Elasticity Effects in Type-II Superconductors

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A phenomenological theory of strain and stress effects in Ginzburg-Landau superconductors is given. The theory provides formulas for the elastic constants and the specific volume in the mixed state, and it is shown how to obtain a complete set of appropriate phenomenological parameters. We also obtain correct expressions for the interaction energy between fluxoids and localized strain fields which are only partly in agreement with the expressions used by previous authors.

INTRODUCTION

By application of thermodynamic principles, the changes in elastic constants and specific volume of type-I superconductors at the transition to the normal state have been related to the stress or strain dependence of the critical magnetic field.[1] Similar relations for the transition at the upper critical field H_{c2} and for the zero-field superconducting state below H_{c1} of type-II superconductors were derived by Hake.[2] In order to obtain a more detailed description of the mixed state between H_{c1} and H_{c2}, we derive in this paper a set of modified Ginzburg-Landau equations with strain-dependent parameters. Solving these equations by means of a perturbation calculation we obtain expressions for the elastic constants and the specific volume in the mixed state. These results provide formulas for determination of the strain dependence of the Ginzburg-Landau parameter and the thermodynamic critical field from measurements of elastic constants.

A perturbation calculation of the free energy gives us correct expressions for the interaction energy between fluxoids and slowly varying strain fields. These expressions provide the correct starting point for calculation of the pinning of fluxoids by the localized strain fields of dislocations and coherent precipitates. The basis for the existing first- and second-order calculations by Kramer and Bauer,[3] by Webb,[4] and by Toth and Pratt[5] is confirmed but additional terms are found. For the second-order calculation the additional term is negligible in the one material for which experimental information on its coefficient exists.

DERIVATION OF THE MODIFIED GINZBURG-LANDAU EQUATIONS

We choose for convenience H_{c2} the upper critical field and κ^2 the square of the Ginzburg-Landau parameter as independent material parameters. Their strain dependence is characterized by the following phenomenological coefficients, which correspond to the first and second derivatives of H_{c2} and κ^2 with respect to the strain ϵ_{ij}:

$$\begin{aligned} a_{ij}&=\partial H_{c2}/H_{c2}\partial\epsilon_{ij},\\ a_{ijkl}&=\partial^2 H_{c2}/H_{c2}\partial\epsilon_{ij}\partial\epsilon_{kl},\\ b_{ij}&=\partial(\kappa^2)/\kappa^2\partial\epsilon_{ij},\\ b_{ijkl}&=\partial^2(\kappa^2)/\kappa^2\partial\epsilon_{ij}\partial\epsilon_{kl}. \end{aligned}\tag{1}$$

These parameters are second- and fourth-rank tensors, respectively, and their symmetry properties are determined by the crystal symmetry of the superconductor. In the case of cubic symmetry, which is of most practical interest, we have $a_{ij}=a\delta_{ij}$, $b_{ij}=b\delta_{ij}$ and the a_{ijkl}'s and b_{ijkl}'s are reduced to two sets of three independent components each. δ_{ij} is the Kroneker delta. For a more detailed discussion of the symmetry properties, see Refs. 1 and 6.

Taking into account all terms up to the second order in ϵ_{ij} we obtain for the free energy of the superconductor[7]

$$\begin{aligned} G=\frac{H_c^2}{4\pi}\int\bigg\{&(H-H_a)^2+\psi^*\left(\frac{i\nabla}{\kappa}+\mathbf{A}\right)^2\psi\\ &-(1+a_{ij}\epsilon_{ij}+\tfrac{1}{2}a_{ijkl}\epsilon_{ij}\epsilon_{kl})\,|\psi|^2\\ &+\tfrac{1}{2}(1+b_{ij}\epsilon_{ij}+\tfrac{1}{2}b_{ijkl}\epsilon_{ij}\epsilon_{kl})\,|\psi|^4\bigg\}dr^3\\ &+\int(\tfrac{1}{2}C_{ijkl}{}^n\epsilon_{ij}\epsilon_{kl}+u_iK_i)\,ds^3+\oint C_{ij}\epsilon_j u_i dr^2. \end{aligned}\tag{2}$$

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[1] D. P. Seraphim and P. M. Marcus, IBM J. Res. Develop. **6**, 94 (1962).

