Tomasch Oscillations in the Density of States of Superconducting Films

T. WOLFRAM
Science Center/Aerospace and Systems Group, North American Rockwell Corporation, Thousand Oaks, California
(Received 21 December 1967)

The mechanism which gives rise to the oscillatory structure in the quasiparticle density of states of superconducting films with spatially varying electron-electron interaction is discussed. It is shown that the Tomasch effect results from processes in which a quasiparticle is condensed into the sea of Cooper pairs, leaving behind a different but degenerate quasiparticle. The tunneling density of states is obtained for a composite two-region superconductor. The nature of the density-of-states structure depends strongly upon the ratio of the energy gaps of the two regions. In the low-energy range, bound eigenstates characterized by the quantization of the difference of the momenta of the degenerate quasiparticles are found for infinite mean free path. Explicit results are presented for the composite film systems In-Al and In-Pb for several values of the electron mean free path.

I. INTRODUCTION

THE observation of periodic oscillations superimposed upon the usual quasiparticle tunneling current versus voltage characteristic of Al-insulator-Pb, Al-insulator-In, and Al-insulator-Sn, Giaever-type tunnel junctions, has been reported by Tomasch.[1–4] Deposition of a thin layer of different material on the back side of the second film leads to considerable enhancement of the amplitude of the Tomasch oscillations. The experimental situation is shown schematically in Fig. 1. Typically, metal A is a very thin evaporated Al film on the order of 10^3 Å in thickness, which is oxidized to form an insulating barrier. The second film, metal B, on the order of 1 to 30 μ, is then deposited. Finally, this second film is overlayed with a thin ($\sim 10^3$ Å) film of dissimilar metal. The tunnel differential conductance depends strongly upon the overlay material. Tomasch has reported results using overlay films of Ag, Al, and Pb.[3,4]

Theoretical studies[5–7] have indicated that the oscillations in the tunnel current and conductance are manifestations of periodic structure in the density of states of the composite film which result from the spatial variation of the energy gap at the overlay interface.

McMillan and Anderson[5,6] and Wolfram and Lehman[7] have formulated very crude models which are capable of accounting for the voltage spacing of the oscillatory structure. These models have shown that mixing of the degenerate quasiparticle states occurs in systems with a nonuniform energy gap and that the density of states for these hybridized quasiparticles contains structure with the Tomasch oscillatory period. The purpose of this paper is to clarify the mechanism underlying the Tomasch effect and to investigate a model sufficiently realistic to predict the amplitude as well as the period of the Tomasch oscillations.

In Sec. II we discuss within the BCS framework the mechanism which gives rise to the mixing of the degenerate quasiparticle states. It is shown that the Tomasch effect is a result of processes in which a quasiparticle interacts with, and is condensed into, the sea of Cooper pairs leaving behind a different but degenerate quasiparticle. In Sec. III the Green's function for a composite system of two dissimilar metals is discussed. It is assumed that the wave functions are continuous across the interface of the two regions. Two different sets of boundary conditions are considered for the surfaces at x_b and x_2.

The case in which the mean free path in the overlay is short is considered in Sec. IV. The average tunneling density of states is calculated for metal B as a function of the energy gap for metal B and for the overlay. A power-series expansion of the tunneling density of states is discussed and interpreted in terms of multiple scattering. Explicit results are obtained for energy gaps corresponding to In overlayed with Al for several values of the mean free path l_1 in metal B. The case for which the energy gap of the overlay is greater than that of metal B is discussed. It is shown in the limit of long mean free path in metal B that bound eigenstates occur at energies less than the overlay energy gap. These bound states are characterized by the quantization of the difference of the momenta corresponding to the two degenerate quasiparticle states. Each discrete energy level corresponding to a bound state contributes a δ-function peak to the density of states when the mean free path l_1 is infinite. For finite l_1 these δ-functions are smeared into resonance peaks of finite width and amplitude. There are no bound states at energies above the overlay energy gap Δ_2 but there is oscillatory struc-

[1] W. J. Tomasch, Phys. Rev. Letters **15**, 672 (1965); **16**, 16 (1966); Phys. Letters **23**, 204 (1966).
[2] W. J. Tomasch and T. Wolfram, Phys. Rev. Letters **16**, 352 (1966).
[3] W. J. Tomasch, Bull. Am. Phys. Soc. **11**, 190 (1966).
[4] W. J. Tomasch, NATO Advanced Institute on Tunneling Phenomena in Solids, Danish Atomic Energy Research Establishment Risö, Denmark, 1967 (to be published).
[5] W. L. McMillan and P. W. Anderson, Phys. Rev. Letters **16**, 85 (1966).
[6] The exact solution of the McMillan-Anderson model discussed by T. Wolfram and M. B. Einhorn, Phys. Rev. Letters **17**, 966 (1966).
[7] T. Wolfram and G. W. Lehman, Phys. Letters **24A**, 101 (1967).

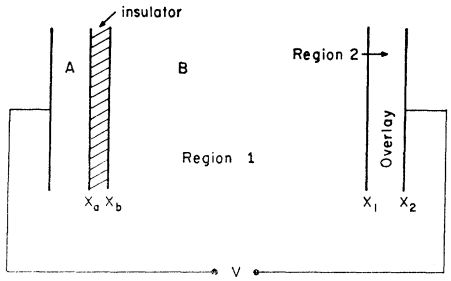

FIG. 1. A schematic of a Giaever-type junction. Film A, typically aluminum, on the order of 1000 Å in thickness, is oxidized to form an insulating barrier extending from x_a to x_b. Superconducting film B extending from x_b to x_2 consists of two regions of dissimilar materials. Region 1 is 1–30 μ in thickness while region 2, the overlay, is on the order of 1000 Å. A bias V is applied across the junction.

ture in the density of states. For a given l_1 the amplitude of the structure in the density of states decreases sharply for $\omega > \Delta_2$. Explicit results are obtained for energy gaps corresponding to In overlayed with Pb. In both cases (In-Al and In-Pb) harmonics of the fundamental Tomasch series are evident.

A brief summary of the results and conclusions is presented in Sec. V. A very simple model for the tunneling process is described in the Appendix.

II. MECHANISM FOR THE TOMASCH EFFECT

In this section we show that the Tomasch effect is a result of processes in which a quasiparticle interacts with, and is condensed into, the sea of Cooper pairs, leaving behind a different but degenerate quasiparticle.

