Stability Limits of the Meissner State and the Mechanism of Spontaneous Vortex Nucleation in Superconductors*

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(Received 27 December 1967)

The limits of metastable existence of the superconducting Meissner state in a magnetic field are found by examining the second variation $\delta^2\Omega$ of the Ginzburg-Landau free energy. No assumptions about boundary conditions are made, and all possible fluctuations are examined. First, confining the fluctuations to one dimension, we show that $\delta^2\Omega$ is positive definite exactly up to that field H_{s2} (first calculated by Ginzburg) at which the Meissner state ceases to exist as a Ginzburg-Landau solution. At H_{s2}, the normal state penetrates spontaneously. Then we take into account arbitrary fluctuations and show that for superconductors with $\kappa\gtrsim0.5$ another instability occurs at a lower field H_{s1}, leading to a new metastable modification of the Meissner state. This new state possesses small vortices with fluxoid quantum zero along the boundary, and is metastable up to a field H_{s3}, which is probably of the order of H_{s2}($H_{s3}=H_{s2}=H_c$ for $\kappa\gg1$). At H_{s3}, the normal state penetrates. Then, in a type-II superconductor with H_{s3} smaller than the upper critical field H_{c2}, spontaneous nucleation of Abrikosov vortices will take place in the normal region without violating fluxoid quantization. This should be the correct mechanism for vortex nucleation in ideal superheating experiments.

I. INTRODUCTION

USUALLY, the Meissner state of a bulk superconductor exists only for applied fields below the thermodynamic critical field H_c or the lower critical field H_{c1}, depending on whether we are dealing with a type-I ($H_c<H_{c1}$) or a type-II ($H_c>H_{c1}$) superconductor. For fields larger than H_c the normal state has a lower free energy than the Meissner state, while for fields larger than H_{c1} the vortex state is energetically favorable. But since both transitions are connected with a finite change of the order parameter,[1] and since all the intermediate states that may occur during this change have a higher free energy than the initial or final state, the transition cannot take place spontaneously, i.e., without some perturbation of the system.[2] Thus, the Meissner state may exist as a metastable state up to higher fields, an effect which is usually called (magnetic) superheating. The metastability exists as long as the Meissner state represents a local minimum of the free energy, i.e., as long as the second variation of the free energy is positive definite.

Superheating was observed for the first time by Garfunkel and Serin,[3] and the first quantitative treatment was presented by Ginzburg[4] on the basis of the Ginzburg-Landau (GL) theory.[5] Although Ginzburg did not consider the transition to the vortex state, his results for bulk specimens are now realized to be correct for all superconductors. He showed that the Meissner state of a superconductor (including the boundary) exists as a solution of the one-dimensional GL equations only for applied fields smaller than a certain maximum field H_{s2}, and he interpreted this field as the superheating field. For superconductors with a very large GL parameter κ, he found $H_{s2}=H_c$, while H_{s2} tends to infinity for $\kappa\to0$. For type-II superconductors, de Gennes and independently Bean and Livingston[6] introduced a different theoretical concept by discussing the electrodynamic surface barrier which opposes the entry of vortices and is produced by the surface supercurrents and the image force in the London approximation. Besides providing only a rough estimate for the superheating field in the high-κ limit, this concept does not, in our opinion, give an adequate picture of the mechanism of spontaneous vortex nucleation. Recently, de Gennes, Matricon, and Saint-James[7] again (but independently) applied Ginzburg's method to both kinds of superconductors and obtained the same results. Fink and Presson[8] found somewhat different numerical values for the superheating field of type-II superconductors. Experimentally, Ginzburg's calculations are confirmed fairly well for $\kappa\gtrsim1$,[9] and for

* Research sponsored in part by the Air Force Office of Scientific Research, Office of Aerospace Research, U.S. Air Force, under AFOSR Contract No. AF49(638)-1389 and in part by the Studienstiftung des deutschen Volkes.

† On leave from Abteilung für Theoretische Festkörperphysik am Institut für Angewandte Physik der Universität Hamburg, West Germany.

[1] This is clear for transitions to the normal state, but it is equally true for the transition to the vortex state (which is of second order with regard to bulk properties), since macroscopic (and even quantized) vortices are created.