[2] R. R. Hake, in Proceedings of the Conference on the Physics of Type-II Superconductivity, Cleveland, Ohio, 1964, Vol. 1, p. 15 (unpublished).

[3] E. F. Kramer and C. L. Bauer, Phil. Mag. **15**, 1189 (1967).

[4] W. W. Webb, Phys. Rev. Letters **11**, 191 (1963).

[5] L. E. Toth and J. P. Pratt, Appl. Phys. Letters **4**, 75 (1964).

[6] See for example C. S. Smith, Solid State Phys. **6**, 215 (1958).

[7] This equation becomes obvious if one goes back to an expansion of the free energy $G_{s0}(n,\epsilon)$ in zero magnetic field in powers of n and ϵ at $n=0$, $\epsilon=0$, where n is the super electron density. The terms of first and second order in n and ϵ are

$$\begin{aligned}\frac{\partial G}{\partial n}n+\frac{1}{2}\frac{\partial^2G}{\partial n^2}n^2+\frac{\partial G}{\partial\epsilon_{ij}}\epsilon_{ij}+\frac{1}{2}\frac{\partial^2G}{\partial\epsilon_{ij}\partial\epsilon_{kl}}\epsilon_{ij}\epsilon_{kl}\\ +\frac{\partial^2G}{\partial n\partial\epsilon_{ij}}\epsilon_{ij}n+\frac{1}{2}\frac{\partial^3G}{\partial\epsilon_{ij}\partial n^2}\epsilon_{ij}n^2+\frac{1}{2}\frac{\partial^3G}{\partial\epsilon_{ij}\partial\epsilon_{kl}\partial n}\epsilon_{ij}\epsilon_{kl}n\\ +\frac{1}{4}\frac{\partial^4G}{\partial\epsilon_{ij}\partial\epsilon_{kl}\partial n^2}\epsilon_{ij}\epsilon_{kl}n^2.\end{aligned}$$

Since we expand at $n=0$, we have $\partial G/\partial\epsilon_{ij}=0$ if the system in the normal state is in equilibrium with respect to ϵ_{ij}. For the same reason $\partial^2G/\partial\epsilon_{ij}\partial\epsilon_{kl}$ is identical with the elasticity tensor *in the normal state*. The connection to Eq. (2) is made by adding the field-dependent typical Ginzburg-Landau terms and by introducing

$$n=-\frac{-\partial G/\partial n}{\partial^2G/\partial n^2}|\psi|^2,\qquad \frac{H_c^2}{4\pi}=\frac{(\partial G/\partial n)^2}{(\partial^2G/\partial n^2)},$$

$$\lambda^2=-\frac{\partial^2G/\partial n^2}{\partial G/\partial n}\frac{mc^2}{4\pi e^{*2}},\qquad \kappa^2=\frac{mc^2}{4\pi\hbar^2e^{*2}}\frac{\partial^2G}{\partial n^2},$$

and

$$H_{c2}=+\sqrt{2}\kappa H_c=-\frac{mc}{\hbar e^*}\frac{\partial G}{\partial n}.$$

Equations (1) are self-evident from the definitions of κ and H_{c2}.

The first integral in Eq. (2) is the Ginzburg-Landau expression for the free energy in Abrikosov's units,[8] with strain effects explicitly included. ϵ_{ij} is defined to be zero in the unconstrained crystal in the normal state. The material parameters κ and $H_c^2=H_{c2}^2/2\kappa^2$ are the values at zero strain. Although the correct measurement of these parameters should be done at constant volume and not as it is usually done in experiments at constant pressure, in fact the difference is very small so that it can always be neglected and the usual values may be taken. The second integral in Eq. (2) contains the elastic energy and the work that is done by volume forces K_i and the third integral, extending over the surface of the superconductor, gives the work done by the applied surface stresses σ_{ij}. C_{ijkl}^n is the tensor of the elastic constants in the normal state, u_i is the displacement vector, and e_j is the surface normal vector. In Eq. (2) and throughout this paper we use the Einstein convention of summation over repeated indices. By variation of G with respect to ψ^* and A, we obtain the modified Ginzburg-Landau equations

$$-\text{curl curl}\mathbf{A}=\mathbf{A}\,|\psi|^2+(i/2\kappa)(\psi^*\nabla\psi-\psi\nabla\psi^*),\qquad(3a)$$

$$(i\nabla/\kappa+\mathbf{A})^2\psi=\psi[1+a_{ij}\epsilon_{ij}+\tfrac{1}{2}a_{ijkl}\epsilon_{ij}\epsilon_{kl} -|\psi|^2(1+b_{ij}\epsilon_{ij}+\tfrac{1}{2}b_{ijkl}\epsilon_{ij}\epsilon_{kl})].\qquad(4a)$$