In this discussion it is important to account for all of the Cooper pairs and therefore we shall discuss the BCS theory[8] strictly within the constraint that the states of the system correspond to a fixed number of electrons (eigenstates of the number operator). The ground state $|0, 2N\rangle$ at zero temperature with $2N$ electrons has N Cooper pairs $(k\uparrow, -k\downarrow)$. The particular electron quasiparticle state $|K\uparrow\rangle$ is a $(2N+1)$-electron state consisting of N Cooper pairs and a single unpaired electron in the Bloch state $K\uparrow$. Starting from the ground state there are two ways to obtain such a $(2N+1)$-electron configuration. If the pair $(K\uparrow, -K\downarrow)$ is empty in the ground state then an electron may be added directly to the Bloch state $K\uparrow$. The amplitude of the pair being empty is the BCS coherence factor u_K. There is also an amplitude v_K such that the pair $(K\uparrow, -K\downarrow)$ is filled. In this case we must first add a Cooper pair to the ground state and then remove an electron from the state $-K\downarrow$. The $(2N+1)$-electron quasiparticle state is a linear combination of these two $(2N+1)$-electron configurations weighted according to their probability amplitudes

$$|K\uparrow\rangle = (u_K C_{K\uparrow}{}^{+} - v_K C_{-K\downarrow} R^{+})\,|0, 2N\rangle, \qquad (1)$$

(the minus sign results from the ordering of the operators). The operator R^+ transforms the N-pair ground state $|0, 2N\rangle$ into an $(N+1)$-pair ground state $|0, 2(N+1)\rangle$.[9] A set of hole-quasiparticle states can be similarly defined. We note that for a fixed number of electrons in the ground state both terms in Eq. (1) add one electron to the system. In many applications it is not important to distinguish between $|0, 2N\rangle$ and $|0, 2(N+1)\rangle$ so that the operator R^+ may be omitted from Eq. (1). It then appears that $|K\uparrow\rangle$ is a linear combination of $(2N+1)$- and $(2N-1)$-electron states, and hence it is often stated that the quasiparticle is a linear combination of electrons and holes. In our discussion it is important to keep track of the number of Cooper pairs in order to understand the mechanism for the Tomasch effect. In this section we shall need only the electron quasiparticle states. The energy E_K for the electron quasiparticle states is given by

$$E_K = (\epsilon_K{}^2 + \Delta^2)^{1/2}, \qquad (2)$$

where ϵ_K is the electron energy measured from the chemical potential μ, $\epsilon_K = (\hbar^2 K^2/2m) - \mu$, and Δ is the energy gap. The four degenerate states $\pm\mathbf{k}^+$, $\pm\mathbf{k}^-$ corresponding to a fixed energy E (see Fig. 2) have

$$|\mathbf{k}^{\pm}| = k^{\pm} = [2m/\hbar^2(\mu \pm \Omega)]^{1/2} \approx k_F \pm \Omega/\hbar v_F,$$
$$\Omega = (E^2 - \Delta^2)^{1/2},$$
$$\mu = \hbar^2 k_F{}^2/2m,$$

and

$$v_F = \hbar k_F/m. \qquad (3)$$

For energies a few millielectron volts above Δ, $k^{\pm}$ is of the order of $k_F (\sim 10^8\ \mathrm{cm}^{-1})$ while $k^+ - k^- \sim 10^4\ \mathrm{cm}^{-1}$. McMillan and Anderson[5] first pointed out that, while an ordinary (Hartree-Fock-type) potential cannot cause scattering between k^+ and k^-, a nonuniform energy gap function $\Delta(x)$ can lead to quasiparticle scat-

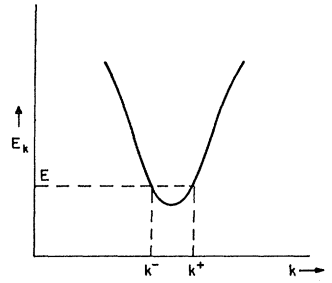

FIG. 2. The quasiparticle energy versus magnitude of the propagation vector. The two propagation constants k^+ and k^-, corresponding to the same energy E, give rise to four degenerate quasiparticle states $\mp\mathbf{k}^{\pm}$.

[8] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957).

[9] J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, Inc., New York, 1964), pp. 24–60.

tering between these states. They showed that scattering, $|k^+\rangle \rightleftarrows |k^-\rangle$, from a perturbation in Δ, would give rise to a quasiparticle interference effect in the local density of states which is periodic in the argument $(k^+-k^-)d$, a distance d in back of the perturbation. A simple one-dimensional picture illustrates this feature. Consider the quasiparticle $|k^+\rangle$ propagating as the plane wave $\exp(ik^+x)$. This wave arrives at the perturbation at $x=d$ with phase k^+d. The reflected wave propagates like $\exp(ik^-x)\ \exp(i\varphi)$, where the phase factor φ must equal $(k^+-k^-)d$ in order that the phase of the incident and reflected waves match at $x=d$. The total wave $\psi(x)$ is then

$$\psi(x)=\exp(ik^+x)+r\exp[i(k^+-k^-)d+ik^-x], \quad (4)$$

where r is the reflection coefficient. The wave density is

$$|\psi(x)|^2=1+r^2+2r\cos(k^+-k^-)(d-x). \quad (5)$$

The oscillatory component will also be manifested in the local density of states (and hence in the tunneling current).

Ordinarily a reflected wave has a k vector and velocity equal and opposite that of the incident wave; scattering from $\mathbf{k}^+\rightarrow-\mathbf{k}^+$, for example. We note from Fig. 2 that the group velocity, $v_g=(1/\hbar)(dE_K/dK)$, at k^- is equal and opposite that at k^+. A wave packet composed of quasiparticle states with wave vectors in a small range about k^+ moves with a velocity equal to the group velocity at k^+. If we were to associate with this packet a momentum mv_g then this wave packet would experience simple reflection into the wave packet (composed of quasiparticles centered about k^-) with velocity $-v_g$ and momentum $-mv_g$. However, from the point of view of the individual electrons involved, this process is not simple reflection. One of the peculiar things about the reflected quasiparticle is that it is propagating in the same direction as the incident quasiparticle and the magnitude of its propagation vector is not the same as that of the incident wave. We may conclude from this that the process involves more than a single electron.

In order to understand the origin of this reflected wave consider the Hamiltonian

$$H=\sum_{k,\sigma}\epsilon_k C_{k\sigma}{}^\dagger C_{k\sigma} + \sum_{1,2,3,4} V(1,2;3,4)C_{k_1\sigma}{}^+C_{k_2-\sigma}{}^\dagger C_{k_3-\sigma}C_{k_4\sigma}, \quad (6)$$

where $C_{k\sigma}{}^\dagger$ creates an electron in the Bloch state k with spin σ and V is the part of the electron-electron potential responsible for superconductivity. The BCS reduced Hamiltonian[9] is obtained by keeping only interaction terms which scatter Cooper pairs (with zero net momentum) into other Cooper pairs:

$$H_{\rm red}=\sum_{k\sigma}\epsilon_k C_{k\sigma}{}^\dagger C_{k\sigma}+\sum_{k,k',\sigma} V_{kk'}C_{-k'-\sigma}{}^\dagger C_{k'\sigma}{}^\dagger C_{k\sigma}C_{-k-\sigma}. \quad (7)$$

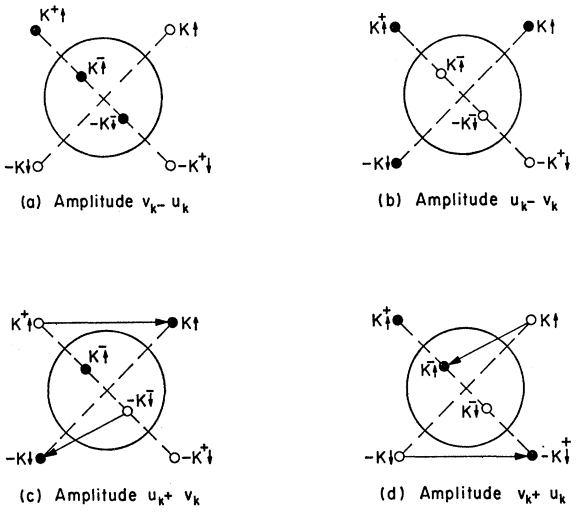

FIG. 3. A schematic representation of the Bloch states which are involved in the scattering from $|k^+\uparrow\rangle$ into $|k^-\uparrow\rangle$. The circles represent the Fermi surface. The two configurations entering into the wave function for the quasiparticle state $|k^+\uparrow\rangle$ are shown in (a) and (b). In addition the occupation of an arbitrary pair state $(k\uparrow, -k\downarrow)$ is specified. The probability amplitude for each configuration is given. A spacially varying electron-electron interaction transforms (a) into (c) and (b) into (d) as indicated by the arrows. The configurations (b) and (d) are those entering into the quasiparticle state $|k^-\uparrow\rangle$.

