bar{A}$, $\bar{B}$, $\bar{C}$ eigenbosons as there are j orbits, say p. We discuss the $\bar{A}$ bosons first.

We know on general grounds that one of these $\bar{A}$ eigenbosons is the zero-energy eigenboson $\hat{N}_{n1}$. Its conjugate boson is $\hat{X}_n$. The other $(p-1)$ eigenbosons are simply derived from Eq. (48). The eigenenergies are

$$\mathcal{E}_{\bar{A}\mu}=2G\Omega,\quad \mu=1,\cdots,(p-1).\qquad (99a)$$

Since these bosons are degenerate, the corresponding coefficients $a_{\mu j}^{(\pm)}$ are not uniquely specified except for the requirements

$$a_{\mu j}^{(+)}=a_{\mu j}^{(-)},\quad \sum_j(2j+1)^{1/2}a_{\mu j}^{(+)}=0,$$
$$\sum_j a_{\mu j}^{(+)}a_{\mu' j}^{(+)}=\delta_{\mu,\mu'};\qquad (99b)$$

for our purposes below this suffices. The conditions of (99b) amount to a statement of orthogonality between the nonzero-energy bosons and $\hat{N}_{n1}$ and $\hat{X}_n$.

Similarly, we have for the $\bar{B}$ bosons the zero-energy $\hat{N}_{p1}$ and its conjugate $\hat{X}_p$. The $(p-1)$ nonzero-energy bosons are specified by

$$\mathcal{E}_{\bar{B}\mu}=2G\Omega,\quad b_{\mu j}^{(+)}=b_{\mu j}^{(-)},\quad \sum_j(2j+1)^{1/2}b_{\mu j}^{(+)}=0,$$
$$\sum_j b_{\mu j}^{(+)}b_{\mu' j}^{(+)}=\delta_{\mu,\mu'},\quad \mu=1,\cdots,(p-1).\qquad (100)$$

The equation for the $\bar{C}$ bosons (53) also admits of ready solution. One of these has the eigenenergy $2GT$ and is just $\hat{T}_{B-}$. The others are given by conditions that parallel (93) and (94);

$$\mathcal{E}_{\bar{C}\mu}=2G\Omega,\quad c_{\mu j}^{(+)}=c_{\mu j}^{(-)},\quad \sum(2j+1)^{1/2}c_{\mu j}^{(+)}=0,$$
$$\sum_j c_{\mu j}^{(+)}c_{\mu' j}^{(+)}=\delta_{\mu,\mu'},\quad \mu=1,\cdots,(p-1).\qquad (101)$$

In order to put H_B' into diagonal form we must write the original set of bosons A, B, C in terms of the

[16] A. de Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963), Appendix.

eigenbosons together with the conjugate bosons $\hat{X}_n$, $\hat{X}_p$. This can be easily accomplished. The $J\neq 0$ parts are already in diagonal form. For the $J=0$ parts a little manipulation gives us

$$A^\dagger(j)=\sum_{\mu=1}^{p-1} a_{\mu j}{}^{(+)}\bar{A}_\mu{}^\dagger+[2(2j+1)]^{1/2}\times[(8U_nV_n\Omega)^{-1}\hat{N}_{n1}+U_nV_n\hat{X}_n], \quad (102a)$$

$$B^\dagger(j)=\sum_{\mu=1}^{p-1} b_{\mu j}{}^{(+)}\bar{B}_\mu{}^\dagger+[2(2j+1)]^{1/2}\times[(8U_nV_n\Omega)^{-1}\hat{N}_{p1}+U_pV_p\hat{X}_p], \quad (102b)$$

$$C^\dagger(j)=\sum_{\mu=1}^{p-1} c_{\mu j}{}^{(+)}\bar{C}_\mu{}^\dagger+[(2j+1)/T]^{1/2}\times(V_nU_p\hat{T}_{B-}-U_nV_p\hat{T}_{B+}). \quad (102c)$$