[2] Here we do not take account of thermodynamic fluctuations.

[3] M. P. Garfunkel and B. Serin, Phys. Rev. **85**, 834 (1952).

[4] V. L. Ginzburg, Zh. Eksperim. i Teor. Fiz. **34**, 113 (1958) [English transl.: Soviet Phys.—JETP **7**, 78 (1958)].

[5] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).

[6] P. G. de Gennes, *Cours: Métaux et Alliages Supraconducteurs* (Faculté des Sciences, Orsay, France, 1963) [English transl.: *Superconductivity of Metals and Alloys* (W. A. Benjamin, Inc., New York, 1966)]; C. P. Bean and J. D. Livingston, Phys. Rev. Letters **12**, 14 (1964).

[7] P. G. de Gennes, Solid State Commun. **3**, 127 (1965); J. Matricon and D. Saint-James, Phys. Letters **24A**, 241 (1967).

[8] H. J. Fink and A. G. Presson, Phys. Letters **25A**, 378 (1967).

[9] A. S. Joseph and W. J. Tomasch, Phys. Rev. Letters **12**, 219 (1964); R. W. de Blois and W. de Sorbo, *ibid.* **12**, 499 (1964); G. Boato, G. Gallinaro, and C. Rizzuto, Solid State Commun. **3**, 173 (1965); J. C. Renard and Y. A. Rocher, Phys. Letters **24A**, 509 (1967); H. J. Fink, A. S. Joseph, and W. J. Tomasch, Phys. Rev. **157**, 315 (1967).

$\kappa \sim 0.1$.[10] Agreement seems to be especially good for large κ.

In this paper, we want to investigate the metastability of the Meissner state in a mathematically rigorous way based on the GL theory (a brief discussion of the high-κ limit was presented before).[11] We essentially discuss the second variation of the free energy in the Meissner state assuming that the boundary conditions for the order parameter are "natural," i.e., are determined by minimization. In Sec. II we state the variational problem and present an expression for the second variation $\delta^2\Omega$ of the free energy, which allows us to account for variations of the order parameter along the boundary of the superconductor. In Sec. III we confine ourselves to one-dimensional GL solutions, as the previous authors have done. Using Jacobi's theory of the second variation, it is shown that $\delta^2\Omega$ is positive definite exactly up to the maximum field H_{s2}, for which the GL equations admit the Meissner solution. There an instability occurs, which leads directly to the normal state. Thus a mathematical basis is provided for the results of the authors of Refs. 4, 7, and 8.

In Sec. IV we take account of fluctuations of the order parameter and the supercurrent along the boundary of the superconductor and show that for materials with $\kappa \gtrsim 0.5$ these fluctuations lead to an instability of the Meissner state at a field H_{s1} lower than H_{s2}. Analyzing the critical fluctuations we gain some insight into the new equilibrium state they lead to. It turns out to be a new metastable modification of the Meissner state which has a layer of small "vortices" along the surface of the superconductor. These vortices are very different from Abrikosov vortices mainly because they have zero fluxoid quantum. For large κ this instability occurs at $H_{s1}=0.745\ H_c$,[11] and thus corresponds to the instability found by Galaiko.[12] He concludes, however, that it leads to spontaneous vortex nucleation by means of thermodynamic fluctuations[13] (in contrast to experiments). We do not support his point of view, but believe that this new metastable state persists up to a field H_{s3} of the order of H_{s2}. For large κ it can be shown that $H_{s3}=H_{s2}=H_c$. At H_{s3} the normal state will penetrate.

Finally in Sec. V we present a picture for the spontaneous formation of Abrikosov vortices by assuming that they can nucleate spontaneously only in the normal region. This seems the only way in which vortices can be formed without ever violating fluxoid quantization.