This form of the Hamiltonian implies not only the pairing principle but also the assumption of translational invariance of the electron-electron potential. A system in which the energy gap varies in space necessarily lacks this invariance. As a result terms involving nonzero momentum-pair operators will be important. Most important to our discussion are terms of the form

$$\sum_k V_{kk^+k^-}\{C_{k\uparrow}{}^\dagger C_{-k\downarrow}{}^\dagger C_{-k^-\downarrow}C_{k^+\uparrow} + C_{k^-\uparrow}{}^\dagger C_{-k^+\downarrow}C_{-k\downarrow}C_{k\uparrow}\}+\text{H.c.}, \quad (8)$$

which are responsible for transitions of the system from the electron quasiparticle state $|k^+\uparrow\rangle$ to $|k^-\uparrow\rangle$. Consider a superconductor in the state $|k^+\uparrow\rangle$. The states pertinent to the discussion are illustrated in Fig. 3. The configurations entering the wave function for the electron quasiparticle state $|k^+\uparrow\rangle$ are shown in 3(a) and 3(b). The first term of Eq. (8) transforms Fig. 3(a) into 3(c) by condensing $k^+\uparrow$ and $-k^-\downarrow$ into the Cooper-pair state $(k\uparrow, -k\downarrow)$, leaving $k^-\uparrow$ unpaired. To the approximation that Δ does not depend upon k, the loss of the binding energy associated with the destruction of $(k^-\uparrow, -k^-\downarrow)$ is compensated by the gain due to the new pair $(k\uparrow, -k\downarrow)$. The second term of Eq. (8) transforms configuration 3(b) into 3(d) by scattering the Cooper pair $(k\uparrow, -k\downarrow)$ into the Bloch states $-k^+\downarrow$ and $k^-\uparrow$. This exchanges the pair $(k\uparrow, -k\downarrow)$ for the pair $(k^+\uparrow, -k^+\downarrow)$ and leaves $k^-\uparrow$ unpaired. The configurations 3(c) and 3(d) are those making up the quasiparticle state

$|k^-\uparrow\rangle$. Since $|k^+\uparrow\rangle$ and $|k^-\uparrow\rangle$ are degenerate and the number of pairs has not changed, the processes conserve energy. The over-all probability of this quasiparticle transition is

$$\sum_k V_{kk^+k^-}u_kv_k\{u_k^+v_k^-+v_k^+u_k^-\}. \tag{9}$$

The fact that $\sum_{k'}V_{kk'u_{k'}v_{k'}}\equiv\Delta_k\approx\Delta$ in BCS theory[9] suggests that

$$\sum_k V_{kk^+k^-}u_kv_k \tag{10}$$

is roughly a measure of the average variation in the energy gap of the system. The coherence factor $u_k^+v_k^-+v_k^+u_k^-=1$ since

$$\begin{pmatrix}u_k\\v_k\end{pmatrix}=\left[\frac{1}{2}\left(1\pm\frac{\epsilon_k}{E_k}\right)\right]^{1/2},$$

$$\epsilon_k^-=-\epsilon_k^+,$$

$$u_k^-=v_k^+,$$

and

$$v_k^-=u_k^+. \tag{11}$$

A similar analysis for scattering from $|k^-\uparrow\rangle$ into $|k^+\uparrow\rangle$ obtains for the Hermitian conjugate, H.c., of Eq. (8). The above result should be contrasted with the process in which the electron in the Bloch state $k^+\uparrow$ above the Fermi surface is scattered into the Bloch state $k^-\uparrow$ below the Fermi surface. This sort of process has the coherence factor $u_k^+u_k^--v_k^+v_k^-=0$. The scattering $|k^+\uparrow\rangle\rightarrow|k^-\uparrow\rangle$ does *not* involve scattering the electron in the Bloch state $k^+\uparrow$ into the Bloch state $k^-\uparrow$. The process results from the electron $k^+\uparrow$ interacting with and being condensed into the sea of Cooper pairs, leaving behind the unpaired electron $k^-\uparrow$.

In the following sections we use the Gorkov theory to calculate the effects of the processes discussed here. In the application of the Gorkov theory, the role of the Cooper pairs is not explicit and consequently the mechanism for the Tomasch effect is obscured. It is for this reason that we have separated the discussion of the mechanism from the calculation.

III. GREEN'S FUNCTION

In the typical Tomasch experiment the second film, metal B, is overlayed with a dissimilar material. In this section we calculate the average tunneling density of states for a two-region film. It is convenient (see Fig. 1) to choose $x_b=-d$, $x_1=0$, and $x_2=d'$. We refer to the region $-d<x<0$ as region 1 and to the region $0<x<d'$ as region 2. The energy gap is Δ_1 in region 1 and Δ_2 in region 2. Δ_1 and/or Δ_2 may be zero so that we include the cases in which one or both of the regions are normal.[10] The case in which region 2 is absent is equivalent to $\Delta_1=\Delta_2$. The Green's functions for the system satisfy Gorkov's equations[11]

$$\{\omega-(p_x^2/2m)\pm\tilde{\mu}\}G_\omega(k_\perp;x,x')-\Delta F_\omega^+(k_\perp;x,x')=\delta(x-x'),$$

$$\{\omega+p_x^2/2m-\tilde{\mu}\}F_\omega^+(k_\perp;x,x')-\Delta G_\omega(k_\perp;x,x')=0, \tag{12}$$

and

$$p_x^2=-\hbar^2\partial^2/\partial x^2.$$

Here $G_\omega(k_\perp;x,x')$ and $F_\omega^+(k_\perp;x,x')$ are defined as the Fourier transform with respect to $k_\perp$ and ω of the correlation functions (see the Appendix)

$$\begin{aligned}G(\mathbf{r},\mathbf{r}';t)&=-i\langle 0\,|\,\psi(\mathbf{r},t)\psi^+(\mathbf{r}',0)\,|\,0\rangle, && t>0\\&=i\langle 0\,|\,\psi^+(\mathbf{r}',0)\psi(\mathbf{r},t)\,|\,0\rangle, && t\leq 0\end{aligned}$$