Variation with respect to u_i yields the modified elasticity equations

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_k}{\partial x_l}\left[C_{ijkl}^n-\frac{H_c^2}{4\pi}(a_{ijkl}\,|\psi|^2-\tfrac{1}{2}b_{ijkl}\,|\psi|^4)\right]\right) -\frac{H_c^2}{4\pi}\left(a_{ij}\frac{\partial\,|\psi|^2}{\partial x_j}-\tfrac{1}{2}b_{ij}\frac{\partial\,|\psi|^4}{\partial x_j}\right) =K_i+\delta(r-s)\,e_j[\sigma_{ij}-(H_c^2/4\pi)(a_{ij}\,|\psi|^2-\tfrac{1}{2}b_{ij}\,|\psi|^4)],\qquad(5a)$$

where $\delta(r-s)$ is a Dirac δ function located at the surface.

For the sake of simplicity we have confined ourselves in Eq. (5a) to symmetries for which a fourth-rank tensor T obeys the relations $T_{ijkl}=T_{jilk}=T_{klij}$. This includes cubic and hexagonal symmetry. The extension to any other symmetry is straightforward. For $|\psi|^2=0$ Eqs. (5a) are the familiar elasticity equations of the normal state.

RELATIONS FOR THE SPECIFIC VOLUME CHANGE

For the macroscopic part of the strain fields, that is slowly varying compared with $|\psi|^2$ in the mixed state, it is convenient to introduce the local averages $\langle u_k\rangle$, $\langle|\psi|^2\rangle$, and $\langle|\psi|^4\rangle$ over a unit cell of the periodic functions $|\psi|^2$ and $|\psi|^4$. Equation (5a) can then be rewritten as

$$\frac{\partial}{\partial x_j}\left(\frac{\partial\langle u_k\rangle}{\partial x_l}\left[C_{ijkl}^n-\frac{H_c^2}{4\pi}(a_{ijkl}\langle|\psi|^2\rangle-\tfrac{1}{2}b_{ijkl}\langle|\psi|^4\rangle)\right]\right) -\frac{H_c^2}{4\pi}\left(a_{ij}\frac{\partial\langle|\psi|^2\rangle}{\partial x_j}-\tfrac{1}{2}b_{ij}\frac{\partial\langle|\psi|^4\rangle}{\partial x_j}\right) =K_i+\delta(r-s)e_j[\sigma_{ij}-(H_c^2/4\pi)(a_{ij}\langle|\psi|^2\rangle-\tfrac{1}{2}b_{ij}\langle|\psi|^4\rangle)].\qquad(5b)$$

In this equation the oscillating parts of u_k, $|\psi|^2$, and $|\psi|^4$ have been omitted because the corresponding terms can be interpreted as rapidly oscillating forces of zero average, which do not influence the average of u_i or ϵ_{ij} on a large scale. Integrating Eq. (5b) with respect to x_j we obtain for uniform $\langle|\psi|^2\rangle$ a difference in strain between the superconducting and the normal state:

$$\epsilon_{ij}^s-\epsilon_{ij}^n=-S_{ijkl}^n(H_c^2/4\pi)(a_{kl}\langle|\psi|^2\rangle-\tfrac{1}{2}b_{kl}\langle|\psi|^4\rangle),\qquad(6a)$$

where S_{ijkl} is the tensor of elastic compliances. Higher-order terms in the very small parameter $S_{ijkl}(H_c^2/4\pi)$ have been omitted in this result.

In the case of cubic symmetry this strain difference is equivalent to a specific volume difference

$$\Delta V/V=3K(H_c^2/4\pi)(a\langle|\psi|^2\rangle-\tfrac{1}{2}b\langle|\psi|^4\rangle),\qquad(6b)$$

where $K=\frac{1}{3}(S_{1111}+2S_{1122})$ is the compressibility.

