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Transport Coefficients of Degenerate Plasma*†

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An improved nonrelativistic calculation, based on the quantum Lenard-Balescu transport equation, is performed for the thermal and electrical conductivity of a plasma of highly degenerate, weakly coupled electrons and nondegenerate, weakly coupled ions. Dynamic shielding in the random-phase approximation is treated correctly, and both electron-ion (*ei*) and electron-electron (*ee*) collisions are included in the thermal conductivity. The argument that *ee* collisions are negligible, because the Pauli exclusion principle limits their effect to $(kT/E_F)^2$ less than that of *ei* collisions, is refuted in the case of the thermal conductivity. For temperatures of about 10^8°K and densities of 10^{26} to 10^{30} electrons/cc, appropriate to red-giant stellar cores, *ee* collisions reduce the thermal conductivity by 25 to 50%. However, *ee* collisions are insignificant in terrestrial solids. The thermal conductivity κ is given by $1/\kappa=1/\kappa_{ei}+1/\kappa_{ee}$, where κ_{ei} and κ_{ee} are conductivities determined by *ei* and *ee* collisions. $\kappa_{ei}\propto Tn/[\ln(1/\lambda_i)+C_{ei}]$, where $\lambda_i^2\ll 1$ is the ion weak-coupling parameter, and the correction C_{ei} involves dynamic shielding. If $\lambda^2\ll 1$ is the electron weak-coupling parameter, and $\gamma\equiv 4\lambda E_F/kT\gg 1$, then $\kappa_{ee}\propto\lambda^3 n^{7/3}T^{-1}$, instead of the usual logarithmic form. If $\gamma\ll 1$, then $\kappa_{ee}\propto T^2 n^{1/3}/[\ln(1/\gamma)+C_{ee}]$, with the temperature dependence contrary to Fermi-liquid theory.

1. INTRODUCTION: LENARD-BALESCU EQUATION

THIS paper presents an improved nonrelativistic calculation, based on the quantum Lenard-Balescu (LB) equation,[1–3] of electrical conductivity, thermoelectric coefficient, and especially thermal conductivity of a plasma of degenerate electrons and nondegenerate ions. The new effects discussed prove significant for the high-temperature, high-density plasma of stellar interiors, but not for metals and semiconductors at terrestrial temperatures.

Electrons and ions are each characterized by a "plasma" or weak-coupling parameter, essentially the ratio of average potential energy to average kinetic energy. For highly degenerate electrons with mass m, number density n, and Fermi energy and momentum E_F and p_F, we define the electron plasma parameter λ as

$$\lambda^2\equiv me^2/\pi\hbar p_F,$$

so that λ is related to the familiar parameter r_s by $\lambda^2=3^{-1/3}\pi^{-5/3}r_s$. The electrons are also characterized by the degeneracy parameter $\alpha\equiv E_F/kT$. For nondegenerate ions with mass M, charge Ze, and number density $n_i=n/Z$, we analogously define the plasma parameter as

$$\lambda_i^2\equiv(9\pi)^{-1/3}n_i^{1/3}Z^2e^2/kT=\tfrac{2}{3}\alpha Z^{5/3}\lambda^2.$$

The quantum-mechanical LB equation for electrons is[1–3]

$$\frac{\partial f_e}{\partial t}+\mathbf{v}_1\cdot\frac{\partial f_e}{\partial \mathbf{x}_1}+\dot{\mathbf{v}}_1\cdot\frac{\partial f_e}{\partial \mathbf{v}_1}=4m^{-2}\int d^3p_2d^3P\sigma_{ee}\delta(E_1+E_2-E_1'-E_2')\{f_e(\mathbf{p}_1')f_e(\mathbf{p}_2')[1-\tfrac{1}{2}(2\pi\hbar)^3f_e(\mathbf{p}_1)]$$
$$\times[1-\tfrac{1}{2}(2\pi\hbar)^3f_e(\mathbf{p}_2)]-f_e(\mathbf{p}_1)f_e(\mathbf{p}_2)[1-\tfrac{1}{2}(2\pi\hbar)^3f_e(\mathbf{p}_1')][1-\tfrac{1}{2}(2\pi\hbar)^3f_e(\mathbf{p}_2')]\}+m^{-2}\int d^3p_2d^3P\sigma_{ei}$$
$$\times\delta(E_1+E_2-E_1'-E_2')\{f_e(\mathbf{p}_1')f_i(\mathbf{p}_2')[1-\tfrac{1}{2}(2\pi\hbar)^3f_e(\mathbf{p}_1)]-f_e(\mathbf{p}_1)f_i(\mathbf{p}_2)[1-\tfrac{1}{2}(2\pi\hbar)^3f_e(\mathbf{p}_1')]\},\quad(1.1)$$

where $f_e(\mathbf{x},\mathbf{p},t)$ and $f_i(\mathbf{x},\mathbf{p},t)$ are the electron and ion distribution functions, normalized to

$$\int d^3p\, f_{e,i}(\mathbf{x},\mathbf{p},t)=n_{e,i}(\mathbf{x},t).$$

The momenta of the two colliding particles before and after collision are $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_1'$, $\mathbf{p}_2'$, respectively, and the momentum transfer is $\mathbf{P}\equiv\mathbf{p}_1'-\mathbf{p}_1=\mathbf{p}_2-\mathbf{p}_2'$. The reduced mass has been set equal to m in the electron-ion (*ei*) collision term, and to $\frac{1}{2}m$ in the electron-electron (*ee*) collision term. Although electrons may have any degree of degeneracy in Eq. (1.1), the ions have been assumed nondegenerate, so that $1-\frac{1}{2}(2\pi\hbar)^3f_i\to 1$.

The cross section σ_{ei} for *ei* collisions is given by

$$\sigma_{ei}(P,\Delta E)=\frac{4m^2e^4Z^2}{|P^2\epsilon(P/\hbar,\Delta E/\hbar)|^2},\quad(1.2)$$

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[1] R. Balescu, Phys. Fluids 4, 94 (1960).
[2] R. Guernsey, Phys. Rev. 127, 1446 (1962).
[3] R. Balescu, *Statistical Mechanics of Charged Particles* (Interscience Publishers, Ltd., London, 1963).

while the ee collision cross section is $\sigma_{ee}=\sigma_{\rm dir}+\sigma_{\rm ex}$, where

$$\sigma_{\rm dir}=\frac{m^2e^4}{|P^2\epsilon(P/\hbar,\Delta E/\hbar)|^2} \tag{1.3}$$

and

$$\sigma_{\rm ex}=\frac{m^2e^4}{P^2(\mathbf{p}_2-\mathbf{p}_1-\mathbf{P})^2|\epsilon(P/\hbar,\Delta E/\hbar)\epsilon(|\mathbf{p}_2-\mathbf{p}_1-\mathbf{P}|/\hbar,(E_2-E_1-\Delta E)/\hbar)|} \tag{1.4}$$

are the cross sections for direct and exchange scattering, respectively. $\Delta E\equiv E_1'-E_1=E_2-E_2'$ is the energy transfer, and $\epsilon(k,\omega)$ is the random-phase-approximation (RPA) dielectric function, given by

$$\epsilon(k,\omega)=1-\frac{4\pi e^2}{k^2}\sum_{\alpha=e,i}Z_\alpha^2 \times\int d^3p\frac{f_\alpha^{(\rm eq)}(\mathbf{p}-\frac{1}{2}\hbar\mathbf{k})-f_\alpha^{(\rm eq)}(\mathbf{p}+\frac{1}{2}\hbar\mathbf{k})}{\hbar\omega-\hbar\mathbf{k}\cdot\mathbf{p}/m_\alpha+i\delta}, \tag{1.5}$$

where $f_{e,i}^{(\rm eq)}$ are the equilibrium distribution functions. The dielectric function is conveniently written in the two forms

$$\epsilon\equiv 1+\lambda^2q^{-2}L(q,V)\equiv 1+\lambda_i^2q^{-2}L'(q,U), \tag{1.6}$$

where $q\equiv P/2p_F$ is a dimensionless momentum transfer, while $V\equiv\Delta E/Pv_F$ and $U\equiv(\Delta E/P)(M/2kT)^{1/2}$ are dimensionless wave velocities normalized to the typical electron and ion speeds, v_F and $\bar{v}_i$, respectively. The first form of ϵ is appropriate when electron shielding is dominant, the second when ion shielding dominates. L has real and imaginary parts L_r and L_i, and is a sum of the electron contribution $L^{(e)}$ and the ion contribution $L^{(i)}$.