We have then, finally,

$$H_B'=-\tfrac{1}{4}G[N(4\Omega-N)-4T^2]-G(\tfrac{2}{3}N-T)+\lambda_N\hat{N}_1+\lambda_\tau\hat{T}_{z1}+\tfrac{1}{4}G\hat{N}_1{}^2+G\hat{T}_{z1}{}^2+G\hat{T}_{B-}\hat{T}_{B+}$$
$$+2G\Omega\sum_{\mu=1}^{p-1}(\bar{A}_\mu{}^\dagger\bar{A}_\mu+\bar{B}_\mu{}^\dagger\bar{B}_\mu+\bar{C}_\mu{}^\dagger\bar{C}_\mu)+2G\Omega\sum_{J\neq0,j\leq j',JM}[A_{JM}{}^\dagger(j,j')A_{JM}(j,j')+B_{JM}{}^\dagger(j,j')B_{JM}(j,j')]$$
$$+2G\Omega\sum_{J\neq0,j,j',J,M}C_{JM}{}^\dagger(j,j')C_{JM}(j,j'). \quad (103)$$

We can regard this result with great satisfaction. The coefficients of the $\hat{N}_1{}^2$, $\hat{T}_{z1}{}^2$ and $(\hat{T}_{B-}\hat{T}_{B+})$ terms agree with the results of the general formulas, Eqs. (72) and (74), applied to this special case. Following the prescription outlined in the previous section, we drop these and the linear terms, and associate N and T with the eigenvalues of the corresponding physical variables to all orders. The ground-state energy, the constant term in Eq. (103), which is good up to and including order Ω, agrees with the exact ground-state energy equation (96).

The excited-state energies also show interesting agreements with the exact values.[16] In terms of the usual seniority S, the reduced isospin t, and the total isospin T', the excitation energy for an even-even nucleus with ground-state $T=T_z$ is

$$G\{T'(T'+1)-T(T+1)+S\Omega-t(t+1)-\tfrac{1}{4}S^2+\tfrac{3}{2}S\}. \quad (104)$$

For low-lying states, $\frac{1}{2}S$ and t are small integers; hence to order Ω, which is all we need for comparison, the excitation energy is

$$G[T'(T'+1)-T(T+1)]+\tfrac{1}{2}S(2G\Omega). \quad (105)$$

We can now compare the exact states[17,18] for the lower-lying excitations with those obtained from the boson approximation. The lowest-lying excitations have $S=2$, $t=1$, $T'=T$, so their excitation energy is $2G\Omega$. There are *two* such states for every (j,j'), $j<j'$ coupled to angular momentum J and projection M. For $j=j'$, $J\neq0$, there are two states for every orbit j, for J even only; there are no states for J odd. For $J=0$, there are $2(p-1)$ states. These states are to be compared with the one-boson excitations given in Eq. (92), which for the degenerate case, all have the same excitation energy, $2G\Omega$. The next set of excited states have $S=2$ and $T'=T+1$, and so have an excitation energy $2GT+2G\Omega$ to order Ω. For $J\neq0$ there is one state for every (j,j') coupled to angular momentum J and projection M. For $J=0$, there are $(p-1)$ states. These states are to be compared with the boson excitations given in Eq. (94), which, for the degenerate case, all have excitation energy $2GT+2G\Omega$. Thus we see agreement in the low-lying states both in excitation energy and angular momentum structure of the states.

[17] K. T. Hecht, Nucl. Phys. **63**, 177 (1965); Phys. Rev. **139**, B794 (1965).

[18] M. Ichimura, Progr. Theoret. Phys. (Kyoto) **33**, 215 (1965); J. N. Ginocchio, Nucl. Phys. **74**, 321 (1965).

X. DEGENERATE-CASE CORRELATED GROUND-STATE WAVE FUNCTION

In this final section we examine the correlations introduced by the quasiboson approximation into the ground-state wave function Ψ_B and compare them with those in the exact solution. Because our main interest has been the isospin problem we examine only the neutron-proton correlations.