and

$$\begin{aligned}F^+(\mathbf{r},\mathbf{r}';t)&=-i\langle 0\,|\,\psi^+(\mathbf{r},t)\psi^+(\mathbf{r}',0)R\,|\,0\rangle, && t>0\\&=i\langle 0\,|\,\psi^+(\mathbf{r}',0)\psi^+(\mathbf{r},t)R\,|\,0\rangle, && t\leq 0\end{aligned} \tag{13}$$

where the operator R transforms the N-pair ground state to an $(N-1)$-pair ground state. In Eq. (12) ω is an energy measured from the chemical potential μ and

$$\tilde{\mu}=[\mu-(\hbar^2/2m)k_\perp^2]^{1/2},$$

$$|\mathbf{k}_\perp|=(k_y^2+k_z^2)^{1/2}. \tag{14}$$

In order to calculate the tunneling density of states we must solve Eq. (12) for the composite system and evaluate $G(k_\perp;x,x')$ in region 1. It is necessary to specify the boundary conditions at the planes $x=-d$, $x=0$, and $x=d'$. For simplicity we shall assume that the chemical potential is uniform over the composite system and that the electron effective mass is the same for both regions. It then follows that the functions F and G and their normal derivatives ($\partial G/\partial x$ and $\partial F/\partial x$) are continuous across the $x=0$ plane. We consider two possible boundary conditions at $x=-d$ and $x=d'$: (1) F and G vanish, or (2) $\partial F/\partial x$ and $\partial G/\partial x$ vanish. The solutions in region 1 [with the $\mp$ corresponding to boundary conditions (1) and (2), respectively] are of the form

$$G_1(k_\perp;x,x')=G_1^0(k_\perp;x,x')\mp G_1^0(k_\perp;x,y_1)+A(x')(\omega+\Omega_1)[\exp(-ik_1^+x)\mp\exp(ik_1^+(x+2d))]+B(x')\Delta_1[\exp(ik_1^-x)\mp\exp(-ik_1^-(x+2d))],$$

$$F_1^+(k_\perp;x,x')=F_1^{0+}(k_\perp;x,x')\mp F_1^{0+}(k_\perp;x,y_1)+A(x')\Delta_1[\exp(-ik_1^+x)\mp\exp(ik_1^+(x+2d))]+B(x')(\omega+\Omega_1)[\exp(ik_1^-x)\mp\exp(-ik^-(x+2d))], \tag{15}$$

[10] The self-consistent requirement on the spatial behavior of the energy gap function is neglected.

[11] L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958) [English transl.: Soviet Phys.—JETP **7**, 505 (1958)].

where $y_1=-x'-2d;$

and in region 2

$$G_2(k_\perp; x, x')=G_2^0(k_\perp; x, x')\mp G_2^0(k_\perp; x, y_2)+C(x')(\omega+\Omega_2)[\exp(ik_2^+x)\mp\exp(-ik_2^+(x-2d'))]$$
$$+D(x')\Delta_2[\exp(-ik_2^-x)\mp\exp(ik_2^-(x-2d'))],$$
$$F_2^+(k_\perp; x, x')=F_2^{0+}(k_\perp; x, x')\mp F_2^{0+}(k_\perp; x, y_2)+C(x')\Delta_2[\exp(ik_2^+x)\mp\exp(-ik_2^+(x-2d'))]$$
$$+D(x')(\omega+\Omega_2)[\exp(-ik_2^-x)\mp\exp(-ik_2^-(x-2d'))], \quad (16)$$

where $y_2=-x'+2d'.$

The functions G^0 and F^{0+} are solutions for the infinite uniform superconductor[6]:

$$G_j(k_\perp; x, z)=-\frac{im}{2\hbar^2\Omega_j}\left(\frac{\omega+\Omega_j}{k_j^+}\exp(ik_j^+\,|\,x-z\,|)+\frac{\omega-\Omega_j}{k_j^-}\exp(-ik_j^-\,|\,x-z\,|)\right),$$

$$F_j^{0+}(k_\perp; x, x')=-\frac{im}{2\hbar^2\Omega_j}\left(\frac{\exp(ik_j^+\,|\,x-z\,|)}{k_j^+}+\frac{\exp(-ik_j^-\,|\,x-z\,|)}{k_j^-}\right), \quad (17)$$

where

$$\Omega_j=(\omega^2-\Delta_j^2)^{1/2}, \qquad \omega>\Delta_j$$
$$=iW_j=i(\Delta_j^2-\omega^2)^{1/2}, \qquad \omega\leq\Delta_j$$

and

$$k_j^\pm=\{(2m/\hbar^2)(\tilde{\mu}\pm\Omega_j)\}^{1/2}, \qquad j=1, 2.$$

In Eqs. (15)–(17) the subscript ω has been omitted from F^+ and G. The coefficients $A(x')$, $B(x')$, $C(x')$, and $D(x')$ are determined by the requirement of continuity at $x=0$. It is easily verified that the Green's functions defined above automatically satisfy the boundary conditions at $x=-d$ and $x=d'$. The $(-)$ sign corresponds to the solutions for which F and G vanish and the $(+)$ sign corresponds to the solutions having $\partial F/\partial x$ and $\partial G/\partial x$ vanish.

In the case of the strong-coupling superconductor, the expressions for F and G should be modified by replacing ω with $Z\omega$ and Δ with φ, where $Z=Z(\omega)$ is the energy renormalization factor and $\varphi=\varphi(\omega)$ is the complex energy gap function according to the Nambu theory.[12] These factors are significant in the case of Pb, as has been shown by McMillian and Anderson.[5]

The function $\rho_B^{(+)}(k_\perp, x)$ (for $-d\leq x\leq 0$), the local density of states, gives the density of available states at x with energy ω and transverse momentum $k_\perp$ and is given by

$$\rho_B^{(+)}(k_\perp; x)=-(1/\pi)\,\mathrm{Im}G_1(k_\perp; x, x')\,|_{x=x'}, \quad (18)$$

where Im indicates the imaginary part. The unperturbed local density of states,

$$\rho_0^{(+)}(k_\perp; x)=-(1/\pi)$$
$$\times\mathrm{Im}\{G_1^0(k_\perp; x, x')\mp G_1^0(k_\perp; x, y_1)\}\,|_{x=x'}, \quad (19)$$

will be the total density of states in the limiting case that $\Delta_1=\Delta_2$.

The finite lifetime of the quasiparticles due to bulk impurity scattering may be accounted for approximately by adding an imaginary part to the propagation constants

$$k_j^\pm\longrightarrow k^\pm\pm i/l_j, \quad (20)$$

where l_j is the average electron mean free path in region j. The case of scattering by magnetic impurities has been discussed by Maki and Griffin[13] and will not be treated here. The strong-coupling superconductors will have an additional energy-dependent imaginary component for $k_j^\pm$ due to the complex nature of $\varphi(\omega)$.

IV. SHORT MEAN PATH IN REGION 2

In this paper, we discuss only the case in which l_2 is sufficiently small that

$$\exp(-2d'/l_2)\ll 1. \quad (21)$$

In the solutions of Eq. (16) the terms proportional to $\exp(-2d'/l_2)$ in region 2 at $x=0$ arise from waves transmitted from region 1 which have been reflected from the $x=d'$ surface and which return to $x=0$. If the mean free path in region 2 is very short, these waves are "damped out" before they can return to region 1. As far as the solutions in region 1 are concerned, $l_2\longrightarrow 0$ is equivalent to $d'\longrightarrow\infty$.