If the electrons are highly degenerate, the Sommerfeld expansion can be used to evaluate $L^{(e)}$ as a series of powers of α^{-2}. The lowest-order, zero-temperature (Lindhard) result is[4]

$$L_r^{(e)}(q,V)=\frac{1}{2}+\frac{1-(q-V)^2}{8q}\ln\left|\frac{1+q-V}{1-q+V}\right| +\frac{1-(q+V)^2}{8q}\ln\left|\frac{1+q+V}{1-q-V}\right|, \tag{1.7}$$

$$\begin{aligned} L_i^{(e)}(q,V)&=\tfrac{1}{2}\pi V, & V&<1-q\\ &=\pi[1-(q-V)^2]/8q, & 1-q&\leqslant V<1+q\\ &=0, & 1+q&\leqslant V\,. \end{aligned} \tag{1.8}$$

In the static, long-wavelength limit, $L^{(e)}(0,0)=1$, giving Fermi-Thomas shielding. For nondegenerate ions, $L^{(i)}$ cannot in general be evaluated in closed form, but the long-wavelength limit

$$L^{(i)}(0,V)=\tfrac{2}{3}\pi^{-1/2}\alpha Z\int_{-\infty}^{\infty}\frac{dx\, xe^{-x^2}}{x-U+i\delta} \tag{1.9}$$

has been computed numerically by Fried and Conte.[5] The static limit $L^{(i)}(0,0)=\frac{2}{3}\alpha$ represents Debye shielding. $L_r^{(i)}(0,V)$ becomes negative at $U\approx 0.9$, reaches a minimum at $U\approx 1.5$, and for $U>2$ is approximated by the first term of an asymptotic series,

$$L_r^{(i)}(0,V)\to-\alpha/3U^2=-m/MV^2. \tag{1.10}$$

The derivation of the quantum LB equation assumes (a) weak coupling, i.e., λ, $\lambda_i\ll 1$, and (b) Born approximation, i.e., $\lambda_{\rm th}\ll r_{\rm ca}$, where $\lambda_{\rm th}\equiv\hbar/\bar{p}_e$ is the electron thermal wavelength, $r_{\rm ca}\equiv e^2/\bar{E}_e$ is the classical distance of closest approach, and $\bar{p}_e$ and $\bar{E}_e$ are the mean momentum and kinetic energy per electron. The Born approximation is always justified in a highly degenerate plasma. The LB equation is interpreted[6] simply as a Boltzmann equation for two-quasiparticle collisions, with the cross section given by Born approximation for the *dynamically shielded* Coulomb interaction. All the many-body correlations lie in the dielectric function. Since the gas of quasiparticles is dilute and weakly interacting, the usual complications of Fermi-liquid theory,[7,8] multi-quasiparticle collisions and quasiparticle interaction energy, can be neglected.

The nature of the LB equation will be clarified by a few comments on the shielding of various types of collisions. Note that $q\leqslant 1$ in all scattering events. Only collisions with $q\ll 1$ are significantly shielded; as a result we shall show that, to lowest order in λ^2 and λ_i^2, $L(q,V)$ can always be replaced by $L(0,V)$. In ee collisions, $V\leqslant 1$, while in ei collisions, where $\Delta E\leqslant 2p_F\bar{v}_i\ll kT$ is small because of the large ei mass ratio, one finds $V\ll 1$.

Shielding is qualitatively quite different in each of the three ranges of V,

(a) $$V^2<m/M\alpha, \tag{1.11a}$$

(b) $$m/M\alpha<V^2<V_0^2, \tag{1.11b}$$

(c) $$V_0^2<V^2, \tag{1.11c}$$

where V_0, the second zero of $L_r(0,V)$ on the real axis, is given approximately by

$$V_0=(mZ/3M)^{1/2}. \tag{1.12}$$

[4] J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **28**, 8 (1954).

[5] B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

[6] H. W. Wyld and D. Pines, Phys. Rev. **127**, 1851 (1962).

[7] L. D. Landau, Zh. Eksperim. i Teor. Fiz. **30**, 1058 (1956); **32**, 59 (1957) [English transls.: Soviet Phys.—JETP **3**, 920 (1957); **5**, 101 (1957)].

[8] A. A. Abrikosov and I. M. Khalatnikov, Rept. Progr. Phys. **22**, 329 (1959).

In region (a) one finds essentially static ion shielding (which dominates static electron shielding because of the large kinetic energy E_F of the degenerate electrons). In region (b), $L_r^{(i)}$ is negative, and in general dominates $L_r^{(e)}$. Consequently, the ion plasma resonance occurs, with dispersion relation

$$\omega^2 \cong \omega_i^2(1-3U^2/\alpha Z),$$

where $\omega_i=(4\pi n_i Z^2e^2/M)^{1/2}$ is the ion plasma frequency. Resonant scattering by exchange of a plasma oscillation is dominant in this regime. In region (c), where $L_r^{(i)}$ is negative but small (because heavy ions cannot participate in high-frequency phenomena), electron shielding is dominant. Although electron shielding is generally dynamic, the electron plasma resonance occurs for $V>1$, which is of no importance in the LB equation. Static electron shielding is a qualitatively reasonable approximation for $V<1$.

Since V is small in *ei* collisions, the dominant shielding is by ions (although it is necessary to include the electron contribution to L_i, i.e., to damping). Electron-electron scattering occurs principally in region (c), where shielding is primarily by electrons, but there is also a small contribution from the ion plasma resonance in region (b).

The transport theory of stellar interiors was originally developed by Marshak,[9] Mestel,[10] and Lee[11] (MML), using an ordinary two-body Boltzmann equation for *ei* scattering, with Born approximation for the *unshielded* Coulomb potential. In order to eliminate the long-range Coulomb divergence, MML arbitrarily cut off the potential at the mean interionic distance $(\frac{4}{3}\pi n_i)^{-1/3}$. Their result for thermal conductivity κ took the form

$$\kappa \propto T/\ln[2p_F/\hbar(\tfrac{4}{3}\pi n_i)^{1/3}] = \tfrac{1}{3}T/\ln(18\pi Z^{1/3}).$$

The logarithm of the ratio of cutoff distance to the mean electron wavelength is characteristic of quantum-mechanical plasma transport coefficients.

Spitzer,[12,13] Landshoff,[14] and others subsequently showed that when $\lambda_i \ll 1$, the appropriate cutoff (shielding) distance is the ionic Debye length $D_i=(kT/4\pi n_i \times Z^2e^2)^{1/2}$, rather than the much smaller $n_i^{-1/3}$. With the argument of the logarithm thus altered, thermal conductivity takes the form

$$\kappa \propto T/[\ln(1/\lambda_i)+C_{ei}].$$

Using the Kubo relations rather than a kinetic equation, Hubbard[15] has recently done a comprehensive study of transport theory of a degenerate electron gas, valid for ion coupling of any strength (but assuming weakly coupled electrons). He has verified that the MML theory is approximately correct for strongly coupled ions ($\lambda_i \gg 1$) and has calculated numerically the transition to the Spitzer form as ions become weakly coupled.