As we have already noted in the general discussion in Sec. VIII, in the pairing-force problem, the eigenbosons with $J\neq0$ do not produce any ground-state correlations; $J\neq0$ bosons do not interact via the pairing force. There are two classes of $J=0$ eigenbosons. There are the $\bar{A}_\mu$, $\bar{B}_\mu$, $\bar{C}_\mu$, $\mu=1,\cdots,(p-1)$, Eqs. (99)-(101), which do not mix the bosons with their conjugates; that is,

$$\bar{A}_\mu=\sum a_{\mu j}{}^{(+)}A(j),\ \bar{B}_\mu=\sum b_{\mu j}{}^{(+)}B(j),\ \bar{C}_\mu=\sum c_{\mu j}{}^{(+)}C(j),\quad \mu=1,\cdots,(p-1). \quad (106)$$

Then $\bar{A}_\mu$, $\bar{B}_\mu$, $\bar{C}_\mu$, $\mu=1,\cdots,(p-1)$, acting on the boson vacuum vanish, so that Ψ_B differs from the boson vacuum only by factors that commute with these bosons. The other class of $J=0$ eigenbosons consists of $\hat{N}_1$, $\hat{T}_{z1}$, and $\hat{T}_{B+}$. The $\hat{N}_1$, $\hat{T}_{z1}$ correlations have already been discussed in terms of packets, Eqs. (85) and (87); in the degenerate case the $g_{jj'}{}^A$, $g_{jj'}{}^B$ of Eq. (85) are given by $[(2j+1)(2j'+1)]^{1/2}/2\Omega$. These correlation factors commute with the other eigenbosons. The correlations related to the $\hat{T}_{B+}$ are what we wish to discuss more fully.

The operator $\hat{T}_{B+}$ [Eq. (62)] in the degenerate case is

$$\hat{T}_{B+}=(2\Omega)^{1/2}(U_nV_p\bar{C}_0^\dagger+V_nU_p\bar{C}_0), \quad (107a)$$

where we have introduced as notation

$$\bar{C}_0=\sum_j \frac{(2j+1)^{1/2}}{(2\Omega)^{1/2}}C(j). \quad (107b)$$

The $\bar{C}_0^\dagger$ can be seen to commute with the $\bar{C}_\mu$, $\mu=1,\cdots,(p-1)$, as well as with all the other eigenbosons. Then the correlation factor in Ψ_B [Eq. (85)] can only involve the $\bar{C}_0^\dagger$, taking the form

$$\exp(-\tfrac{1}{2}g^c\bar{C}_0^\dagger\bar{C}_0^\dagger). \quad (108a)$$

The condition ($\hat{T}_{B+}\Psi_B=0$) fixes the constant g^c as

$$g^c=U_nV_p/V_nU_p. \quad (108b)$$

As an alternative procedure for finding the correlation factor, we could have straightforwardly solved the set of simultaneous linear equations for the $g_{jj'}{}^c$ like those shown in (86); the result is, of course, identical. Having obtained the correlation factor in either way, we want to compare it with the exact ground-state solution.

The exact ground-state wave function for the degenerate case can be written in the compact form[17]

$$\psi_e(T,T_z=T) =\mathfrak{N}[S_n^\dagger S_p^\dagger-(S_{np}^\dagger)^2]^{\frac{1}{4}(N-2T)}(S_n^\dagger)^T|0\rangle. \quad (109)$$

The $\mathfrak{N}$ is a normalization constant, $|0\rangle$ is the particle vacuum, and the correlated-pair-creation operators are

$$S_n^\dagger=\tfrac{1}{2}\sum_{jm}(-1)^{j-m}a_{jm}^\dagger a_{j-m}^\dagger,$$
$$S_p^\dagger=\tfrac{1}{2}\sum_{jm}(-1)^{j-m}b_{jm}^\dagger b_{j-m}^\dagger, \quad (110)$$
$$S_{np}^\dagger=\tfrac{1}{2}\sum_{jm}(-1)^{j-m}a_{jm}^\dagger b_{j-m}^\dagger.$$

The pairing Hamiltonian in the degenerate case is just the interaction between these correlated pairs. In fact, from Eq. (1) it can be seen immediately that it is

$$H=-2G(S_n^\dagger S_n+S_p^\dagger S_p+2S_{np}^\dagger S_{np}). \quad (111)$$

The factors in Ψ_e [Eq. (109)] have simple interpretations. The factor $[S_n^\dagger S_p^\dagger-(S_{np}^\dagger)^2]$ is a correlated four-nucleon cluster with total isospin zero, as well as zero angular momentum. The $(S_n^\dagger)^T$ are the two nucleon pairs that carry all the isospin.