II. STATEMENT OF THE VARIATIONAL PROBLEM

The (magnetic) Gibb's free energy of a superconductor as derived by Ginzburg and Landau[5] is

$$\Omega=\int d^3r\left[\tfrac{1}{2}(1-F^2)^2+(\kappa^{-1}\nabla F)^2+F^2\mathbf{Q}^2+(\mathbf{H}_0+\mathrm{curl}\mathbf{Q})^2\right]. \tag{1}$$

Here F means the absolute value of the complex order parameter $\psi=F\ \exp(i\varphi)$; the "superfluid velocity" is $\mathbf{Q}=\nabla\varphi/\kappa-\mathbf{A}$ (note: $\mathrm{curl}\mathbf{Q}=-\mathrm{curl}\mathbf{A}=-\mathbf{H}$); and $\mathbf{H}_0$ is the applied magnetic field. We are using the usual dimensionless units where lengths are measured in units of the London penetration depth λ_L, fields in units $\sqrt{2}H_c$ and energies in units $H_c^2/4\pi$. The complete first variation of Ω is given by

$$\begin{aligned}\delta\Omega=2\int d^3r\{ & f[F(F^2+\mathbf{Q}^2-1)-(\nabla/\kappa)^2F] \\ & =+\mathbf{q}\cdot[F^2\mathbf{Q}+\mathrm{curl\ curl}\mathbf{Q}]\} \\ & +2\int d\mathbf{S}\cdot[f\kappa^{-2}\nabla F+\mathbf{q}\times(\mathbf{H}_0+\mathrm{curl}\mathbf{Q})].\end{aligned} \tag{2}$$

We have integrated by parts and made use of the fact that curl $\mathbf{H}_0=0$. The variations of F and $\mathbf{Q}$ are called f and $\mathbf{q}$. The condition $\delta\Omega=0$ for all f and $\mathbf{q}$ leads to the GL equations

$$(\nabla/\kappa)^2F=F(F^2+\mathbf{Q}^2-1); \tag{3}$$

$$\mathrm{curl\ curl}\mathbf{Q}=-F^2\mathbf{Q}; \tag{4}$$

and the natural boundary conditions

$$\partial F/\partial\mathbf{n}=0; \qquad H_0=-\mathrm{curl}\mathbf{Q}. \tag{5}$$

($\partial/\partial\mathbf{n}$ means the normal derivative at the boundary.) In deriving this last result, F and $\mathbf{Q}$ were varied even at the boundary. In our opinion this is the right procedure, since in the region where the GL theory is applicable (that is, for $T\to T_c$) the coherence length and the London penetration depth become very large. Then the region, which is influenced by the boundary, is negligible compared to them. This just means that the bulk free energy itself determines the boundary conditions by minimization. But now the boundary conditions themselves can become unstable.

In order to test GL solutions for stability we must examine the second variation of Ω, given by

$$\begin{aligned}\delta^2\Omega[f,\mathbf{q}]=\int d^3r\{ & [3F^2+\mathbf{Q}^2-1]f^2+(\kappa^{-1}\nabla f)^2 \\ & +4Ff\mathbf{Q}\cdot\mathbf{q}+F^2\mathbf{q}^2+(\mathrm{curl}\mathbf{q})^2\}.\end{aligned} \tag{6}$$

[10] J. Feder, S. R. Kiser, and F. Rothwarf, Phys. Rev. Letters **17**, 87 (1966); J. P. Burger, J. Feder, S. R. Kiser, F. Rothwarf, and C. Valette, in *Proceedings of the Tenth International Conference on Low-Temperature Physics, Moscow, 1966* (Proizvodstrenno-Izdatel'skii Kombinat, VINITI, Moscow, 1967); F. W. Smith and M. Cardona, Solid State Commun. **5**, 345 (1967); Phys. Letters **24A**, 247 (1967); R. Doll and P. Graf, Phys. Rev. Letters **19**, 897 (1967).

[11] L. Kramer, Phys. Letters **24A**, 571 (1967).

[12] V. P. Galaiko, Zh. Eksperim. i Teor. Fiz. **50**, 717 (1966) [English transl.: Soviet Phys.—JETP **23**, 475 (1966)].

[13] V. P. Galaiko, Zh. Eksperim. i Teor. Fiz. **50**, 1322 (1966) [English transl.: Soviet Phys.—JETP **23**, 878 (1966)].