All of these authors assumed static shielding, precluding calculation of the subdominant correction C_{ei}, which depends upon dynamic effects. We shall calculate C_{ei} in this paper.

Langer,[16] in a series of papers directed toward terrestrial metals, has developed the transport theory of a degenerate electron gas subject to scattering by dilute impurities, including electron-electron correlations to all orders, at zero temperature. However, he as well as the previous authors, neglected *ee* scattering as a mechanism for thermal resistance, since heuristic reasoning indicates that the Pauli exclusion principle drastically restricts the phase space for *ee* collisions, reducing the probability of *ee* collisions by order α^{-2} (and entirely preventing *ee* collisions at absolute zero temperature). We shall see that these arguments are not entirely correct, and that under conditions prevalent in red giant stellar cores, *ee* collisions can reduce thermal conductivity by 25 to 50%. It is also of interest that the contribution of *ee* collisions, for sufficiently great degeneracy, takes an anomalous, nonlogarithmic form. There is, however, no significant correction to the electrical conductivity due to *ee* collisions, because *ee* collisions conserve electrical current.

2. PHYSICAL BASIS OF TRANSPORT IN A DEGENERATE PLASMA

In this section we shall anticipate the principal results of the paper by means of a simple analysis of the contribution of typical collisions to thermal resistivity (the reciprocal of thermal conductivity). In particular, we wish to explain (a) the unexpected significance of *ee* collisions, (b) the unusual form of the *ee* contribution to thermal conductivity (nonlogarithmic λ dependence for $\gamma \equiv 4\alpha\lambda \gg 1$, and proportionality to T^2 for $\gamma \ll 1$, instead of T^{-1} as predicted by the Abrikosov-Khalatnikov[8] Fermi-liquid theory), and (c) the negligibly small contribution of both direct *ee* scattering with large momentum transfer $P \sim p_F$ and exchange *ee* scattering. We shall see that the effects of the Pauli exclusion principle are somewhat more complex in a plasma, where interactions are predominantly weak and long range, than in a molecular gas. Throughout this section, we use the static shielding form of the cross sections.

The Pauli principle requires that an electron with momentum $p_1 \cong p_F$ can collide only with another electron whose momenta before and after collision, p_2 and p_2', both lie on the thermally "smeared-out" Fermi surface, i.e., $p_F(1-\alpha^{-1}) \lesssim p_2,\ p_2' \lesssim p_F(1+\alpha^{-1})$. Since the ratio of the momentum transfer P to the thermal width

[9] R. E. Marshak, Ann. N. Y. Acad. Sci. **41**, 49 (1941).
[10] L. Mestel, Proc. Cambridge Phil. Soc. **46**, 331 (1950).
[11] T. D. Lee, Astrophys. J. **111**, 625 (1950).
[12] R. S. Cohen, P. Routly, and L. Spitzer, Phys. Rev. **80**, 230 (1950).
[13] L. Spitzer and R. Härm, Phys. Rev. **89**, 977 (1953).
[14] R. Landshoff, Phys. Rev. **76**, 904 (1949); **82**, 442 (1951).
[15] W. B. Hubbard, Astrophys. J. **146**, 858 (1966).
[16] J. S. Langer, Phys. Rev. **120**, 714 (1960); **124**, 997 (1961); **124**, 1003 (1961); **127**, 5 (1962); **128**, 110 (1962).

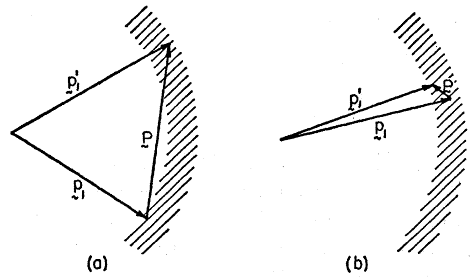

FIG. 1. The shaded area is the thermal width of the Fermi surface, of order $p_F\alpha^{-1}$. An electron has momenta $\mathbf{p}_1$ before and $\mathbf{p}_1'$ after collision; $\mathbf{P}$ is the collisional momentum transfer. (a) Type-I collision, $P\gg p_F\alpha^{-1}$. (b) Type-II collision, $P\lesssim p_F\alpha^{-1}$.

$p_F\alpha^{-1}$ of the Fermi surface is a crucial parameter, we distinguish between type-I collisions with $P\gg p_F\alpha^{-1}$ and type-II collisions with $P\lesssim p_F\alpha^{-1}$ (see Fig. 1). Figure 1(b) indicates that in a type-II collision, the Pauli principle demands only that $\mathbf{p}_2$ lie in the Fermi surface (which then implies that $\mathbf{p}_1'$ and $\mathbf{p}_2'$ also lie in the Fermi surface). Consequently, a fraction of order α^{-1} of all possible type-II collisions are permitted by the Pauli principle. On the other hand, for type-I collisions, the requirements that $\mathbf{p}_2$ and $\mathbf{p}_2'$ lie in the Fermi surface are independent, so that only a fraction of order $\alpha^{-2}(P/p_F)^{-1}$ of all possible collisions are permitted.

In a gas of strongly interacting molecules, typical collisions are type I with $P\sim p_F$, so that the Pauli principle restricts the phase space by order α^{-2}. But in a plasma, the dominant collisions are direct scatterings with $P\sim p_F\lambda$. If $\gamma\ll 1$, these collisions are type II, so that the Pauli principle restricts the phase space by only a factor α^{-1}. In the absence of degeneracy, stronger collisions, with $P\sim p_F$, would contribute to the subdominant term [smaller by order $1/\ln(1/\lambda)$]. Since the probability of such type-I collisions is reduced by order α^{-2}, one finds that only $P\lesssim p_F\alpha^{-1}$ contributes to the subdominant term, to lowest order in α^{-2}, i.e., the upper limit of integrations over P effectively lies at $p_F\alpha^{-1}$, rather than at p_F.

For $\gamma\gg 1$, the dominant collisions are type I, so that the exclusion principle reduces the collision probability by a factor $\alpha^{-2}\lambda^{-1}$. Since the probability of stronger collisions with $P\sim p_F$ is further reduced by order λ, it turns out that the contribution of these collisions is smaller by order λ^2, rather than merely $1/\ln(1/\lambda)$, and is again negligible.

Note that the exclusion principle imposes no restrictions on *ei* collisions, since energy transfer is much less than kT.

The significance of *ee* collisions in thermal resistance can be understood as follows. Collisions can reduce heat flow in either of two ways: (1) deflection of colliding electrons, or (2) transfer of energy from the faster to the slower particle. In the case of *ei* collisions in a plasma, both mechanisms are rather ineffective. Since the dominant scattering is small angle, elastic deflection of electrons occurs slowly, while the large *ie* mass ratio essentially prevents energy transfer. Similarly, in a *nondegenerate* plasma, *ee* scattering is predominantly small angle, and thus also $\Delta E\ll kT$; again, neither mechanism is efficient. It turns out that *ee* and *ei* collisions are of comparable importance in a nondegenerate plasma.

In the case of *ee* collisions in a sufficiently degenerate plasma, ΔE is small compared to the electron kinetic energy E_F, but is typically of order kT. Since the distribution function varies on the scale of kT, energy redistribution by *ee* collisions becomes an important mechanism for thermal resistance over a significant range of high temperatures and densities. For even greater degeneracy, the Pauli principle restrictions on phase space choke off *ee* collisions.