We next manipulate the ψ_e into a form that will permit a more direct comparison with the boson result. This desired form is

$$\Psi_e=\mathfrak{N}(\Theta)\times(S_n^\dagger)^{\frac{1}{4}(N+2T)}(S_p^\dagger)^{\frac{1}{4}(N-2T)}|0\rangle, \quad (112)$$

where Θ is an operator. To do this write Ψ_e in expanded form

$$\Psi_e=\mathfrak{N}\sum_{z=0}^{\frac{1}{4}(N-2T)}\frac{(-1)^z}{z![\frac{1}{4}(N-2T)-z]!} \times(S_{np}^\dagger)^{2z}(S_n^\dagger)^{\frac{1}{4}(N+2T)-z}(S_p^\dagger)^{\frac{1}{4}(N-2T)-z}|0\rangle. \quad (113)$$

Introducing the conjugate operators S_n, S_p, and S_{np}, and noting the commutation relations

$$[S_n,S_n^\dagger]=(\Omega-\hat{N}_n),\ [S_p,S_p^\dagger]=(\Omega-\hat{N}_p),\ [S_n,S_p^\dagger]=0=[S_p,S_n^\dagger], \quad (114)$$

we can rewrite Ψ_e as

$$\Psi_e=\mathfrak{N}\left\{\sum_{z=0}^{\frac{1}{4}(N-2T)}\frac{(-1)^z[\frac{1}{4}(N+2T)-z]![\Omega-\frac{1}{4}(N+2T)]![\Omega-\frac{1}{4}(N-2T)]!}{z![\frac{1}{4}(N+2T)]![\frac{1}{4}(N-2T)]![\Omega-\frac{1}{4}(N+2T)+z]![\Omega-\frac{1}{4}(N-2T)+z]!}(S_{np}^\dagger)^{2z}(S_nS_p)^z\right\} \times(S_n^\dagger)^{\frac{1}{4}(N+2T)}(S_p^\dagger)^{\frac{1}{4}(N-2T)}|0\rangle, \quad (115)$$

which is the desired form; the factor in the curly bracket is the operator Θ. The factor

$$(S_n^\dagger)^{\frac{1}{4}(N+2T)}(S_p^\dagger)^{\frac{1}{4}(N-2T)}|0\rangle,$$

which is a product of a neutron part and a proton part, is an exact eigenfunction of (H_n+H_p). The BCS solution for this Hamiltonian (H_n+H_p) when projected for neutron number $\frac{1}{4}(N+2T)$ and proton number $\frac{1}{4}(N-2T)$ is just equal to this factor. The quasiboson approximation to the unprojected BCS solution improves it and brings it closer to the exact solution. For our purposes and to our order, we can regard this factor as the product BCS solution with some A and B correlation factors. The C correlations, those associated with $\hat{T}_{B+}$, are contained in the first factor Θ.

To compare with the previous analysis, BCS and quasiboson, we write the operators in Θ in terms of the quasiparticle operators. Using the notation of Eq. (32) for the combinations of bilinear operators, we have

$$S_{np}^{\dagger}=(\tfrac{1}{2}\Omega)^{1/2}\left[U_nU_p\sum_j\left(\frac{2j+1}{2\Omega}\right)^{1/2}\mathcal{C}^{\dagger}(j)-V_nV_p\sum_j\left(\frac{2j+1}{2\Omega}\right)^{1/2}\mathcal{C}(j)\right]-\tfrac{1}{2}U_nV_p\sum_j\tau_+(j)-\tfrac{1}{2}U_pV_n\sum_j\tau_-(j)\,,$$

$$S_n=\Omega U_nV_n+\Omega^{1/2}\left[U_n^2\sum_j\left(\frac{2j+1}{4\Omega}\right)^{1/2}\mathcal{A}(j)-V_n^2\sum_j\left(\frac{2j+1}{4\Omega}\right)^{1/2}\mathcal{A}^{\dagger}(j)\right]-U_nV_n\sum_j\mathfrak{N}_{\alpha}(j)\,, \tag{116}$$

$$S_p=\Omega U_pV_p+\Omega^{1/2}\left[U_p^2\sum_j\left(\frac{2j+1}{4\Omega}\right)^{1/2}\mathcal{B}(j)-V_p^2\sum_j\left(\frac{2j+1}{4\Omega}\right)^{1/2}\mathcal{B}^{\dagger}(j)\right]-U_pV_p\sum_j\mathfrak{N}_{\beta}(j)\,.$$