If, on inserting the Meissner solution for F and $\mathbf{Q}$, this quadratic functional is positive definite, the solution is stable against small fluctuations. Thus, in order to find the stability limit of the Meissner state, we have to minimize $\delta^2\Omega$ with respect to f and $\mathbf{q}$, and find out when the minimum becomes negative (a quadratic functional has either the minimum zero or $-\infty$). At this point there exist (nontrivial) variations f and $\mathbf{q}$ which yield $\delta^2\Omega[f, \mathbf{q}]=0$, so that $\delta^2\Omega$ is positive semidefinite. These "critical variations" clearly are solutions of the Euler-Lagrange equations for $\delta^2\Omega$, which are called the Jacobi equations for Ω.[14]

For simplicity, consider a superconductor occupying the half-space $x\geq 0$. The magnetic field is applied in the z direction. In the usual Meissner state $\mathbf{Q}$ has only a y component (simply denoted by Q) and all quantities depend on x only. Specializing Eq. (6) to this geometry, it is easily seen that all variations in the z direction make a positive contribution. Thus we may assume translational invariance in the z direction and confine $\mathbf{q}$ to the x-y plane $[\mathbf{q}\equiv(q_x, q_y)]$.

It is natural to expand f and $\mathbf{q}$ in a Fourier series with respect to y. Again, upon inserting a general Fourier expansion into $\delta^2\Omega$, one sees that in order to find the minimum we may restrict ourselves to the specialized expansions

$$f=\sum_{k\geq 0}\tilde{f}(k, x)\cos ky;$$
$$q_x=\sum_{k\geq 0}\tilde{q}_x(k, x)\sin ky;$$
$$q_y=\sum_{k\geq 0}\tilde{q}_y(k, x)\cos ky. \qquad (7)$$

Inserting Eq. (7) into Eq. (6) we obtain

$$\delta^2\Omega=\sum_{k\geq 0}\int_0^\infty dx\{[3F^2+Q^2+(k/\kappa)^2-1]\tilde{f}^2+\kappa^{-2}\tilde{f}'^2$$
$$+4FQ\tilde{f}\tilde{q}_y+F^2[\tilde{q}_x^2+\tilde{q}_y^2]+[\tilde{q}_y'-k\tilde{q}_x]^2\}. \qquad (8)$$

(The prime means the derivative with respect to x. Factors in front of the integral were omitted since they are of no importance.) As there is no coupling between different modes, only a single mode contributes to the instability and the sum may be omitted. The critical wave number k is determined by minimizing $\delta^2\Omega$ with respect to it. Minimization with respect to $\tilde{q}_x$ leads to the simple relation

$$\tilde{q}_x=(F^2+k^2)^{-1}k\tilde{q}_y', \qquad (9)$$

which finally gives us

$$\delta^2\Omega=\int_0^\infty dx\{[3F^2+Q^2+(k/\kappa)^2-1]\tilde{f}^2+\kappa^{-2}\tilde{f}'^2$$
$$+4FQ\tilde{f}\tilde{q}_y+F^2\tilde{q}_y^2+(F^2+k^2)^{-1}F^2\tilde{q}_y'^2\}. \qquad (10)$$

[14] See, e.g., I. M. Gelfand and S. V. Fomin, *Calculus of Variations* (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963).

Unfortunately, the general minimization of this expression is quite a complicated numerical procedure (although perhaps not hopeless), since F and Q can only be obtained by numerical integration of the GL equations.

III. CASE $k=0$

First of all, consider the case $k=0$, where f and $\mathbf{q}$ depend on x only. The problem is one dimensional and $\mathbf{q}$ has only a y component. On integrating Eq. (10) by parts we obtain

$$\delta^2\Omega=\int dx\{f[(3F^2+Q^2-1)f-\kappa^{-2}f''+2FQq_y]$$
$$+q_y[F^2q_y-q_y''+2FQf]\}-\kappa^{-2}ff'|_{x=0}-q_yq_y'|_{x=0}. \qquad (11)$$