We now proceed to calculate, in several typical collisions, the component $\Delta Q'$ of energy flux change $\Delta\mathbf{Q}$ parallel to ∇T. We anticipate Sec. 3 by noting that in the presence of heat flux with no particle flux, to lowest order in α^{-2} the electron distribution function takes the form

$$f_e(\mathbf{x},\mathbf{p},t)=2(2\pi\hbar)^{-3}[f^-(p^2)-f^-(p^2)f^+(p^2)\mathbf{p}\cdot\nabla TA(p^2)/mT],$$

where f^- and f^+ are the Fermi functions defined in Eqs. (3.2), and $A(p^2)$ is an even function of $p^2-p_F^2$. Thus, for directions where $\mathbf{p}\cdot\nabla T>0$, the anisotropic part of the distribution consists of equal numbers of particles, with $p^2=p_F^2+(\delta p)^2$, and holes, with $p^2=p_F^2-(\delta p)^2$, or vice versa if $\mathbf{p}\cdot\nabla T<0$.

The dynamics of *ei* collisions is quite simple, since the electron can be regarded as scattering elastically off a fixed force center. In Fig. 2, two typical, equally probable scattering events are shown, one involving an electron with momentum $p_{1a}=p_F(1+\alpha^{-1})^{1/2}$, the other a hole with momentum $p_{1b}=p_F(1-\alpha^{-1})^{1/2}$. P, typically of order λ_ip_F, is the same in each case.

The loss in parallel energy flux resulting from the electron scattering of Fig. 2(a) is $\Delta Q_a'=-P^2p_{1a}/4m^2$, while in the hole scattering of Fig. 2(b), $\Delta Q_b'=+P^2p_{1b}/4m^2$.

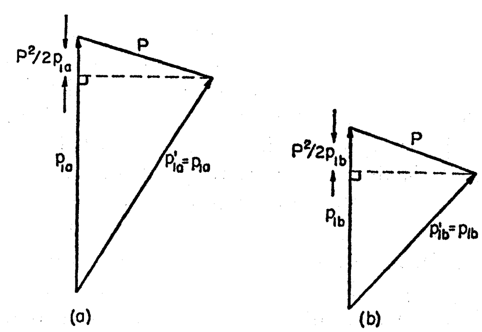

FIG. 2. Two equally probable scattering events. The temperature gradient is vertical in each case. (a) Elastic scattering of an electron off a stationary ion. Momenta before and after collision are $\mathbf{p}_{1a}$ and $\mathbf{p}_{1a}'$, respectively, and $\mathbf{P}$ is the momentum transfer. $p_{1a}=p_{1a}'=p_F(1+\alpha^{-1})^{1/2}$. (b) Elastic scattering of a hole off an ion. The momentum transfer P is the same, but the momenta of the hole before and after collision are $p_{1b}=p_{1b}'=p_F(1-\alpha^{-1})^{1/2}$.

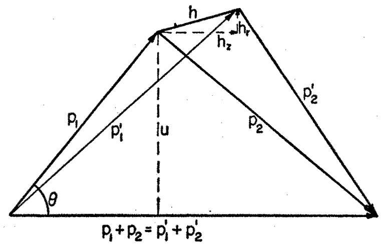

FIG. 3. An electron-electron collision. The momenta of the two electrons before collision are $\mathbf{p}_1$ and $\mathbf{p}_2$. The plane containing the momenta after collision, $\mathbf{p}_1'$ and $\mathbf{p}_2'$, has been rotated about $\mathbf{p}_1+\mathbf{p}_2$ until it coincides with the plane of $\mathbf{p}_1$ and $\mathbf{p}_2$.

Thus the net energy flux change in the two collisions is

$$\Delta Q_{ei}'=-P^2(p_{1a}-p_{1b})/4m^2\cong-(P^2/2p_F^2)\alpha^{-1}v_FE_F. \quad (2.1)$$

We now consider ee collisions (see Fig. 3). Throughout this section, we assume for simplicity that $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_1'$, and $\mathbf{p}_2'$ are all coplanar and that $\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_1'+\mathbf{p}_2'$ is *perpendicular* to ∇T. The component $\mathbf{u}$ of $\mathbf{p}_1$ perpendicular to $\mathbf{p}_1+\mathbf{p}_2$ is taken to be the same in each ee collision considered in this section, as is $\mathbf{p}_1+\mathbf{p}_2$. Both are of order p_F. In the regime $\gamma\gg1$, three typical ee collisions are defined by the following values of p_1, p_2, p_1', and p_2':

$$p_{1a}=p_F(1+\alpha^{-1})^{1/2},\quad p_{2a}=p_F(1-\alpha^{-1})^{1/2},\quad p_{1a}'=p_{2a}'=p_F; \quad (2.2a)$$

$$p_{1b}=p_{2b}=p_F,\quad p_{1b}'=p_F(1-\alpha^{-1})^{1/2},\quad p_{2b}'=p_F(1+\alpha^{-1})^{1/2}; \quad (2.2b)$$

$$p_{1c}=p_F(1+\alpha^{-1})^{1/2},\quad p_{2c}=p_F(1-\alpha^{-1})^{1/2},\quad p_{1c}'=p_F(1+2\alpha^{-1})^{1/2},\quad p_{2c}'=p_F(1-2\alpha^{-1})^{1/2}. \quad (2.2c)$$

Each of the collisions [(2.2a) and (2.2b)] results in

$$-\Delta Q_{ee}'\cong\alpha^{-1}(u/m)(p_F^2/2m)\sim\text{order } \alpha^{-1}v_FE_F. \quad (2.3)$$

Comparison with Eq. (2.1) reveals that $\Delta Q_{ee}'$ is larger than $\Delta Q_{ei}'$ by a factor $(P/p_F)^{-2}$, typically of order λ^{-2}. The large value of $\Delta Q_{ee}'$ is due to the significant energy transfer $\Delta E\sim kT$ in collisions between highly degenerate electrons.

Note that the collision (2.2c) would cancel the effect of collision (2.2a) if the two events were equally probable. However, (2.2c) is forbidden by the Pauli principle, since the final momenta are far from the Fermi surface.

Note also that, in collisions (2.2a) and (2.2b), the energy flux $\mathbf{Q}$ is almost entirely perpendicular to ∇T, and the changes in both the perpendicular component of $\mathbf{Q}$ and the absolute value $|\mathbf{Q}|$ are very small, of order $\alpha^{-2}v_FE_F$. However, thermal resistance is due to the much larger parallel component $\Delta Q'$, of order $\alpha^{-1}v_FE_F$. This situation has not been clearly understood in the past.[17]

We can now use these considerations to estimate the relative magnitudes of the contributions to thermal resistivity from ei and ee collisions, for $\gamma\gg1$. It is well known that ei collision integrals can be put in the form

$$I_{ei}\propto\int_0^{p_F}\frac{dP\,P^3}{(P^2+p_F^2\lambda_i^2)^2}\sim\ln(1/\lambda_i). \quad (2.4)$$

Since the contribution of a typical ee collision to thermal resistivity is greater than that of an ei collision with the same value of P, by a factor of order p_F^2/P^2, while phase-space considerations make the ei collision probability greater by order α^2P/p_F, the analogous ee form is[18]

$$I_{ee}\propto\alpha^{-2}p_F^3\int_0^{p_F}\frac{dP}{(P^2+p_F^2\lambda^2)^2}\sim\alpha^{-2}\lambda^{-3}\sim\frac{I_{ei}}{\alpha^2\lambda^3\ln(1/\lambda_i)}. \quad (2.5)$$