We keep only the leading term of the quasiboson approximation to each of the above quantities. This amounts to discarding everything but the first terms in each expression (which are clearly of smaller order in Ω) and replacing $\mathcal{C}^{\dagger}(j)$, $\mathcal{C}(j)$ by the bosons $C^{\dagger}(j)$, $C(j)$:

$$S_{np}^{\dagger}\to(\tfrac{1}{2}\Omega)^{1/2}(U_nU_p\bar{C}_0^{\dagger}-V_nV_p\bar{C}_0)\,,\quad S_n\to\Omega U_nV_n\,,\quad S_p\to\Omega U_pV_p\,. \tag{117}$$

Then,

$$\Theta\to\sum_{z=0}^{\frac{1}{4}(N-2T)}\frac{(-1)^z[\frac{1}{4}(N+2T)-z]![\Omega-\frac{1}{4}(N+2T)]![\Omega-\frac{1}{4}(N-2T)]!}{2^zz![\frac{1}{4}(N+2T)]![\frac{1}{4}(N-2T)]![\Omega-\frac{1}{4}(N+2T)+z]![\Omega-\frac{1}{4}(N-2T)+z]!}\Omega^{3z}(U_nU_p)^{3z}(V_nV_p)^z \times[\bar{C}_0^{\dagger}-(V_nV_p/U_nU_p)\bar{C}_0]^{2z}\,. \tag{118}$$

This form for Θ can be considerably simplified. Thus in the ratio of factorials

$$[\tfrac{1}{4}(N+2T)-z]!/[\tfrac{1}{4}(N+2T)]!=\{\tfrac{1}{4}(N+2T)[\tfrac{1}{4}(N+2T)-1]\cdots[\tfrac{1}{4}(N+2T)-z+1]\}^{-1} =[\tfrac{1}{4}(N+2T)]^{-z}[1+(\text{terms of order } z^2/\Omega)]\,,$$

the terms of order z^2/Ω can be shown to lead to a contribution of order $1/\Omega$ relative to the leading term, and so can be dropped. The reason is that the z's are really limited in the sum to order Ω^0 by the weighting factors. To prove the point most simply we can use the form we shall presently obtain, converting the sum to an integral, and easily examine the Ω dependence of this leading term versus the terms that were dropped. By similar reasoning the sum in (118) can be extended to infinity. Then, using the expressions for the U's and V's [Eq. (95a)],

$$\Theta\cong\frac{1}{[\frac{1}{4}(N-2T)]!}\sum_{z=0}\frac{1}{z!}\left(-2\frac{U_nU_p}{V_nV_p}\right)^z(\tfrac{1}{2}V_p)^{2z} \times\left[\bar{C}_0^{\dagger}-\frac{V_nV_p}{U_nU_p}\bar{C}_0\right]^{2z}\,. \tag{119}$$

We rewrite this expression in normal form for the $\bar{C}_0^{\dagger}$, $\bar{C}_0$. Since Θ acts on the C_0-boson vacuum, only the normal terms involving all creation operators will contribute. It can easily seen be that

$$\left[\bar{C}_0^{\dagger}-\frac{V_nV_p}{U_nU_p}\bar{C}_0\right]^{2z}\to\sum_{k=0}^{z}\frac{(2k)!}{(z-k)!(2k)!} \times\left(\frac{-V_nV_p}{2U_nU_p}\right)^{z-k}[\bar{C}_0^{\dagger}]^{2k}\,, \tag{120}$$

ignoring terms in normal order that contain $\bar{C}_0$'s. Then

$$\Theta\cong\frac{1}{[\frac{1}{4}(N-2T)]!}\sum_{k=0}f(k)[\bar{C}_0^{\dagger}]^{2k}\,, \tag{121a}$$

where

$$f(k)=\frac{1}{(2k)!}\left(\frac{-2V_nV_p}{U_nU_p}\right)^k\sum_{z=k}\frac{(2k)!}{z!(z-k)!}(\tfrac{1}{2}V_p)^{2z}\,. \tag{121b}$$