Substituting for a and b the definitions Eq. (1), we obtain at zero magnetic field the same result as for type-I superconductors[1]:

$$(\Delta V/V)_{H=0}={}^3K(H_c/4\pi)(\partial H_c/\partial V).\qquad(6c)$$

CHANGE OF ELASTIC CONSTANTS

The natural way to solve Eqs. (3), (4), and (5b) is a perturbation calculation with the solutions of the unmodified Ginzburg-Landau equations and the elasticity equations of the normal state as zero-order solutions. The perturbation parameters are ϵ_{ij} for the calculation of $|\psi|^2$ and $\mathbf{A}$, and $S_{ijkl}(H_c^2/4\pi)$ for the calculation of ϵ_{ij}. $S_{ijkl}(H_c^2/4\pi)$ is always very small (of the order of 10^{-7}) and by the requirement that ϵ_{ij} be also small compared to 1, we are only restricted to the validity range of linear elasticity theory, and the lowest nonvanishing order of the perturbation is always a very good approximation.

In order to obtain the linear response of the system to applied volume forces and surface stresses, we have only to calculate the first-order perturbations of $\langle|\psi|^2\rangle$ for some selected values of the applied magnetic field.

(a) $H_a<H_{c1}$:

$$\langle|\psi|^2\rangle=|\psi|^2=1+a_{kl}\epsilon_{kl}-b_{kl}\epsilon_{kl},$$

[8] N. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].

TABLE I. Coefficients of the strain dependence of H_{c2} and κ^2 in Pb-Tl alloys at 4.2°K from experimental data of Alers.[a]

	$(\Delta C_L/C_L)_{H_{c2}}$ [b]	$(C^s-C^n)/C^n$	$(\partial C/\partial H_a)_{H_{c2}}H_{c2}/C$	a_{11}	$a_{1111}-a_{1122}$	$b_{1111}-b_{1122}$
Pb–1.7 at.% Tl	6×10^{-6}			11.9		
Pb–6 at.% Tl	4.3×10^{-6}	5.7×10^{-6}	6.6×10^{-6}	11.8	11.6	1.7
Pb–17.4 at.% Tl	≈0	5.5×10^{-6}	7.6×10^{-6}	≈0	13.6	3.3

[a] See Refs. 11–13.
[b] $C_L=\frac{1}{2}(C_{1111}+C_{1122}+2C_{1212})$.
[c] $C=\frac{1}{2}(C_{1111}-C_{1122})$.

and therefore

$$a_{ij}\frac{\partial|\psi|^2}{\partial x_j}-\tfrac{1}{2}b_{ij}\frac{\partial|\psi|^4}{\partial x_j}=(a_{ij}-b_{ij})(a_{kl}-b_{kl})\frac{\partial^2u_k}{\partial x_j\partial x_l}.$$

Thus the system responds according to Eq. (5b) with an effective elastic tensor

$$(C_{ijkl}^s)_{H=0}=C_{ijkl}^n-(H_c^2/4\pi)\times[a_{ijkl}-\tfrac{1}{2}b_{ijkl}+(a_{ij}-b_{ij})(a_{kl}-b_{kl})]. \quad (7a)$$

This can be rewritten by means of the definitions (1) as

$$(C_{ijkl}^s-C_{ijkl}^n)_{H=0}=-(1/8\pi)[\partial^2(H_c^2)/\partial\epsilon_{ij}\partial\epsilon_{kl}], \quad (7b)$$

which is again the same relation as for type-I superconductors.