The requirement of continuity at $x=0$ leads to the solutions

$$A(x')=A\{\alpha\beta\Delta_1\exp(-ik_1^-\,|\,x'\,|)$$
$$-\alpha^2R_1\exp(-2ik_1^-d)\exp(ik_1^+\,|\,x'\,|)$$
$$\mp\beta^2R_1\exp(ik_1^+\,|\,x'+2d\,|)$$
$$\pm\Delta_1\alpha\beta\exp(-2ik_1^-d)\exp(-ik_1^-\,|\,x'+2d\,|)\},$$

[12] Y. Nambu, Phys. Rev. **117**, 648 (1960).

[13] K. Maki and A. Griffin, Phys. Rev. **150**, 356 (1966).

and

$$B(x')=A\{\alpha\beta R_1\exp(ik_1^+\,|\,x'\,|)$$
$$-\alpha^2\Delta_1\exp(2ik_1^+d)\exp(-ik_1^-\,|\,x'\,|)$$
$$\mp\beta^2\Delta_1\exp(-ik_1^-\,|\,x'+2d\,|)$$
$$\pm\alpha\beta R_1\exp(2ik_1^+d)\exp(ik_1^+\,|\,x'+2d\,|)\},\quad(22)$$

where

$$A=[im/(\hbar^2\tilde{k}_F\Omega_1R_1\beta^2)][1-(\alpha/\beta)^2\exp(2i(k^+-k^-)d)]^{-1},$$
$$\alpha=R_2\Delta_1-R_1\Delta_2,$$
$$\beta=R_1R_2-\Delta_1\Delta_2,$$
$$R_j=\omega+\Omega_j.\quad(23)$$

In Eqs. (22) and (23) we have made the approximation that $k_j^{\pm}\approx\tilde{k}_F$ in amplitude factors but *not in phase factors*. This is an excellent approximation for $k_\perp\ll k_F$, a condition well satisfied for electrons which are able to tunnel from metal A to metal B. Also in Eqs. (22) and (23), for compactness, we have not introduced the mean-free-path parameters. The replacement indicated by Eq. (20) should be made when using Eqs. (22) and (23).

If the density of states is averaged over the variation in the film thickness as well as the variation in the insulator surface, one obtains

$$\langle\rho_B^{(+)}(k_\perp;-d)\rangle=\langle\rho_0^{(+)}\rangle$$
$$-(2/\pi)\,\mathrm{Im}[AR_1\{\alpha\beta\Delta_1\exp(-2d/l_1)\exp(i(k^+-k^-)d)$$
$$-\alpha^2\omega\exp(-4d/l_1)\exp(2i(k^+-k^-)d)\}],\quad(24)$$

where

$$\langle\rho_0^{(+)}\rangle\simeq\mathrm{Re}(m\omega/\pi\hbar^2\tilde{k}_F\Omega_1).$$

In obtaining Eq. (24) we have assumed that r, the distance over which we average $\rho^{(+)}$, is large compared to k_F^{-1} but small compared to $(k^+-k^-)^{-1}$, so that

$$r^{-1}\int_0^r dx\exp[\pm ik_1^{\pm}x]\approx 0,$$
$$r^{-1}\int_0^r dx\exp[\pm i(k_1^+-k_1^-)x]\approx 1.\quad(25)$$

This condition is well satisfied for ω corresponding to energies in the millivolt range if 10^{-8} cm$\ll r\ll 10^{-4}$ cm. It is usually supposed that variations in film thickness are on the order of 10^{-6} cm. It is important to note that $\langle\rho_B^{(+)}(k_\perp,-d)\rangle$ does not depend upon which boundary condition was used at the free surfaces.[14]

When $|(\alpha/\beta)^2\exp(-4d/l_1)|<1$ we can write $\langle\rho_B^{+}(k_\perp,-d)\rangle$ in a form which can be interpreted in terms of multiple scattering.[15] The coefficient A can be expanded in a power series with the result that

$$\langle\rho_B^{(+)}(k_\perp;-d)\rangle$$
$$=\mathrm{Re}\left\{\langle\rho_0^{(+)}\rangle\left[1-\sum_{n=1}^{\infty}\left(\frac{2\Delta_1}{\omega}\psi^{2n-1}-2\psi^{2n}\right)\right]\right\},\quad(26)$$

where

$$\psi=(\alpha/\beta)\exp[i(k^+-k^-)d-2d/l_1].$$

We may interpret Eqs. (26) in terms of a multiple scattering picture. A wave traveling from $x=-d$ and scattering from the discontinuity in the energy gap at $x=0$ has a reflection coefficient (α/β) and the reflected wave returns to $x=-d$ with a phase $(k^+-k^-)d$. Since a distance of $2d$ is traveled in a medium with mean free path l_1 the amplitude is reduced by $\exp(-2d/l_1)$. We must recognize, however, that if the incident wave represented a single-particle Green's function (G wave) then the reflected wave represents

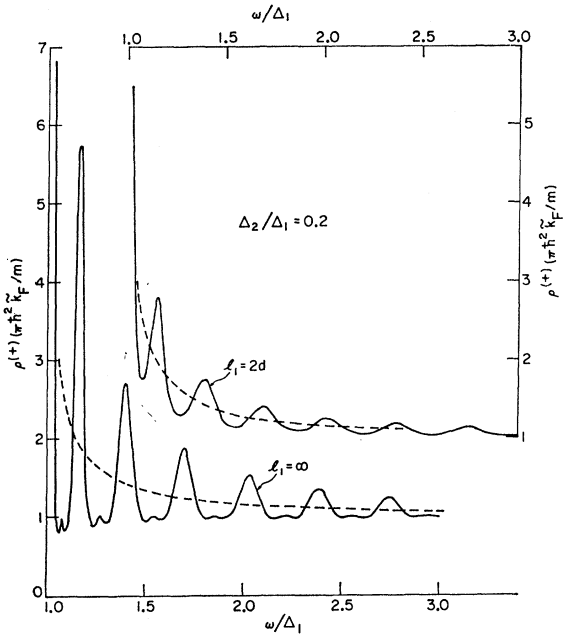

FIG. 4. The averaged density of quasiparitcle states near $x=-d$ with transverse momentum $\hbar\tilde{k}_\perp[\tilde{k}_F=(k_F^2-k_\perp^2)^{1/2}]$ for a composite film having $\Delta_2/\Delta_1=0.2$. The lower curve corresponds to an infinite mean free path in region 1 ($l_1=\infty$) while the upper curve is for $l_1=2d$, where d is the thickness of region 1. The quantity $2m\Omega_1d/\hbar^2k_F$ is taken to be $16\Omega_1/\Delta_1$. For $k_\perp\ll k_F$ the parameters here correspond approximately to a 10-μ-thick In film with an Al overlay. The dashed lines indicate the unperturbed density of states.

[14] This result holds independent of the magnitude of l_2.