The physical situation is quite different if $\gamma\ll1$. The dominant collisions with $P\sim p_F\lambda$ are then type II, with $\Delta E\sim\lambda E_F\ll kT$. Two typical ee collisions, analogous to (2.2a) and (2.2c), are defined by

$$p_{1a}=p_F(1+\alpha^{-1})^{1/2},\quad p_{2a}=p_F(1-\alpha^{-1})^{1/2},\quad p_{1a}'=p_{1a}(1-\lambda)^{1/2},\quad p_{2a}'=p_{2a}(1+\lambda)^{1/2}, \quad (2.6a)$$

and

$$p_{1b}=p_F(1+\alpha^{-1})^{1/2},\quad p_{2b}=p_F(1-\alpha^{-1})^{1/2},\quad p_{1b}'=p_{1b}(1+\lambda)^{1/2},\quad p_{2b}'=p_{2b}(1-\lambda)^{1/2}. \quad (2.6b)$$

Because of Pauli-principle phase-space restrictions, the probability π_b of collision (2.6b) is slightly smaller than the probability π_a of collision (2.6a), i.e., $(\pi_a-\pi_b)/\pi_a\sim\alpha P/p_F$ [whereas (2.2c), analogous to (2.6b) for $\gamma\gg1$, is essentially forbidden by the Pauli principle]. In collision (2.6a), $\Delta Q_a'\sim-(P/p_F)v_FE_F$, while $-\Delta Q_b'$ for (2.6b) differs from $\Delta Q_a'$ only by order P/p_F. However, taking account of the two slightly different probabilities, we find a net

$$\Delta Q_{ee}'\sim-(P/p_F)^2\alpha v_FE_F \quad (2.7)$$

for the two collisions. Comparing Eqs. (2.7) and (2.1), we see that $\Delta Q_{ee}'$ is larger than $\Delta Q_{ei}'$ by order α^2. The exclusion principle reduces the phase space for type-II ee collisions by order α^{-1}. Thus, by comparison with Eq. (2.4),[18]

$$I_{ee}\propto\int_0^{p_F\alpha^{-1}}\frac{dP\,P^3\alpha}{(P^2+p_F^2\lambda^2)^2}\sim\alpha\ln(1/\gamma)\sim\alpha I_{ei}\frac{\ln\lambda_i}{\ln\gamma}, \quad (2.8)$$

where we have noted, in accordance with previous discussion, that type-I collisions with $P\gg p_F\alpha^{-1}$ should not be included in the integral. Note that the contribution of ee collisions is larger than that of ei collisions by order α, directly contradicting the usual oversimplified argument from the Pauli principle.

[17] See the last paper of Ref. 16.

[18] Static shielding of ei collisions is predominantly by ions, but only electrons can statically shield ee collisions. Therefore λ_i appears in I_{ei}, while λ appears in I_{ee}.

This discussion of the case $\gamma \ll 1 \ll \alpha$ can easily be modified to apply to nondegenerate electrons, the only significant change being that factors of α go over to order unity. The contributions of *ee* and *ei* collisions are then seen to be of the same order.

Finally, we wish to discuss exchange *ee* scattering. The factor $(P^2+\lambda^2\bar{p}_e^2)^{-2}$ in the direct scattering cross section is replaced by

$$(P^2+\lambda^2\bar{p}_e^2)^{-1}[(\mathbf{p}_2-\mathbf{p}_1-\mathbf{P})^2+\lambda^2\bar{p}_e^2]^{-1} \sim (P^2+\lambda^2\bar{p}_e^2)^{-1}\bar{p}_e^{-2},$$

since $(\mathbf{p}_2-\mathbf{p}_1-\mathbf{P})^2$ is generally order $\bar{p}_e^2$ when P is small. Thus, in the nondegenerate case,

$$I_{ex}\propto\int_0^{\bar{p}_e}\frac{dP\,P^3}{\bar{p}_e^2(P^2+\lambda^2\bar{p}_e^2)}\sim O(1)\sim\frac{I_{ee}}{\ln(1/\lambda)}. \quad (2.9)$$

However, when electrons are degenerate with $\gamma\ll 1$, the upper limit of integration is lowered by phase-space considerations, so that

$$I_{ex}\propto\alpha\int_0^{p_F\alpha^{-1}}\frac{dP\,P^3}{p_F^2(P^2+\lambda^2p_F^2)}\sim\alpha^{-1}\sim\frac{I_{ee}}{\alpha^2\ln(1/\gamma)}, \quad (2.10)$$

which is thus negligible. Finally, for degenerate electrons with $\gamma\gg 1$,

$$I_{ex}\propto p_F\alpha^{-2}\int_0^{p_F}\frac{dP}{p^2+\lambda^2p_F^2}\sim\alpha^{-2}\lambda^{-1}\sim\lambda^2 I_{ee}, \quad (2.11)$$

which is again negligible. Therefore, exchange scattering is always insignificant in a highly degenerate electron gas, because phase-space considerations suppress strong collisions with $P\sim p_F$.

3. CHAPMAN-ENSKOG SOLUTION

In this section we review the Uhlenbeck-Uehling-Chapman-Enskog solution[19,20] of the quantum statistical Boltzmann equation, establishing notation and introducing approximations based on the neglect of higher orders in $(m/M)^{1/2}$.

Since transport by ions is negligible compared with the light, highly degenerate electrons, ions may be assumed to be in local equilibrium, i.e.,

$$f_i(\mathbf{x},\mathbf{p},t)=n_i(2\pi MkT)^{-3/2}f^0(p^2;\,\mathbf{x},t),$$

where f^0 is the local Boltzmann distribution:

$$f^0(p^2;\,\mathbf{x},t)=\exp[-p^2/2MkT(\mathbf{x},t)].$$

The electrons are assumed to be close to local Fermi equilibrium;

$$f_e(\mathbf{x},\mathbf{p},t)=2(2\pi\hbar)^{-3}[f^-(p^2;\,\mathbf{x},t)-f^-f^+\Phi(\mathbf{x},\mathbf{p},t)], \quad (3.1)$$

$$f^-(p^2;\,\mathbf{x},t)=\{1+\exp[(p^2/2m-\mu(\mathbf{x},t))/kT(\mathbf{x},t)]\}^{-1}, \quad (3.2\text{a})$$

$$f^+=1-f^-, \quad (3.2\text{b})$$

$$\partial f^-/\partial(p^2/2m)=-f^-f^+/kT,$$

where μ is the chemical potential.

We proceed in standard fashion to linearize in Φ and separate terms in Φ linear in ∇T and $\mathbf{E}'$, where $-e\mathbf{E}'\equiv-e\mathbf{E}-\nabla P/n$ is the total force acting on an electron, due to electric field $\mathbf{E}$ and pressure gradient ∇P. We find that

$$\Phi(\mathbf{x},\mathbf{p},t)=(\mathbf{p}/m)\cdot\nabla TA(p^2)/T+(\mathbf{p}\cdot e\mathbf{E}'/mkT)D(p^2), \quad (3.3)$$

where $A(p^2)$ and $D(p^2)$ satisfy the linear integral equations

$$f^-f^+\{[p^2/2m-(5/3)\bar{E}_e]/kT\}\mathbf{p}=\mathbf{I}(pA), \quad (3.4)$$

$$f^-f^+\mathbf{p}=\mathbf{I}(\mathbf{p}D), \quad (3.5)$$

and the collision integrals $\mathbf{I}$, $\mathbf{I}_{ee}$, and $\mathbf{I}_{ei}$ are defined by

$$\mathbf{I}(\mathbf{p}A)\equiv\mathbf{I}_{ee}(\mathbf{p}A)+\mathbf{I}_{ei}(\mathbf{p}A), \quad (3.6\text{a})$$

$$\mathbf{I}_{ee}(\mathbf{p}A)\equiv 8m^{-2}(2\pi\hbar)^{-3}\int d^3p_2\,d^3p_1'\sigma_{ee}\times\delta(E_1+E_2-E_1'-E_2')f_1^-f_2^-f_1'^+f_2'^+\times[\mathbf{p}_1A_1+\mathbf{p}_2A_2-\mathbf{p}_1'A_1'-\mathbf{p}_2'A_2'], \quad (3.6\text{b})$$

$$\mathbf{I}_{ei}(\mathbf{p}A)\equiv n_i(2\pi MkT)^{-3/2}m^{-2}\int d^3p_2d^3P\,\sigma_{ei}\times\delta(E_1+E_2-E_1'-E_2')f_1^-f_1'^+f_2^0\times[\mathbf{p}_1A_1-\mathbf{p}_1'A_1']. \quad (3.6\text{c})$$