(b) H_a near H_{c2}: For simplicity we assume that the second-order coefficient of an expansion of $\langle|\psi|^2\rangle$ in powers of $(1-H_a/H_{c2})$ is negligibly small compared with the first-order coefficient $2\kappa^2/(2\kappa^2-1)\beta$. This was implicitly shown to be true by Lasher[9] in a calculation of the free energy and the magnetic moment near H_{c2}. We have, then

$$\langle|\psi|^2\rangle=\left[\left(1-\frac{2\kappa^2}{2\kappa^2-1}b_{kl}\epsilon_{kl}\right)\left(1-\frac{H_a}{H_{c2}}\right)+a_{kl}\epsilon_{kl}\right]\times2\kappa^2/(2\kappa^2-1)\beta,$$

$$\langle|\psi|^4\rangle=\beta\langle|\psi|^2\rangle^2,$$

with $\beta=1.16$ for the triangular flux line lattice.[10] Inserting this in (5b) and keeping only the linear terms in $(1-H_a/H_{c2})$, we get the effective elastic tensor

$$C_{ijkl}^s=C_{ijkl}^n-\frac{H_c^2}{4\pi}\left[a_{ijkl}\left(1-\frac{H_a}{H_{c2}}\right)+a_{ij}a_{kl}-a_{ij}b_{kl}\left(1-\frac{H_a}{H_{c2}}\right)\frac{4\kappa^2}{2\kappa^2-1}\right]\times\frac{2\kappa^2}{(2\kappa^2-1)\times1.16}. \quad (8)$$

Equation (8) indicates a jump of the elastic constants at H_{c2},

$$(\Delta C_{ijkl})_{H=H_{c2}}=(H_c^2/4\pi)a_{ij}a_{kl}[2\kappa^2/(2\kappa^2-1)\times1.16], \quad (9)$$

and a change of the elastic constants with field,

$$\left(\frac{\partial C_{ijkl}}{\partial H}\right)_{H=H_{c2}}=\frac{1}{H_{c2}}\times\frac{H_c^2}{4\pi}\left[a_{ijkl}-\frac{4\kappa^2}{2\kappa^2-1}a_{ij}b_{kl}\right]\times2\kappa^2/(2\kappa^2-1)\times1.16. \quad (10)$$

Equation (9) is correct independent of our assumption that $\langle|\psi|^2\rangle$ is linear in $(1-H_a/H_{c2})$.

If κ and the thermodynamic critical field H_c are known for a material, all the coefficients introduced in Eq. (1) can be obtained by elasticity measurements and a specific volume measurement. For cubic symmetry, $a_{11}^2=a_{22}^2=a_{33}^2$ are directly given by the jump of the bulk modulus at H_{c2} as shown in Eq. (9). According to Eq. (6b) a measurement of the volume difference between the normal state and the zero-field superconducting state provides then $b_{11}=b_{22}=b_{33}$. After the a_{ij}'s and b_{ij}'s are known, any pair of coefficients a_{ijkl} and b_{ijkl} is obtained through Eqs. (7a) and (10) by a measurement of the field dependence near H_{c2} and the total change between the normal and the zero-field superconducting state of the elastic constant C_{ijkl} with the same indices i, j, k, l. This method is of interest from the experimental point of view because direct measurement of the strain dependence of H_{c2} and κ is very difficult.

The only existing experiments from which some of our parameters can be obtained are the measurements of the sound velocity in Pb-Tl alloys by Alers.[11,12] From his data we obtain the numbers given in Table I.

INTERACTIONS BETWEEN FLUXOIDS AND ELASTIC DEFECTS

We start with an exact solution ψ_0, A_0, ϵ_{0ij} of the modified Ginzburg-Landau equations (3) and (4) and the modified elasticity equations (5a). The introduction of an internal strain source like a dislocation or other defects imposes new boundary conditions on the elasticity equations. Writing the total strain field ϵ_{ij} under these boundary conditions as $\epsilon_{ij}=\epsilon_{0ij}+\eta_{ij}$,

[9] G. Lasher, Phys. Rev. **140**, A523 (1965).
[10] W. H. Kleiner, L. M. Roth, and S. H. Autler, Phys. Rev. **133**, A1227 (1964).
[11] G. A. Alers, in Proceedings of the Conference on the Physics of Type-II Superconductivity, Cleveland, Ohio, 1964, Vol. 1, p. 82 (unpublished).
[12] G. A. Alers (private communication).

we obtain for η_{ij} the equation

$$(\partial/\partial x_j)[\eta_{kl}(C_{ijkl}-(H_c^2/4\pi)(a_{ijkl}|\psi|^2-\tfrac{1}{2}b_{ijkl}|\psi|^4))]-(H_c^2/4\pi)(\partial/\partial x_j)[\epsilon_{0kl}(a_{ijkl}(|\psi|^2-|\psi_0|^2)$$
$$-\tfrac{1}{2}b_{ijkl}(|\psi|^4-|\psi_0|^4))]-(H_c^2/4\pi)(\partial/\partial x_j)[a_{ij}(|\psi|^2-|\psi_0|^2)-\tfrac{1}{2}b_{ij}(|\psi|^4-|\psi_0|^4)]=0. \quad (5c)$$