These $f(k)$ have simple properties. By direct examination it can be shown that

$$f(k+1)=\frac{1}{(k+1)}\left(\frac{-U_nV_p}{2V_nU_p}\right)f(k)\,, \tag{122a}$$

so that

$$f(k)=\frac{1}{k!}\left(\frac{-U_nV_p}{2V_nU_p}\right)^kf(0)\,. \tag{122b}$$

The quantity $f(0)$ is a convergent series that we do not evaluate because it will be absorbed into the normalization constant. Finally,

$$\Theta\cong\frac{1}{[\frac{1}{4}(N-2T)]!}f(0)\sum_{k=0}\frac{1}{k!}\left(\frac{-U_nV_p}{2V_nU_p}\right)^k[\bar{C}_0^{\dagger}]^{2k} =\left\{\frac{1}{[\frac{1}{4}(N-2T)]!}f(0)\right\}\exp\left(-\frac{1}{2}\frac{U_nV_p}{V_nU_p}\bar{C}_0^{\dagger}\bar{C}_0^{\dagger}\right)\,. \tag{123}$$

The resemblance to the quasiboson result [Eq. (108)] is unmistakable.

We have thus shown that, to leading order, the neutron-proton correlations in the exact wave function are the same as those introduced by the quasiboson approximation.

ACKNOWLEDGMENTS

We wish to thank A. K. Kerman, L. Kisslinger, and J. da Providencia for discussions.

APPENDIX

We here carry out the solution of the simultaneous equations (70a)–(70d) for the conjugate boson χ_n and for ω_n.

Equation (70c) requires that we equate the commutator

$$[\mathcal{H}_B,\hat{\chi}_n]=2\sum_j E_{nj}\chi_{nj}[A^\dagger(j)+A(j)] -G[\sum_{j'}(2j'+1)^{1/2}(U_{nj'}^2-V_{nj'}^2)\chi_{nj'}] \times[\sum(2j+1)]^{1/2}(U_{nj}^2-V_{nj}^2)[A^\dagger(j)+A(j)], \quad (A1)$$

with

$$2\omega_n\hat{N}_{n1}=2\omega_n\sum_j[2(2j+1)]^{1/2} \times U_{nj}V_{nj}[A^\dagger(j)+A(j)]. \quad (A2)$$

Matching coefficients of $(A^\dagger(j)+A(j))$, we obtain

$$\chi_{nj}=[2(2j+1)]^{1/2}(\omega_n/E_{nj}) \times[\Gamma(U_{nj}^2-V_{nj}^2)+U_{nj}V_{nj}], \quad (A3)$$

where

$$\Gamma\equiv\frac{G}{2\omega_n}\sum[\tfrac{1}{2}(2j'+1)]^{1/2}(U_{nj'}^2-V_{nj'}^2)\chi_{nj'}. \quad (A4)$$

Substituting (A3) into (A4) and using the identities

$$(U_{nj}^2-V_{nj}^2)=1-4U_{nj}^2V_{nj}^2,\qquad 2U_{nj}V_{nj}=\Delta_n/E_{nj}, \quad (A5)$$

along with the gap equation (34b), we obtain for Γ

$$\Gamma_n=\frac{\sum_{j'}(2j'+1)(U_{nj'}^2-V_{nj'}^2)U_{nj'}^2V_{nj'}^2}{4\sum_j(2j+1)U_{nj}^3V_{nj}^3}. \quad (A6)$$

The normalization equation (70a) gives us a result for ω_n:

$$\omega_n=\Delta_n/8[\sum_j(2j+1)U_{nj}^3V_{nj}^3](4\Gamma_n^2+1). \quad (A7)$$

Similar results hold for χ_p, ω_p, with $n\to p$.