($\tilde{f}$, $\tilde{\mathbf{q}}$ and f, $\mathbf{q}$ are identical for $k=0$.) As explained before, the critical fluctuations minimize $\delta^2\Omega$ and thus are solutions of its Euler-Lagrange equations

$$\kappa^{-2}f''-[3F^2+Q^2-1]f=2FQq_y;$$
$$q_y''-F^2q_y=2FQf. \qquad (12)$$

One may look upon these equations also as the variational equations[14] (in physical literature often called "perturbation equations") of the GL equations, which in one dimension obtain the simple form

$$\kappa^{-2}F''=F(F^2+Q^2-1); \qquad Q''=F^2Q. \qquad (13)$$

Suppose the solutions of Eqs. (13) with the boundary conditions of the Meissner state ($F\to 1$, $Q\to 0$ for $x\to\infty$ and $F'(0)=0$, $Q'(0)=-H_0$) were given numerically for different possible applied fields H_0. Instead of characterizing the solutions by H_0 they may be specified by the boundary value of the order parameter $F(0)\equiv F_0$. So we have a one-parameter family of Meissner solutions $F(x; F_0)$, $Q(x; F_0)$, and we now easily see that

$$f(x)=\partial F(x; F_0)/\partial F_0;$$
$$q_y(x)=\partial Q(x; F_0)/\partial F_0 \qquad (14)$$

are solutions to Eqs. (12) with the boundary conditions f, $q_y\to 0$ for $x\to\infty$ and $f'(0)=0$. Inserting this result into Eq. (11) we obtain

$$\delta^2\Omega=-\frac{\partial Q}{\partial F_0}\frac{\partial Q'}{\partial F_0}\bigg|_{x=0}=\frac{\partial Q}{\partial F_0}\bigg|_{x=0}\frac{\partial H_0}{\partial F_0}. \qquad (15)$$

For low fields $\delta^2\Omega>0$ holds since then both derivatives are negative. However, the numerical calculations show[4] that the curve $H_0(F_0)$ is not monotonic but exhibits a maximum at a certain value $F_0=F_m$ (lying between $1/\sqrt{2}$ and 0). This field is the maximum field H_{s2} up to which the Meissner solution exists calculated by the authors of Refs. 4, 7, and 8. We have plotted it in Fig. 1. But according to Eq. (15) $\delta^2\Omega$ changes sign exactly at this field, and thus F and Q will slip into some other

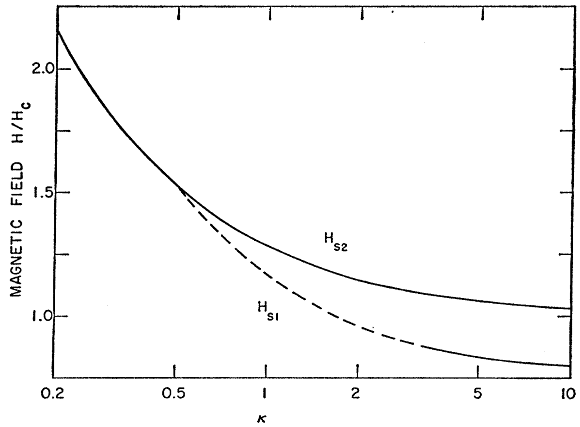

FIG. 1. Superheating fields H_{s2} (taken from Ref. 4) and H_{s1} as a function of κ.

equilibrium state.[15] The result also tells us that there exists such a state which is a solution of the one-dimensional GL equations. This clearly is the normal state.

For $\kappa\gg1$, terms of order κ^{-2} may be neglected in our case, thus obtaining from Eqs. (13) $F^2=1-Q^2$ and

$$Q''=(1-Q^2)Q;\qquad Q'^2=Q^2-Q^4/2(=H^2). \quad (16)$$

There is only one Jacobi equation:

$$q_y''=(1-3Q^2)q_y,$$

which has the solution $q_y=Q'$. From Eq. (11) we easily deduce

$$\delta^2\Omega=H_0Q''(0).$$

According to Eqs. (16), the condition $Q''(0)=0$ leads to $Q(0)=1$ and $Q'(0)=1/\sqrt{2}$. So we have

$$H_{s2}=1/\sqrt{2}=H_c. \quad (17)$$

In the high-κ limit the order parameter reaches zero at the boundary for $H_0=H_{s2}(F_m=0$ in this case).