By inserting $\epsilon=\epsilon_0+\eta$ in Eqs. (3) and (4), new equations for A and ψ are obtained. In these equations we consider the terms containing η_{ij} as a perturbation. The solution is obtained as a perturbation series

$$\psi=\psi_0+\sum_{n=1}^{\infty}\varphi_n,$$

$$A=A_0+\sum_{n=1}^{\infty}\alpha_n,$$

where φ_n and α_n are proportional to the nth power of η_{ij}. The equations for φ_1 and α_1 are, for example,

$$-\text{curl curl}\alpha_1=\alpha_1|\psi_0|^2+2A_0\,\text{Re}(\varphi_1\psi_0)+(1/\kappa)\,\text{Im}(\psi_0{}^*\nabla\varphi_1+\varphi_1{}^*\nabla\psi_0) \quad (3b)$$

and

$$[(i\nabla/\kappa)+A_0]^2\varphi_1+2A_1[(i\nabla/\kappa)+A_0]\psi_0=\varphi_1[1+a_{ij}\epsilon_{0ij}+\tfrac{1}{2}a_{ijkl}\epsilon_{0ij}\epsilon_{0kl}]-(\varphi_1|\varphi_0|^2+2\psi_0\,\text{Re}(\varphi_1\psi_0))$$
$$\times[1+b_{ij}\epsilon_{0ij}+\tfrac{1}{2}b_{ijkl}\epsilon_{0ij}\epsilon_{0kl}]+\eta_{ij}[(a_{ij}+a_{ijkl}\epsilon_{0kl})-|\psi_0|^2(b_{ij}+b_{ijkl}\epsilon_{0kl})]\psi_0. \quad (4b)$$

Inserting $\epsilon=\epsilon_0+\eta$, $\psi=\psi_0+\sum\varphi_n$, and $A=A_0+\sum\alpha_n$ in the free-energy equation (2), we obtain the energy of the solution with no defect and additional terms which we arrange in rising powers of η. Using the fact that A_0, ψ_0, ϵ_{0ij} are a solution of Eqs. (3), (4), and (5), we obtain the first-order term

$$\Delta E_1=\int\eta_{0ij}\left[C_{ijkl}{}^n\epsilon_{0kl}-\frac{H_c^2}{4\pi}(a_{ij}|\psi_0|^2+\tfrac{1}{2}b_{ij}|\psi_0|^4)\right]d^3r. \quad (11)$$

We have neglected higher orders of the very small parameter $S_{ijkl}(H_c^2/4\pi)$ in this result, and consequently for η_{ij} the strain field of the defect in normal material η_{0ij} may be taken, because the corrections from the terms containing $|\psi|^2$ in Eq. (5c) are also of the order $S_{ijkl}(H_c^2/4\pi)$.

For the second-order energy contribution we obtain a very lengthy expression containing φ_1, α_1, ϵ_0, and η. Since a solution of Eqs. (3b) and (4a) for φ_1 and α_1 is not known and is very difficult to obtain, we give here only a second-order formula for the simple case where for symmetry reasons the strain field of the defect η_{ij} is a pure shear. Then $a_{ij}\eta_{ij}$ and the cross products $a_{ijkl}\epsilon_{0ij}\eta_{kl}$ and $b_{ijkl}\epsilon_{0ij}\eta_{kl}$ are zero, and therefore the first-order corrections φ_1 and α_1 are zero. The second-order energy contribution in this case is

$$\Delta E_2=\int\eta_{ij}\eta_{kl}(\tfrac{1}{2}C_{ijkl}{}^n-(H_c^2/8\pi)(a_{ijkl}|\psi_0|^2-\tfrac{1}{2}b_{ijkl}|\psi_0|^4))d^3r, \quad (12)$$

where η_{ij} is the solution of Eq. (5c) with $\psi=\psi_0$. Higher orders of $S_{ijkl}(H_c^2/4\pi)$ have again been neglected in this result.