[15] It is important to recognize that there does not exist a one-to-one correspondence between the nth term of Eq. (26) and the nth term of a power-series expansion in the perturbation $(\Delta_1-\Delta_2)$. In order to obtain the correct energy dependence for the coefficient of $\exp[in(k^+-k^-)d]$ it is necessary to sum *all* terms in the perturbation expansion which oscillate as $\exp[in(k^+-k^-)d]$ regardless of the power of $(\Delta_1-\Delta_2)$ which multiplies the term.

a pair correlation function (F^+ wave). Conversely, if the incident wave is an F^+ wave then the reflected wave is a G wave. Reflection from a free surface at $x=-d$, on the other hand, reflects $G\rightarrow G$ and $F^+\rightarrow F^+$. We can understand physically why scattering from an energy gap involves the $G\rightarrow F^+$ conversion if we consider the mechanism involved in the scattering process. In Sec. II it was shown that in the scattering process the incident unpaired electron is condensed into a Cooper pair. The unpaired electron is described by G but after it is condensed it must be described by F^+. The other mechanism involved in scattering from $k^+\rightleftarrows k^-$ was the destruction of one incident Cooper pair in such a way that one of the electrons of this pair becomes a quasiparticle. This incident pair is described by an F^+ wave but after scattering the remaining unpaired electron is described by a G wave.[16] Now consider multiple scattering: A G wave, for example, can scatter into F^+ at $x=0$, travel to $x=-d$, reflect as an F^+ wave, travel back to $x=0$, and be converted again into a G wave.

The density of states $\rho^{(+)}$ depends only upon G, therefore (1) a G wave must be reflected from the gap perturbation an even number of times in order to contribute to $\rho^{(+)}$, and (2) an F^+ wave must be reflected an odd number of times by a gap perturbation in order to contribute to $\rho^{(+)}$.[17]

In Eq. (26) the series of waves ψ^{2n} represents the contribution to $\rho^{(+)}$ from the incident G^0 wave, while the terms ψ^{2n-1} represent the contributions from the incident F^{0+} wave.

A. $\Delta_1>\Delta_2$

The behavior of $\rho^{(+)}$ is quite different, depending upon whether $\Delta_1>\Delta_2$ or $\Delta_2>\Delta_1$. A case for which $\Delta_1>\Delta_2$ is illustrated in Fig. 4, where $\langle\rho_B{}^{(+)}(k_\perp,-d)\rangle$ is plotted for two values of the mean free path l_1. The parameters are $(k^+-k^-)d\approx(2m\Omega_1/\hbar^2\tilde{k}_F)d=16\Omega_1/\Delta_1$ and $\Delta_2/\Delta_1=0.2$, corresponding approximately to an In-Al film with $d\sim10\times10^{-4}$ cm.

The weak harmonic series associated with the $n=2$ term is not resolved but causes considerable distortion of the oscillations. Oscillations in the conductance of Al-insulator-In-Al Giaever tunnel junctions of large amplitude have been reported by Tomasch.[3,4] The amplitude of the structure corresponding to $l_1=2d$ in Fig. 4 is comparable to the amplitude of the conductance oscillations observed by Tomasch.

B. $\Delta_1<\Delta_2$

In the case that region 1 is a normal metal ($\Delta_1=0$) there is no F^{0+} and hence only the even-numbered harmonics of the Tomasch oscillations occur in $\rho^{(+)}$. Rowell and McMillan[18] first reported the observation of effects associated with the $n=2$ term for an Ag-Pb film. This effect has also been observed by Tomasch for films with $\Delta_1\neq0$, where it appears as a weak second series of oscillations.[3,4]

The case for which $\Delta_1<\Delta_2$ is anomalous in the range $\Delta_1<\omega<\Delta_2$ since quasiparticles in region 1 cannot propagate in region 2. This means that the reflection coefficient (α/β) must have modulus unity. That this is in fact the case is easily verified making use of the fact that $R_2R_2{}^*=\Delta_2{}^2$ for $\omega<\Delta_2$.

In the case that the mean free path in region 1 is very long ($l_1\rightarrow\infty$) we can expect the formation of *bound eigenstates* for $\omega<\Delta_2$. The energy of these bound eigenstates is determined by the poles of A. If we write $(\alpha/\beta)=e^{i\theta}$ then we find

$$\tan\theta=-W_2\Omega_1/(\Delta_1\Delta_2-\omega^2), \tag{27}$$

and the condition for the vanishing of the denominator of A is

$$\tan(k^+-k^-)d=-\tan\theta. \tag{28}$$

Equation (28) shows that the bound states are characterized by the quantization of the difference of the momenta associated with the degenerate quasiparticle states. This kind of quantization is characteristic of superconducting systems with boundary conditions imposed on F and $\partial F/\partial x$.[7] If we think of the quasiparticle eigenstate as an admixture of the k^+ and k^- waves it can be described as a beat modulated wave. The rapidly oscillating wave with wavelength of the order of $\tilde{k}_F{}^{-1}$ is modulated by an envelope wave with wavelength of the order of $(k^+-k^-)^{-1}$. These eigenstates are characterized by quantization of the envelope momentum. Wolfram and Lehman[7] have shown that a theory of the Tomasch oscillation can be constructed on the basis of this type of quantization. For infinite l_1

$$\langle\rho_B{}^{(+)}\rangle\propto\sum_j\delta(\omega-\omega_j)\qquad(\omega<\Delta_2), \tag{29}$$

where the ω_j satisfy Eq. (28). We note also that in this case there exists *no power-series expansion of the factor A* for $\omega<\Delta_2$ and thus the usual expansion of the Green's function into a Born series is not valid.

For finite l_1 the bound states become scattering resonances and the δ function are smeared out. This smearing process can be followed in Fig. 5, where we

[16] The conversion of G and F^+ by this mechanism is basic to the proximity effect whereby an energy gap is induced in a normal metal in contact with a superconductor.

[17] This is most easily seen in the 2-dimensional Nambu representation, where the Green's-function matrix $\hat{G}$ has the single-particle Green's functions on the diagonals and the pair-correlation functions on the off diagonals. In this representation the gap perturbation is proportional to the off-diagonal matrix

$$\tau_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

τ_1 on $\hat{G}$ carries G into F^+ and F^+ into G.

[18] J. M. Rowell and W. L. McMillan, Phys. Rev. Letters **16**, 453 (1966).

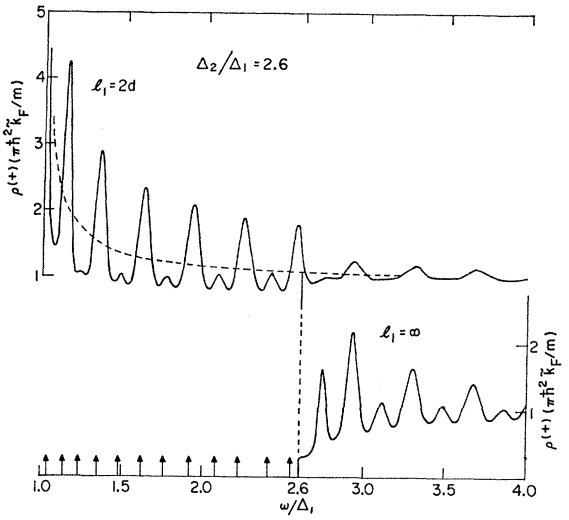

FIG. 5. The averaged density of quasiparticle states near $x=-d$ with transverse momentum $\hbar k_\perp[\tilde{k}_F=(k_F^2-k_\perp^2)^{1/2}]$ for a composite film having $\Delta_2/\Delta_1=2.6$. The lower curve corresponds to an infinite mean free path in region 1 ($l_1=\infty$) while the upper curve is for $l_1=2d$, where d is the thickness of region 1. The quantity $2m\Omega_1 d/\hbar^2\tilde{k}_F$ is taken to be $16\Omega_1/\Delta_1$. For $k_\perp \ll k_F$ the parameters here correspond approximately to a 10-μ-thick In film with a Pb overlay. The arrows indicate the energies of bound eigenstates. Note that the upper curve has a shifted vertical scale. The dashed line in the upper curve indicates the unperturbed density of states.