IV. CASE $k\neq0$

We now proceed by taking into account variations of the order parameter and the superfluid velocity in the y direction, i.e., allowing for a finite k in the second variation (10). First consider the case $\kappa\rightarrow\infty$. Then we can find a k with $k\gg1$ and $k\kappa^{-1}\ll1$. Neglecting small terms and inserting $F^2=1-Q^2$ into Eq. (10), we obtain

$$\delta^2\Omega=\int_0^\infty dx\{2(F\tilde f+Q\tilde q_y)^2+(1-3Q^2)\tilde q_y^2\}. \quad (18)$$

Thus $\delta^2\Omega$ ceases to be positive definite as soon as $Q^2\geq\frac{1}{3}$ somewhere. From Eqs. (16) we find that this instability occurs at an applied field[12,11]

$$H_{s1}=(\tfrac{1}{3}-\tfrac{1}{18})^{1/2}=0.745H_c. \quad (19)$$

Our aim is now to determine the new equilibrium state this instability leads to and to find out what happens for smaller κ. For both purposes we make some rather rough approximations. Consider the minimizing equations of $\delta^2\Omega$ which determine the critical variations. They are given by

$$\kappa^{-2}\tilde f''-[3F^2+Q^2+(k/\kappa)^2-1]\tilde f=2FQ\tilde q_y; \quad (20)$$

$$(d/dx)(F^2(F^2+k^2)^{-1}\tilde q_y')-F^2\tilde q_y=2FQ\tilde f; \quad (21)$$

$$k\int_0^\infty dx\{\kappa^{-2}\tilde f^2-(F^2+k^2)^{-2}F^2\tilde q_y'^2\}=0. \quad (22)$$

The last equation is obtained by minimizing with respect to k. It has one solution $k=0$, considered before, and a second solution which is coupled to Eqs. (20) and (21). For a rough approximation we may insert the variations from the case $k=0$ into Eq. (22) and solve for k. In the high-κ limit this can be done by using the results from the end of the preceding section ($q_y=Q'$). We obtain $k^2=b\kappa$, where $b\approx1.26$ for $H_0=H_{s1}$ and $b\rightarrow0$ for $H_0\rightarrow H_c$. Restoring the units this leads to

$$k=b^{1/2}(\xi\lambda_L)^{-1/2}. \quad (23)$$

Thus at the onset of the instability the critical wavelength lies between the GL coherence length ξ and the London penetration depth λ_L. For increasing fields the wavelength increases and becomes of order λ_L for $H_0\rightarrow H_c$.

For decreasing κ, the field H_{s1} increases, and it would be interesting to know whether H_{s1} lies below $H_{s2}(k=0)$ for all κ or not. In order to settle this question we examined the case $\kappa\ll1$, where the GL equations may be solved by successive approximations.[4] No instability with respect to $k\neq0$ seems to occur then. To find the κ at which the instability disappears [i.e., where the $H_{s2}(\kappa)$ curve joins the $H_{s1}(\kappa)$ curve], the following equation has to be solved for κ:

$$\int_0^\infty dx\{\kappa^{-2}f^2-F^{-2}q_y'^2\}=0. \quad (24)$$

It is obtained by dividing Eq. (22) through by k and then setting $k=0$. We have solved Eq. (24) in a rough approximation by using the variations from the high-κ limit and obtained $\kappa\approx0.5$. Thus the curve $H_{s1}(\kappa)$ should look qualitatively as shown in Fig. 1.

We proceed to find the approximate form of the critical variations. For simplicity, we consider the case $\kappa\gg1$, assuming that the qualitative results will hold quite generally. By omitting small terms and setting $F^2=1-Q^2$ we obtain from Eqs. (20) and (21)

$$\tilde q_y''-\frac{2QQ'}{1-Q^2}\tilde q_y'-k^2\frac{1-3Q^2}{1-Q^2}\tilde q_y=0. \quad (25)$$

[15] Our discussion may not seem complete, since some other solutions of Eqs. (12) could bring about a change of sign of $\delta^2\Omega$ already at lower fields. Insertion of $f=F'$ and $q_y=Q'$ into Eqs. (12) shows that this is another solution which does not change our result. But now we have taken into account the two linearly independent solutions with f, $q_y\rightarrow0$ inside the superconductor (which certainly holds).