We now move on to relating the ω's to the second derivatives of the variational energy as given in Eq. (79b). Our BCS wave function is a product of a BCS-neutron wave function and a BCS-proton wave function. Thus the BCS energy is the sum of a BCS-neutron energy and BCS-proton energy:

$$E_4(N,T)=E_4^{(n)}(N_n)+E_4^{(p)}(N_p). \quad (A8)$$

Since we also have a separation in variables,

$$N=N_n+N_p,\qquad T=\tfrac{1}{2}(N_n-N_p), \quad (A9)$$

the first derivatives are separable;

$$\frac{\partial E_4}{\partial N}=\frac{1}{2}\left(\frac{\partial E^{(n)}}{\partial N_n}+\frac{\partial E^{(p)}}{\partial N_p}\right),\qquad T\frac{\partial E_4}{\partial T^2}=\frac{1}{2}\left(\frac{\partial E_4^{(n)}}{\partial N_n}-\frac{\partial E_4^{(p)}}{\partial N_p}\right), \quad (A10)$$

as are also the second derivatives;

$$\frac{\partial^2}{\partial(N)^2}E_4=T^2\frac{\partial^2E_4}{\partial(T^2)^2}+\frac{1}{2}\frac{\partial E_4}{\partial(T^2)} =\frac{1}{4}\left[\frac{\partial^2E_4^{(n)}}{\partial(N_n)^2}+\frac{\partial^2E_4^{(p)}}{\partial(N_p)^2}\right], \quad (A11)$$

$$T\frac{\partial^2E}{\partial N\partial(T^2)}=\frac{1}{4}\left[\frac{\partial^2E_4^{(n)}}{\partial(N_n)^2}-\frac{\partial^2E_4^{(p)}}{\partial(N_p)^2}\right].$$

Our problem is then completely reduced to solving for the second derivative of the BCS energy for one kind of particle, say for neutrons.

The neutron BCS energy is

$$E_4^{(n)}=\sum\epsilon_j(2j+1)V_{nj}^2-\Delta_n^2/2G. \quad (A12)$$

The derivative with respect to N_n is

$$\frac{\partial E_4^{(n)}}{\partial N_n}=\sum_j\epsilon_j(2j+1)2V_{nj}\frac{\partial V_{nj}}{\partial N_n} -\Delta_n\sum(2j+1)\frac{\partial}{\partial N_n}U_{nj}V_{nj}. \quad (A13)$$

The normalization condition $(U_{nj}^2+V_{nj}^2=1)$ gives us

$$\frac{\partial U_{nj}V_{nj}}{\partial N_n}=\left[\frac{U_{nj}^2-V_{nj}^2}{U_{nj}}\right]\frac{\partial V_{nj}}{\partial N_n},$$

which, using the BCS condition (16), becomes

$$\frac{\partial U_{nj}V_{nj}}{\partial N_n}=2\frac{(\epsilon_j-\lambda_N-\frac{1}{2}\lambda_\tau)}{\Delta_n}V_{nj}\frac{\partial V_{nj}}{\partial N_n}.$$

Substituting into (A13), the energy derivative is just the neutron Lagrange multiplier defined by $\lambda_n=\lambda_N+\frac{1}{2}\lambda$:

$$\frac{\partial E_4^{(n)}}{\partial N_n}=\lambda_n\sum_j(2j+1)2V_{nj}\frac{\partial V_{nj}}{\partial N_{nj}}=\lambda_n\,, \qquad \text{(A14)}$$

where we have used the derivative of the number equation

$$1=\frac{\partial N_n}{\partial N_n}=\sum_j(2j+1)2\frac{\partial N_{nj}^2}{\partial N_n}$$
$$=\sum(2j+1)2V_{nj}\frac{\partial V_{nj}}{\partial N_n}. \qquad \text{(A15)}$$

Then,

$$\frac{\partial^2 E_4^{(n)}}{\partial N_n^2}=\frac{\partial\lambda_n}{\partial N_n}.$$