The transport coefficients S_{ij} are often defined by

$$\mathbf{J}=eS_{11}[e\mathbf{E}+T\nabla(\mu/T)]+eS_{12}\nabla T/T, \quad (3.7\text{a})$$

$$\mathbf{Q}=-S_{21}[e\mathbf{E}+T\nabla(\mu/T)]-S_{22}\nabla T/T, \quad (3.7\text{b})$$

where $\mathbf{J}$ and $\mathbf{Q}$ are electrical and thermal current, respectively. However, we shall find it more convenient to use the alternative definition

$$\mathbf{J}=e^2S_{11}'\mathbf{E}'+eS_{12}'\nabla T/T, \quad (3.8\text{a})$$

$$\mathbf{Q}=-eS_{21}'\mathbf{E}'-S_{22}'\nabla T/T-(5/3)(\mathbf{J}/e)\bar{E}_e. \quad (3.8\text{b})$$

The S_{ij} and S_{ij}' are related by

$$S_{11}=S_{11}',$$
$$S_{12}=S_{12}'+(5/3)\bar{E}_eS_{11}',$$
$$S_{22}=S_{22}'+(10/3)\bar{E}_eS_{12}'+[(5/3)\bar{E}_e]^2S_{11}'.$$

By the Onsager relation, $S_{12}=S_{21}$ and $S_{12}'=S_{21}'$. Electrical conductivity (at constant temperature) is $\sigma=e^2S_{11}$, while thermal conductivity κ, defined by

[19] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1961).
[20] E. A. Uehling and G. E. Uhlenbeck, Phys. Rev. **43**, 552 (1933); E. A. Uehling, *ibid.* **46**, 917 (1934).

TABLE I. The subdominant correction C_{ei} to the electron-ion collision integral, as a function of αZ and A/Z. The dependence on A/Z is seen to be very slight. If shielding is assumed static, $C_{ei}=\frac{1}{2}$.

αZ	$A/Z=1$	$A/Z=2$
5	0.346	0.346
10	0.222	0.221
20	0.137	0.136
40	0.0853	0.0845
60	0.0662	0.0653
80	0.0561	0.0552
100	0.0497	0.0489
200	0.0360	0.0353
400	0.0280	0.0274
600	0.0250	0.0244
800	0.0232	0.0227
1000	0.0221	0.0216

$Q=-\kappa\nabla T$ with $J=0$, is

$$\kappa=(S_{11}'S_{22}'-S_{12}'^2)/TS_{11}'=(S_{11}S_{22}-S_{12}^2)/TS_{11}. \quad (3.9)$$

The primed transport coefficients have the advantage that convective heat flux $-(5/3)\bar{E}_eJ/e$ is separated out; consequently, S_{12}' vanishes at high degeneracy, while S_{12} does not.

Defining the bracket relations

$$\{\mathbf{X}(\mathbf{p}),\mathbf{Y}(\mathbf{p})\}\equiv\{\mathbf{X}(\mathbf{p}),\mathbf{Y}(\mathbf{p})\}_{ei}+\{\mathbf{X}(\mathbf{p}),\mathbf{Y}(\mathbf{p})\}_{ee}, \quad (3.10a)$$

$$\{\mathbf{X}(\mathbf{p}),\mathbf{Y}(\mathbf{p})\}_{ee(ei)}=\{\mathbf{Y}(\mathbf{p}),\mathbf{X}(\mathbf{p})\}_{ee(ei)} \equiv 2(2\pi\hbar)^{-3}m^{-2}\int d^3p\mathbf{X}\cdot\mathbf{I}_{ee(ei)}(\mathbf{Y}), \quad (3.10b)$$

one can show that

$$S_{11}'=\tfrac{1}{3}\{\mathbf{p}D,\mathbf{p}D\}/kT, \quad (3.11)$$

$$S_{12}'=\tfrac{1}{3}\{\mathbf{p}A,\mathbf{p}D\}, \quad (3.12)$$

$$\kappa=\tfrac{1}{3}k[\{\mathbf{p}A,\mathbf{p}A\}-\{\mathbf{p}A,\mathbf{p}D\}^2/\{\mathbf{p}D,\mathbf{p}D\}]. \quad (3.13)$$

The equations for A and D are solved by polynomial expansion. Let

$$A(P^2)=\sum_0^\infty a_jP_j(p^2),$$

$$D(p^2)=\sum_0^\infty d_jP_j(p^2),$$

$$\alpha_k\equiv\{\mathbf{p}A,\mathbf{p}P_k\}=2(2\pi\hbar)^{-3}m^{-2}(kT)^{-1}\int d^3pp^2f^-f^+ \times[p^2/2m-(5/3)\bar{E}_e]P_k(p^2), \quad (3.14)$$

$$\delta_k\equiv\{\mathbf{p}D,\mathbf{p}P_k\}=2(2\pi\hbar)^{-3}m^{-2} \times\int d^3pp^2f^-f^+P_k(p^2), \quad (3.15)$$

$$a_{jk}=a_{jkei}+a_{jkee}=\{\mathbf{p}P_j,\mathbf{p}P_k\}, \quad (3.16a)$$

$$a_{jkei}=\{\mathbf{p}P_j,\mathbf{p}P_k\}_{ei}, \quad (3.16b)$$

$$a_{jkee}=\{\mathbf{p}P_j,\mathbf{p}P_k\}_{ee}, \quad (3.16c)$$

where $P_j(p^2)$ is a polynomial of order j in p^2, and a_j and d_j are unknown coefficients. Equations (3.4) and (3.5) then lead to the following equations for the coefficients a_j and d_j:

$$\sum_{j=0}^\infty a_ja_{jk}=\alpha_k, \quad \sum_{j=0}^\infty d_ja_{jk}=\delta_k. \quad (3.17)$$

Approximate solutions are found by truncating these equations:

$$\sum_{j=0}^n a_ja_{jk}=\alpha_k, \quad \sum_{j=0}^n d_ja_{jk}=\delta_k. \quad (3.18)$$

If the polynomials P_j are chosen properly, one expects the solutions $a_k^{(n)}$, $d_k^{(n)}$ of the truncated equations to converge quickly to the exact a_k and d_k as $n\to\infty$. This truncation procedure is also shown by Uhlenbeck and Uehling[20] to be the result of a variational principle.