It is not difficult to get some idea of $\tilde{q}_y$ by remembering that the last term of this equation changes sign somewhere near the boundary of the superconductor if the applied field is larger than H_{s1}. Let us assume that $Q^2=\frac{1}{3}$ at $x=x_0$. Then $\tilde{q}_y$ looks about as shown in Fig. 2. We have also plotted $\tilde{q}_x$ as determined from Eq. (9) and the amplitudes of the current variations $\mathbf{j}$ as determined from Maxwell's equation curl curl $\mathbf{q}=-\mathbf{j}$. This leads to

$$j_x=(1-Q^2)\tilde{q}_x \sin ky; \qquad j_y=(1-3Q^2)\tilde{q}_y \cos ky, \tag{26}$$

where k is determined by Eq. (22). Thus j_y changes sign at x_0 while j_x does not. In Fig. 3 we have plotted the two-dimensional pattern of the current variations. They form small vortices along the surface of the superconductor with alternating direction of rotation.

What happens to these critical variations as the critical point is passed? It turns out that the higher-order terms in the free energy make a negative contribution so that the amplitudes of the variations are not bounded at the beginning. They will grow in time as soon as H_{s1} is reached. Strictly speaking, we cannot describe the further development of the system with the GL theory since it gives us only the time independent equilibrium states. But it is not hard to guess what will happen: The amplitudes grow until at some point the order parameter reaches zero. Then a new metastable equilibrium state will be established, which can be visualized as a superposition of the usual translationally invariant screening currents of the Meissner state and these surface vortices. Certainly they will now be somewhat distorted by higher-order terms. We have tacitly taken account of this in Fig. 2 by adjusting the boundary conditions in the right way. This would not be possible in the linear approximation. Figure 2 should exhibit the approximate proportions of the amplitudes of $\mathbf{q}$ and $\mathbf{j}$ as compared to the translationally invariant parts Q and J at an applied field $H_0=0.9\ H_c$ for large κ. At lower fields the amplitudes are larger while they tend to zero for $H_0\to H_{s2}$ (this is strictly true only for $\kappa\to\infty$).

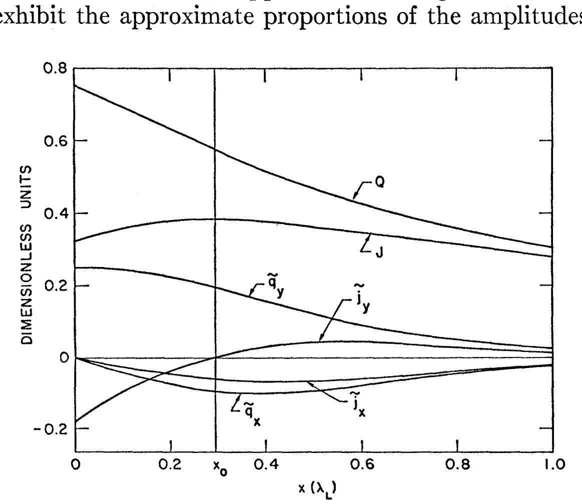

FIG. 2. Amplitudes of the critical-velocity variations $\tilde{\mathbf{q}}=(\tilde{q}_x,\tilde{q}_y)$ and the critical-current variations $\tilde{\mathbf{j}}=(\tilde{j}_x,\tilde{j}_y)$ for $\kappa\gg1$ and $H_0=0.9\ H_c$ in qualitative form. The translationally invariant parts $Q(=Q_y)$ and $J(=J_y)$ have been added for comparison.

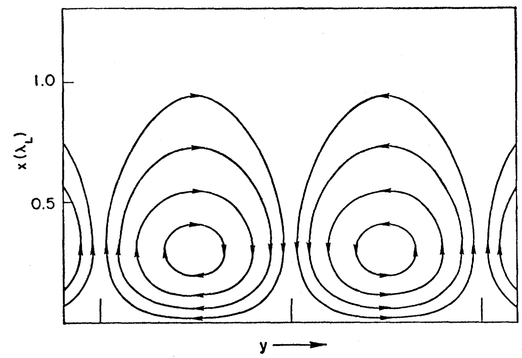

FIG. 3. Approximate form of the critical-current variations $\mathbf{j}$.