have plotted $\langle\rho_B^{(+)}\rangle$ for the parameters $2m\Omega_1/\hbar^2\tilde{k}_F=16\Omega_1/\Delta_1$ and $\Delta_2/\Delta_1=2.6$, corresponding approximately to the case of an In-Pb film with $d=10\times10^{-4}$ cm. The locations of the bound eigenstates (corresponding to $l_1=\infty$) are indicated by the short vertical arrows at the base of Fig. 5. A δ function in the density of states is to be associated with each of these bound states. The value $\omega=\Delta_2/\Delta_1=2.6$ is the threshold energy above which transmission into region 2 is possible. In the case of finite l_1, the δ-function peaks are smeared out into sharp resonance peaks. The harmonic is clearly resolved here even in the presence of damping. Two features are to be noted. First, we see that the amplitude of the harmonic grows with increasing energy relative to the fundamental; and second, we note that the amplitude of both series are sharply reduced above threshold. Tomasch has noted this sort of cutoff effect above threshold in Al-insulator-In-Pb films.[4] When the mean free path is long the scattering expansion [Eq. (26)] converges slowly and it is more reasonable to describe the system by introducing a small amount of damping into the bound-state description. In fact, one may regard all Tomasch oscillations as scattering resonances associated with the formation of quantized envelope states. If l_1 is small then the expansion of A converges rapidly and one need only keep the first few terms of Eq. (26).

V. SUMMARY AND CONCLUSIONS

In the preceding sections we have discussed the mechanisms which give rise to the Tomasch oscillations. It was shown that a spatially varying electron-electron interaction gives rise to quasiparticle-pair interactions which cause transitions between the degenerate quasiparticle states. The interference between the components causes large scale structure to appear in the density of states. Gorkov's equations were employed to make explicit calculations for the two-region composite superconductor. Only the case in which the overlay had a very short mean free path was treated in detail. This situation can probably be obtained for an Al or Ag overlay. On the other hand, there appears to be evidence that In and Pb films anneal at room temperature[19] so that the electron mean free path is long in region 1. In the case of the In overlayed with Pb the situation is complicated. The mean free path of the quasiparticle in the Pb may be short because Pb is a strong-coupling superconductor and the intrinsic quasiparticle lifetime is short. Experimental difficulties occur with the In-Pb system which appear to be associated with diffusion. If diffusion occurs the mean free path in region 1 will be severely reduced and correspondingly the amplitude of the oscillatory structure will be reduced.

The results of a situation in which l_2 is large can be anticipated. (1) Additional oscillatory factors involving $\exp[i(k_2^+-k_2^-)d']$ will appear. If d' is small compared to d such terms will be nearly constant over the low-energy range for which the Tomasch structure is large. (2) Bound states will appear for all ratios of Δ_2/Δ_1 and at all energies when both l_1 and l_2 are infinite. The low-energy states will be characterized by approximate quantization of $k_1^+-k_1^-$ and the high-energy states by the usual quantization of either k^+ or k^-. For finite l_1 and l_2 the high-energy discrete spectrum will be washed out, but the low-energy spectrum will show resonance structure.

In order to make an accurate comparison between the results of this theory and experiment it is necessary to calculate the differential conductance. This involves a numerical calculation of the derivative of the convolution of the density functions of metals A and B (this problem is briefly discussed in the Appendix). Numerical calculation of the conductance is presently being carried out and the results will be reported in a subsequent publication.

ACKNOWLEDGMENTS

The author wishes to express his appreciation to W. F. Hall and G. W. Lehman for helpful suggestions and to W. J. Tomasch for many valuable discussions.

APPENDIX: TUNNELING DENSITY OF STATES

In this section we discuss a very simple model for the tunneling of electrons between two thin films. The tunnel junction shown in Fig. 1 consists of two metallic

[19] W. J. Tomasch (private communication).

films separated by a very thin insulating barrier. The films are infinite in the y and z directions and finite in the x direction. At zero temperature a net current of electrons flows from A to B because of the applied bias V. Electrons in A making collisions with the insulating barrier at x_a have a small probability T of appearing at x_b in metal B. This tunneling process can be approximated by assuming that the systems A and B are connected by a coupling term[20] in the Hamiltonian of the form

$$H_T=\sum_{k_\perp}T(k_\perp)\psi(k_\perp, x_a)\psi^\dagger(k_\perp, x_b)+\text{c.c.}, \qquad \text{(A1)}$$

where $\psi^\dagger(k_\perp, x)$ creates an electron at x with transverse momentum $\hbar\mathbf{k}_\perp(\mathbf{k}_\perp=(k_y, k_z))$.

In Eq. (A1) it is assumed that $k_\perp$ is a constant of the motion of the electrons and is conserved in the tunneling process. At zero temperature electrons tunnels only from A→B. The systems A and B are in their ground states $|\,0_A\rangle$ and $|\,0_B\rangle$ initially. After an electron tunnels system A is in an excited state $|\,n\rangle$ corresponding to the loss of one electron while system B is in the excited state $|\,m\rangle$ corresponding to the addition of an electron. According to the "Golden Rule," the rate at which the electron transition occurs is

$$W_{AB}=(2\pi/\hbar)\sum_{mn}|\,\langle m\,|\,\langle n\,|\,H_T\,|\,0_A\rangle\,|\,0_B\rangle\,|^2\delta(E_m+E_n-V)$$
$$=(2\pi/\hbar)\sum_{k_\perp k_\perp'}T(k_\perp)T(k_\perp')^*\sum_{mn}\langle 0_A\,|\,\psi^+(k_\perp, x_a)\,|\,n\rangle\langle n\,|\,\psi(k_\perp', x_a)\,|\,0_A\rangle$$
$$\times\langle 0_B\,|\,\psi(k_\perp, x_b)\,|\,m\rangle\langle m\,|\,\psi^+(k_\perp', x_b)\,|\,0_B\rangle\delta(E_m+E_n-V). \qquad \text{(A2)}$$