In the case of Boltzmann statistics, the Sonine polynomials used by Chapman and Cowling[19] are most convenient, since they have both suitable orthogonality relations and a useful generating function. The subsequent algebra is then simplified by the fact that $\delta_k=0$ for $k\neq0$, and $\alpha_k=0$ for $k\neq1$. For Fermi statistics, Sonine polynomials are no longer appropriate, but we can ensure that $\delta_k=0$ for $k\neq0$ and $\alpha_k=0$ for $k>1$, by defining the P_k such that each P_k, $k\geqslant1$, is orthogonal to $P_0\equiv1$, and each P_k, $k\geqslant2$, is orthogonal to $[p^2/2m-(5/3)\bar{E}_e]/kT$, where two functions $X(p^2)$ and $Y(p^2)$ are orthogonal if

$$\int d^3pf^-(p^2)f^+(p^2)p^2X(p^2)Y(p^2)=0.$$

To complete the definition of the polynomials, we require that for $j>k\geqslant2$, P_j be orthogonal to P_k. If we define

$$t\equiv(p^2/2m-E_F)/kT\cong[p^2/2m-(5/3)\bar{E}_e]/kT,$$

and use the Sommerfeld expansion to evaluate the orthogonality integrals for high electron degeneracy, we then find, to lowest order[21] in α^{-1},

$$P_1=t, \quad (3.19)$$

$$P_2=t^2-\tfrac{1}{3}\pi^2, \quad (3.20)$$

$$P_3=t^3-(7/5)\pi^2t, \quad (3.21)$$

$$\delta_0=8\pi(2\pi\hbar)^{-3}m^{-1}kTp_F^3, \quad (3.22)$$

$$\alpha_0=4\pi^3(2\pi\hbar)^{-3}m^{-1}kTp_F^3\alpha^{-1}, \quad (3.23)$$

$$\alpha_1=(8/3)\pi^3(2\pi\hbar)^{-3}m^{-1}kTp_F^3. \quad (3.24)$$

Solving Eq. (3.18), we find, as in Chapman and Cowling,[22] the following determinantal expressions for

[21] Small corrections of order α^{-1} in these quantities appear in the transport coefficients only in order α^{-2}.

[22] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1961), Chap. 8.

the n-polynomial approximation to the transport coefficients:

$$\kappa=\tfrac{1}{3}k\frac{\begin{vmatrix} & & \delta_0 & \alpha_0\\ & & 0 & \alpha_1\\ a_{ij} & & \vdots & 0\\ & & \vdots & \vdots\\ & & 0 & 0\\ \delta_0 0 & \cdots 0 & 0 & 0\\ \alpha_0\alpha_1 0 & \cdots 0 & 0 & 0\end{vmatrix}}{\begin{vmatrix} & \delta_0\\ a_{ij} & 0\\ & \vdots\\ & 0\\ \delta_0 0\cdots 0 & 0\end{vmatrix}}\,, \tag{3.25}$$

$$S_{11}'=\tfrac{1}{3}(kT)^{-1}\frac{\begin{vmatrix} & \delta_0\\ a_{ij} & \vdots\\ & 0\\ \delta_0 0\cdots 0 & 0\end{vmatrix}}{|a_{ij}|}\,, \tag{3.26}$$

$$S_{12}'=\tfrac{1}{3}\frac{\begin{vmatrix} & \delta_0\\ a_{ij} & 0\\ & \vdots\\ & 0\\ \alpha_0\alpha_1 0\cdots 0 & 0\end{vmatrix}}{|a_{ij}|}\,, \tag{3.27}$$

where

$$|a_{ij}|\equiv\begin{vmatrix} a_{11} & a_{12}\cdots & a_{1n}\\ a_{21} & & \vdots\\ \vdots & & \\ a_{2n} & \cdots & a_{nn}\end{vmatrix}.$$

We now adopt *part* of the Lorentz model to obtain a Lorentzian approximation for the *ei* collision integral. Since $M\gg m$, it follows that $\bar{p}_e\ll\bar{p}_i$ and $\bar{v}_e\gg\bar{v}_i$, so that the maximum energy transfer in typical *ei* collisions is of order

$$\Delta E_{\max}=2\bar{p}_e\bar{v}_i=6kT(\bar{p}_e/\bar{p}_i)\ll kT.$$

The functions f^0, f^-, f^+, A, and D all vary with energy on the scale of kT. Thus we approximate E_1' by E_1 in $f_1'^+$, A_1', and D_1', reducing Eq. (3.6c) for $\mathbf{I}_{ei}$ to the simplified form

$$\mathbf{I}_{ei}(\mathbf{p}A)=-n_i m^{-2}(2\pi MkT)^{-3/2}f_1^-f_1^+A_1\int d^3p_2\, f_2^0 \times\int d^3P\,\mathbf{P}\sigma_{ei}(P,p_2)\delta(E_1+E_2-E_1'-E_2')\,. \tag{3.28}$$

The Lorentzian approximation neglects terms of order $\Delta E_{\max}/kT$, which is of order $(m/M)^{1/2}$ when both electrons and ions are nondegenerate. For degenerate electrons, $\Delta E_{\max}/kT\sim p_F/\bar{p}_i>6(m/M)^{1/2}$, but $\Delta E_{\max}/kT$ is still small as long as ions are nondegenerate.

The Lorentzian approximation assumes much less than the complete Lorentz model. We do not neglect either *ee* collisions or energy dependence in the cross section (i.e., dynamic shielding). Since $\hbar\omega_i/\Delta E_{\max}\sim\lambda_i\ll 1$, dynamic effects, particularly resonant exchange of an ion plasma oscillation during *ei* scattering, contribute significantly to the subdominant term of the expansion in λ_i.

The Lorentzian approximation not only facilitates the calculation of $\mathbf{I}_{ei}$, but also simplifies the structure of the Chapman-Enskog solution for highly degenerate electrons. The Lorentzian expression (3.28) will be shown in Sec. 4 to be of the form const$\times(f^-f^+A+\text{order}\,\alpha^{-1})$. Since P_k is orthogonal to P_0 for $k\geqslant 1$, this indicates that a_{0kei} is order α^{-1}. Moreover, $\mathbf{I}_{ee}(pP_0)=a_{0kee}=0$ by momentum conservation. Thus, to lowest order[21] in α^{-1}, a_{0k} may be set equal to zero. Equations (3.26) and (3.27) then reduce exactly to

$$S_{11}'=\tfrac{1}{3}(kT)^{-1}\delta_0^2/a_{00}=(64/3)\pi^2(2\pi\hbar)^{-6}\times m^{-2}kTp_F^6/a_{00}\,, \tag{3.29}$$

$$S_{12}'=\tfrac{1}{3}\delta_0\alpha_0/a_{00}=(64/3)\pi^4(2\pi\hbar)^{-6}\times m^{-1}k^3T^3p_F^4/a_{00}\,, \tag{3.30}$$

i.e., electrical conductivity and thermoelectric coefficient, according to the Lorentzian approximation at great degeneracy, are given exactly by the first polynomial and are completely independent of *ee* collisions.

The expression (3.25) for κ reduces to

$$\kappa^{[1]}=\tfrac{1}{3}k\alpha_1^2/a_{11}=(64/27)\pi^6(2\pi\hbar)^{-6}\times m^{-2}k^3T^2p_F^6/a_{11}\,, \tag{3.31a}$$

$$\kappa^{[2]}=\kappa^{[1]}(1-a_{12}^2/a_{11}a_{22})^{-1}\,, \tag{3.31b}$$

$$\kappa^{[3]}=\kappa^{[1]}(a_{11}a_{22}a_{33}-a_{11}a_{23}^2)/(a_{11}a_{22}a_{33}+2a_{12}a_{23}a_{31}-a_{11}a_{23}^2-a_{22}a_{31}^2-a_{33}a_{12}^2)\,, \tag{3.31c}$$

for truncation at one, two, or three polynomials, respectively.