DISCUSSION

ΔE_1 can be interpreted as the first-order interaction energy between a fluxoid or a configuration of fluxoids characterized by ψ_0, A_0, ϵ_{0ij} and an elastic defect characterized by its strain field η_{0ij}. ΔE_2 contains the elastic energy of the defect itself, and therefore the energy $\Delta E_{2\infty}$ of the defect at a large distance from the fluxoid configuration has to be subtracted from ΔE_2 to obtain the second-order interaction energy. We compare now our general results with the explicit calculations of the interaction between defects and single fluxoids by previous authors. The expression in brackets in Eq. (11) can be interpreted as the stress field σ_{0ij} of a fluxoid. The first-order interaction energy is then

$$\Delta E_1=\int\eta_{0ij}\sigma_{0ij}d^3r.$$

This form, and the appropriate calculation of σ_{0ij} from Eq. (5a), is identical with the formalism used by Kramer and Bauer in their calculation of the first-order interaction between a fluxoid and an edge dislocation. But their result for σ_{ij} is only obtained from Eq. (5a) if the coefficient b_{ij} is assumed to be zero. Whether or not the corresponding difference in ΔE_1 is significant cannot be decided because no measurements are available from which a_{ij} and b_{ij} can be determined separately. Equation (5c) with $\psi=\psi_0$ and Eq. (12) are identical with the problem of a defect in a medium with local variations $\delta C_{ijkl}(\mathbf{r})$ of the elastic constants C_{ijkl}, where in our case $\delta C_{ijkl}=-(H_c^2/4\pi)(a_{ijkl}|\psi_0|^2-\tfrac{1}{2}b_{ijkl}|\psi_0|^4)$. The formalism used by Webb[4] to calculate the interaction between a fluxoid and a screw dislocation and, following the same scheme, by Toth and Pratt[5] to calculate the interaction between a fluxoid and the shear stress field around a coherent precipitate, correspond to use of our Eqs. (5c) and (12) with the additional assumption $b_{ijkl}=0$. Alternatively these calculations may be regarded as relying on the assumption that $|\psi|^2=|\psi|^4$ everywhere. Then the same factor $a_{ijkl}-\frac{1}{2}b_{ijkl}$ appears in Eqs. (5c) and (12) and in Eq. (7a) for $C_{ijkl}^s-C_{ijkl}^n$. Under either of these assumptions, the appropriate coefficients are available for niobium, vanadium, and tantalum from the measurements by Alers and Waldorf[13] on the changes of elastic constants between the normal and superconducting states $C_{ijkl}^s-C_{ijkl}^n$. More recent measurements by Alers on some lead-indium alloys have shown that in fact the terms in b_{ijkl} are much smaller than the terms in a_{ijkl} for these alloys (see Table I). However, the coefficients may be quite different in other materials and should be measured for each material separately. The basis of the aforementioned detailed calculations is confirmed by our more general approach although additional terms have been found. It should be pointed out however that if the strain field of a defect has dilatational components, the second-order interaction involves terms containing the first-order corrections φ_1, α_1 of ψ_0 and A_0 which are then nonzero. These terms can be comparable in magnitude with those given in Eq. (12). Since the first- and second-order interactions seem to be of the same magnitude for small fluxoid-defect distances,[3] reliable values of the total interaction energy for these defects, i.e., essentially for all defects *except* screw dislocations, cannot be obtained from the existing calculations. (Since our equations indicate no exact cancellation between the various terms contributing to the total energy, these calculations may then still give the right order of magnitude.)

[13] G. A. Alers and D. L. Waldorf, Phys. Rev. Letters **6**, 677 (1961).

VALIDITY OF THE METHOD

It should be emphasized that the validity of our calculations is not only restricted by the limitations of the two-parameter local Ginzburg-Landau theory but also by our implicit assumption that the local value of the free-energy parameters depends on only the local value of the strain and not on its derivatives. We expect this to hold only if the strain is nearly constant over regions of the size of the coherence length. Thus for instance the calculation of the interaction force between a fluxoid and a dislocation by means of this method is not correct if the fluxoid is near the dislocation core, but nevertheless may give a good estimate of the true interaction.

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