Solving for this derivative is facilitated by using the expressions for the U_{nj}, V_{nj} given in Eq. (34a) in terms of the gap Δ_n and λ_n. Then, taking the derivative of the gap equation (34b) and the equation for the number of neutrons (A15), we have two equations linear in $\partial\lambda_n/\partial N_n$ and $\partial\Delta_n/\partial N_n$:

$$\left[\sum_j(2j+1)\frac{(\epsilon_j-\lambda_n)}{E_{nj}^3}\right]\frac{\partial\lambda_n}{\partial N_n}-\Delta_n\left[\sum_j\frac{(2j+1)}{E_{nj}^3}\right]\frac{\partial\Delta_n}{\partial N_n}=0\,,$$

$$\tfrac{1}{2}\Delta_n^2\left[\sum_j\frac{(2j+1)}{E_{nj}^3}\right]\frac{\partial\lambda_n}{\partial N_n} \qquad \text{(A16)}$$
$$+\tfrac{1}{2}\Delta_n\left[\sum_j(2j+1)\frac{(\epsilon_j-\lambda_n)}{E_{nj}^3}\right]\frac{\partial\Delta_n}{\partial N_n}=1\,.$$

These two equations are easily solved, and using the identities in (A5), can be written simply in terms of ω_n and Γ_n:

$$\frac{\partial\lambda_n}{\partial N_n}=\frac{\partial^2E_4^{(n)}}{\partial N_n^2}=2\omega_n\,,$$
$$\frac{\partial\Delta_n}{\partial N_n}=4\Gamma_n\omega_n\,. \qquad \text{(A17)}$$

Similar equations hold for the derivatives of $E_4^{(p)}$ with respect to N_p, with of course, $n\to p$ and $\lambda_\tau\to-\lambda_\tau$. Using these results in Eq. (A11), we finally arrive at the expressions in (79b).

PHYSICAL REVIEW VOLUME 170, NUMBER 4 20 JUNE 1968

Exponentially Velocity-Dependent Potential in the Shell Model of O^{18}

P. S. Ganas*

Daily Telegraph Theoretical Department, School of Physics, University of Sydney, Sydney, New South Wales, Australia

(Received 2 February 1968)

The low-lying energy levels of O^{18} are calculated in the harmonic-oscillator shell model using as residual interaction the exponentially velocity-dependent singlet-even potential of Tabakin and Davies in combination with the triplet-odd potential of Green. The main result is that the energy levels so calculated agree closely with the energy levels of Kuo and Brown calculated without core polarization from the hard-core Hamada-Johnston nucleon-nucleon potential.

1. INTRODUCTION

The suggestion originally made by Peierls[1] that the hard core in nucleon-nucleon potentials may be replaced by a velocity-dependent potential has received much attention over the years. The most extensively discussed velocity-dependent potentials are ones which have a quadratic dependence on the relative momentum:

$$V(\mathbf{r},\mathbf{p})=-V_1(r)+p^2V_2(r)+V_2(r)p^2.$$

A nucleon-nucleon potential of this form was found to be consistent with the nucleon-nucleon scattering data. Green,[2] for example, showed that the singlet and triplet phase shifts in the region 100–300 MeV as well as the low-energy data could be fitted well by several sets of values of the potential parameters. Using the potentials so determined he calculated the energy per particle of infinite nuclear matter applying perturbation theory; he found that the convergence rate was relatively slow. The applicability of Green's potentials in the nuclear shell model has been investigated by Ganas and McKellar,[3] referred to hereafter as GM. They find that, in considering the energy spectra of O^{18} and F^{18}, the ground state as calculated from a velocity-dependent potential

* Address after July 31: Physics Department, University of Florida, Gainesville, Fla.

[1] R. E. Peierls, in *Proceedings of the International Conference on Nuclear Structure, Kingston, Canada, 1960,* edited by D. A. Bromley and E. W. Vogt (North-Holland Publishing Co., Amsterdam, 1960), p. 7.

[2] A. M. Green, Nucl. Phys. 33, 218 (1962).

[3] P. S. Ganas and B. H. J. McKellar, Nucl. Phys. (to be published); P. S. Ganas, thesis, University of Sydney, 1967 (unpublished).