We note that $\langle 0_A\,|\,\psi^\dagger(k_\perp, x_a)\,|\,n\rangle$ will vanish unless the state $|\,n\rangle$ has transverse momentum $k_\perp$ and $\langle n\,|\,\psi(k_\perp, x_a)\,|\,0_A\rangle$ will vanish unless $\langle n\,|$ has transverse momentum $k_\perp'$ (since we have assumed the transverse momentum is a constant of the motion). Therefore, the product vanishes unless $k_\perp=k_\perp'$. Then, writing

$$\delta(E_m+E_n-V)=\int_{-\infty}^{\infty}d\omega\delta(\omega-E_m)\delta(\omega-V+E_m), \qquad \text{(A3)}$$

we have

$$W_{AB}=\frac{2\pi}{\hbar}\int_{-\infty}^{\infty}d\omega\{\sum_{k_\perp}|\,T(k_\perp)\,|^2\sum_n|\,\langle 0_A\,|\,\psi^+(k_\perp, x_a)\,|\,n\rangle\,|^2\delta(\omega-V+E_m)\sum_m|\,\langle 0_B\,|\,\psi(k_\perp, x_a)\,|\,m\rangle\,|^2\delta(\omega-E_n)\}. \qquad \text{(A4)}$$

The two sums (over n and m) may be identified in terms of the Green's functions for A and B. The Green's function is defined as

$$G(\mathbf{r}, \mathbf{r}', t)=-i\langle 0\,|\,\psi(\mathbf{r}, t)\psi^\dagger(\mathbf{r}', 0)\,|\,0\rangle, \qquad t>0$$
$$=i\langle 0\,|\,\psi^\dagger(\mathbf{r}', 0)\psi(\mathbf{r}, t)\,|\,0\rangle, \qquad t\leq 0 \qquad \text{(A5)}$$

where $\langle 0\,|$ is the ground state. Since

$$\psi(\mathbf{r}, t)=\exp(iHt/\hbar)\psi(\mathbf{r}, 0)\exp(-iHt/\hbar), \qquad \text{(A6)}$$

Eq. (A5) may be written as

$$G(r, r', t)=-i\sum_m\langle 0\,|\,\psi(r0)\,|\,m\rangle\langle m\,|\,\psi^\dagger(r'0)\,|\,0\rangle\exp(-iE_mt/\hbar), \qquad t>0$$
$$=i\sum_n\langle 0\,|\,\psi^\dagger(r'0)\,|\,n\rangle\langle n\,|\,\psi(r, 0)\,|\,0\rangle\exp(iE_nt/\hbar) \qquad t\leq 0. \qquad \text{(A7)}$$

The transformed Green's function is

$$G_\omega(k_\perp, x, x')\,|_{x=x'}=\left\{\int_{-\infty}^{\infty}d\omega\exp(i\omega t/\hbar)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d(y-y')d(z-z')\exp[ik_\perp((y-y'), (z-z'))]G(r, r't)\right\}|_{x=x'}$$
$$=\hbar\sum_m\frac{|\,\langle 0\,|\,\psi(k_\perp, x)\,|\,m\rangle\,|^2}{\omega-E_m+i\delta}+\hbar\sum_n\frac{|\,\langle 0\,|\,\psi^\dagger(k_\perp, x)\,|\,n\rangle\,|^2}{\omega+E_n+i\delta} \qquad \text{(A8)}$$

[20] This tunneling Hamiltonian is essentially equivalent to the usual momentum space operator (see, for example, Ref. 9). $\Sigma_{k_ak_b}T_{k_ak_b}C_{k_a}C_{k_b}^+$+c.c. if T depends only upon $k_\perp$. The local property of the tunneling process has been emphasized by McMillan *et al.* (see Refs. 5 and 18).

and

$$\mathrm{Im}G_\omega(k_\perp, x, x')\,|_{x=x'} = -\pi\sum_m |\,\langle 0\,|\,\psi(k_\perp, x)\,|\,m\rangle\,|^2\delta(\omega-E_m), \qquad \omega>0$$
$$= -\pi\sum_n |\,\langle 0\,|\,\psi^\dagger(k_\perp, x)\,|\,n\rangle\,|^2\delta(\omega+E_n) \qquad \omega\leq 0. \tag{A9}$$

Using Eq. (A9) we find that

$$W_{AB} = 2\pi\int_{-\infty}^{\infty} d\omega\sum_{k_\perp} |\,T(k_\perp)\,|^2\rho_A^{(-)}(k_\perp, \omega-V, x_a)\rho_B^{(+)}(k_\perp, \omega, x_b), \tag{A10}$$

where

$$\rho^{(-)}(k_\perp, \omega, x) = -(1/\pi)\ \mathrm{Im}G_\omega(k_\perp; x, x')\,|_{x=x'}, \qquad \omega\leq 0$$
$$\rho^{(+)}(k_\perp, \omega, x) = -(1/\pi)\ \mathrm{Im}G_\omega(k_\perp; x, x')\,|_{x=x'} \qquad \omega>0. \tag{A11}$$

$\rho^{(-)}$ can be interpreted as the density of electrons at x with energy ω and transverse momentum $k_\perp$, and $\rho^{(+)}$ is the density of states available for adding an electron at x with energy ω and transverse momentum $k_\perp$.

The tunneling current will be proportional to W_{AB}. The metal-insulator interface will be irregular and diffuse and the film thickness will vary so that in an experiment one measures an average W_{AB}. In the simplest model one supposes that tunneling from different sections of the film and from different points x_a to x_b are incoherent so that as an approximation we may average the quantities $\rho_A^{(-)}$ and $\rho_B^{(+)}$ independently.

In Sec. IV we employ a simple linear averaging procedure for the density of states near $-d$:

$$\langle\rho_B^{(+)}(k_\perp, \omega, -d)\rangle = r^{-1}\int_{-d-r/2}^{-d+r/2}\rho_B^{(+)}(k_\perp, \omega, -x)\,dx, \tag{A12}$$

where r should be on the order of the variation in the film thickness. Furthermore, Harrison[21] has shown that $|\,T(k_\perp)\,|^2$ decreases exponentially with increasing $k_\perp$ so that the major contribution to the sum on $k_\perp$ comes for $k_\perp\approx 0$. Because of the fact that $\tilde{k}_F$ changes very slowly with $k_\perp$ for $k_\perp\ll k_F$, the density of states $\langle\rho_B^{(+)}(k_\perp; -d)\rangle$ will depend weakly upon $k_\perp$. Consequently, as a first approximation, the current will be proportional to

$$I\propto\int_{-\infty}^{\infty} d\omega\langle\rho_A^{(-)}(k_\perp, \omega-V, x_a)\rangle\langle\rho_B^{(+)}(k_\perp, \omega, x_b)\rangle\,|_{k_\perp=0}, \tag{A13}$$

and the differential conductance will be dI/dV. The density of electrons $\rho_A^{(-)}$ for metal A may be taken to be the simple ($T=0$) BCS function

$$\rho_A^{(-)}(k_\perp, \omega-V, x_a) = \mathrm{Re}\left\{\frac{m\,|\,\omega-V\,|}{\pi\hbar^2\tilde{k}_F[(\omega-V)^2-\Delta_A^2]^{1/2}}\right\}, \qquad \omega-V\leq 0$$
$$= 0, \qquad \omega-V>0 \tag{A14}$$

where Δ_A is the energy gap of metal A. In the case that $\Delta_A=0$ the differential conductance divided by the normal-state differential conductance will be proportional to $\rho_B^{(+)}$. For nonzero Δ_A, the ratio of the differential conductances will still be qualitatively proportional to $\rho_B^{(+)}$. However, preliminary numerical results indicate that there are significant differences.

[21] W. A. Harrison, Phys. Rev. **123**, 85 (1961).