It should be emphasized that these variations have nothing to do with Abrikosov vortices and do not, in our opinion, represent nuclei for them. The surface vortices have no singularity of the superfluid velocity at their center and do not carry a fluxoid quantum. Nucleation of Abrikosov vortices must be based on an entirely different mechanism. As the field is increased above H_{s1}, the surface vortices and their mutual distance become larger. At some field H_{s3} which probably does not differ much from H_{s2}, the state ceases to be metastable and a transition to the normal state takes place similar to that discussed in the preceding section. In the case $\kappa\to\infty$ we have $H_{s3}=H_{s2}=H_c$. This is easy to understand since then the translationally invariant part of the order parameter becomes zero at the surface and the amplitudes of the oscillations vanish.

The surface vortices represent a state with lower energy already at fields somewhat smaller than H_{s1}, but their formation can be delayed up to H_{s1}. Nevertheless, in actual experiments the transition might take place at a lower field.

V. SPONTANEOUS VORTEX NUCLEATION IN TYPE-II SUPERCONDUCTORS

We showed that the Meissner state of a superconductor with $\kappa\gtrsim0.5$ suffers a strange modification before its limit of metastability is reached. Furthermore, we made plausible that at this limit a transition to the normal state takes place. How do vortices penetrate a type-II superconductor? The simple answer: In the absence of any perturbation vortices will never penetrate spontaneously a superconducting region, because this process could only be achieved by a change of the fluxoid quantum number in the superconductor and therefore a high-energy barrier would have to be passed.[16] Super-

[16] N. Byers and C. N. Yang, Phys. Rev. Letters **7**, 46 (1961); F. Bloch, Phys. Rev. **137**, A787 (1965).

conducting states with different numbers of vortices behave like different discrete and stationary quantum states, so that a transition between them can only be achieved by some perturbation.

In carefully designed superheating experiments the Meissner state will always become unstable with respect to the normal state. The penetration of the normal state is allowed since no flux penetrates the superconducting region. The screening current system is pushed inside the superconductor without losing its connectedness. Then, if the applied field is still below the upper critical field H_{c2}, spontaneous vortex nucleation will immediately take place in the normal region. Thus the flux is not carried into the superconductor but vortices grow out of the normal state, trapping their flux quantum from the beginning on.

This picture of vortex nucleation implies that the superheating field is larger than or equal to the thermodynamic critical field H_c for all temperatures and superconductors independent of the applicability of the GL theory.[11] Our assumptions also lead to the conclusion that not only the Meissner state but every vortex state is metastable against a change of the number of flux lines. Thus, if one starts from the equilibrium mixed state with an arbitrary number of Abrikosov vortices, similar superheating effects should be observed.

VI. CONCLUDING REMARKS

In Sec. II we found the stability limits of a family of numerically known solutions of the GL equations without any additional calculations by restricting the permissible variations to one dimension. It is a general feature of Jacobi's theory[14] that this can always be done. Each solution of the Jacobi equations [Eqs. (12)] is embedded into a family of solutions of the Euler-Lagrange equations [the one-dimensional GL equations (13)] in such a way that the former can be obtained by differentiating the latter with respect to some parameter characterizing the individual solution. This theory is in fact applicable to any numerical solution of a variational problem and characterizes its stability. However, the method cannot in general be used when only a restricted class of solutions of the variational problem is available and a wider class of variations is admitted in the stability problem. This is why it could not be used in Sec. IV, where two-dimensional variations were considered.

The discussion in Sec. IV partly suffered from the fact that we had to investigate the nonequilibrium behavior of a system with a theory which is essentially time-independent. It might be worthwhile to attack the problem with a time-dependent generalization of the GL theory. The result shows that one must be very careful when assuming some rather "obvious" symmetry property of superconducting states. Perhaps there exist other cases where something similar happens (e.g., current-carrying superconductor, surface sheath). Unfortunately, we do not know of any experiment to detect this new metastable Meissner state, since the field outside the superconductor is not distorted by the surface vortices. Only the penetration depth varies periodically by a small amount.

It was pointed out in Sec. V that a superconducting region in equilibrium should always be metastable against a change of flux in its interior. Therefore, we can explain supercooling of the mixed state in much the same way as superheating. Our model provides a surface mechanism for hysteresis in ideal type-II superconductors without reference to the superconducting surface sheath.

ACKNOWLEDGMENT

I would like to thank Professor Alexander L. Fetter for his interest in this work and for valuable discussions.