All of the physics lies in the calculation of a_{ij}. Equations (3.16) for a_{ij} are conveniently rewritten as

$$a_{ij}=8\pi(2\pi\hbar)^{-3}m^{-1}kTp_F\int_{-\infty}^{\infty}dt\;\mathbf{p}P_i(t)\cdot\mathbf{I}(\mathbf{p}P_j)\,. \tag{3.32}$$

4. ELECTRON-ION COLLISIONS

In this section, we calculate $\mathbf{I}_{ei}(\mathbf{p}A)$ and a_{jkei}, given by Eqs. (3.28) and (3.32). We note that a first approximation is easily obtained by using the static dielectric function in the cross section. We then have

$$\mathbf{I}_{ei}^{\text{static}}(\mathbf{p}A)=8me^4Z_in_if_1^-f_1^+A_1$$
$$\times\int d^3p_1'(\mathbf{p}_1-\mathbf{p}_1')\delta(p_1^2-p_1'^2)[P^2+(2p_F\lambda_iZ^{1/3})^2]^{-2}$$
$$\cong 4\pi n_ime^4Z^2(\mathbf{p}_1/p_1^3)f_1^-f_1^+A_1$$
$$\times[\ln(Z^{1/3}/\lambda_i)-\tfrac{1}{2}+\text{order }\lambda_i^2]. \quad (4.1)$$

Even with dynamic shielding, the calculation of $\mathbf{I}_{ei}$ is fairly simple because of the Lorentzian approximation. Performing exactly the integration over p_2 in Eq. (3.28) yields

$$\mathbf{I}_{ei}(\mathbf{p}A)=n_iMm^{-2}(2\pi MkT)^{-1/2}f_1^-f_1^+A_1$$
$$\times\int d^3P\sigma_{ei}(\mathbf{P})\frac{\mathbf{P}}{P}\exp\left[-\frac{M}{2P^2kT}\left(\frac{\mathbf{p}_1\cdot\mathbf{P}}{m}+\frac{P^2}{2m}+\frac{P^2}{2M}\right)^2\right].$$

By symmetry, only the projection $\mathbf{p}_1\cdot\mathbf{P}/p_1\cong P^2/2p_1$ of $\mathbf{P}$ on $\mathbf{p}_1$ contributes to this integral. Thus, in spherical coordinates for $\mathbf{P}$ about $\mathbf{p}_1$,

$$\mathbf{I}_{ei}(\mathbf{p}A)=\pi n_iMm^{-2}(2\pi MkT)^{-1/2}(\mathbf{p}_1/p_1^2)f_1^-f_1^+A_1$$
$$\times\int_0^\infty dPP^3\int_0^\pi d\psi\,\sin\psi\sigma_{ei}(P,\psi)\exp[-(p_1P\cos\psi/m$$
$$+P^2/2m+P^2/2M)^2M/2P^2kT], \quad (4.2)$$

where ψ is the angle between $\mathbf{P}$ and $\mathbf{p}_1$. Noting that

$$\Delta E=(2p_1P\cos\psi+P^2)/2m,$$
$$U=(M/2\pi kT)^{1/2}(2p_1\cos\psi+P)/2m,$$
$$B\equiv p_1(M/2kT)^{1/2}/m\cong(\alpha M/m)^{1/2},$$

we have

$$\mathbf{I}_{ei}(\mathbf{p}A)=\pi^{1/2}n_im^{-1}(\mathbf{p}_1/p_1^3)f_1^-f_1^+A_1\left\{\int_{-B}^{B}dU\right.$$
$$\times e^{-U^2}\int_0^{2p_1-2m\bar{v}_iU}dP\,P^3\sigma_{ei}(P,U)+\int_{-\infty}^{-B}dU$$
$$\left.\times e^{-U^2}\int_{-2p_1-2m\bar{v}_iU}^{2p_1-2m\bar{v}_iU}dP\,P^3\sigma_{ei}(P,U)\right\}. \quad (4.3)$$

We use Eq. (1.2) for σ_{ei}, replacing $L'(q,U)$ by $L'(0,U)$. Since $L'(q,U)\cong L'(0,U)+\text{order }q^2$ for $q\ll1$, and shielding is significant only for $q\ll\lambda_i\ll1$, this results only in neglect of order λ_i^2. The integration over P in Eq. (4.3) can then be done exactly, yielding

$$\mathbf{I}_{ei}(\mathbf{p}A)=4\pi^{1/2}mn_iZ^2e^4(\mathbf{p}_1/p_1^3)f_1^-f_1^+A_1$$
$$\times(I_1+I_2+I_3+I_4+I_5), \quad (4.4a)$$

$$I_1=\ln(Z^{1/3}/\lambda_i)\int_{-B}^{B}dU\,e^{-U^2}, \quad (4.4b)$$

$$I_2=\frac{1}{4}\int_{-B}^{B}dU\,e^{-U^2}$$
$$\times\ln\left[\frac{[(1-B^{-1}U)^2+\lambda_i^2Z^{-2/3}L_r']+\lambda_i^4Z^{-4/3}L_i'^2}{L_r'^2+L_i'^2}\right], \quad (4.4c)$$

$$I_3=-\frac{1}{2}\int_{-B}^{B}dU\,e^{-U^2}\frac{L_r'}{|L_i'|}$$
$$\times\left[\tan^{-1}\frac{(1-B^{-1}U)^2+\lambda_i^2Z^{-2/3}L_r'}{\lambda_i^2Z^{-2/3}|L_i'|}\right.$$
$$\left.-\tan^{-1}\frac{L_r'}{|L_i'|}\right], \quad (4.4d)$$

$$I_4=\frac{1}{4}\int_{-\infty}^{-B}dU\,e^{-U^2}$$
$$\times\ln\left[\frac{[(1-B^{-1}U)^2+\lambda_i^2Z^{-2/3}L_r']^2+\lambda_i^4Z^{-4/3}L_i'^2}{[(1+B^{-1}U)^2+\lambda_i^2Z^{-2/3}L_r']^2+\lambda_i^4Z^{-4/3}L_i'^2}\right], \quad (4.4e)$$

$$I_5=-\frac{1}{2}\int_{-\infty}^{-B}dU\,e^{-U^2}\frac{L_r'}{|L_i'|}$$
$$\times\left[\tan^{-1}\left(\frac{(1-B^{-1}U)^2+\lambda_i^2Z^{-2/3}L_r'}{\lambda_i^2Z^{-2/3}|L_i'|}\right)\right.$$
$$\left.-\tan^{-1}\left(\frac{(1+B^{-1}U)^2+\lambda_i^2Z^{-2/3}L_r'}{\lambda_i^2Z^{-2/3}|L_i'|}\right)\right]. \quad (4.4f)$$

Although Eqs. (4.4a)–(4.4f) seem very complicated, we can now make obvious approximations that reduce $\mathbf{I}_{ei}$ to very simple form. Since $\alpha M/m>10^4$, both e^{-U^2} and $e^{-U^2}L_r'/|L_i'|$ are utterly negligible for $U^2>\alpha M/m$. Neglecting this region of integration, as well as terms of order λ_i^2 in I_2 and I_3, we can rewrite $\mathbf{I}_{ei}$ as

$$\mathbf{I}_{ei}=4\pi n_ime^4Z^2f_1^-f_1^+A_1(\mathbf{p}_1/p_1^3)$$
$$\times[\ln(Z^{1/3}/\lambda_i)-C_{ei}], \quad (4.5)$$

where

$$C_{ei}=\pi^{-1/2}\int_0^\infty dU\,e^{-U^2}[\tfrac{1}{2}\ln(L_r'^2+L_i'^2)$$
$$+(L_r'/L_i')\cot^{-1}(L_r'/L_i')]. \quad (4.6)$$

The subdominant term C_{ei} depends on the two parameters αZ and A/Z, where A is the ionic weight, but the dependence on A/Z is very slight (about 2%) over the physically significant range $1\leqslant/AZ\lesssim2$. Numerical results for C_{ei} are given in Table I. Note that the dominant (logarithmic) term of $\mathbf{I}_{ei}$ is given correctly by the static shielding result (4.1), but that C_{ei} is not.

Using Eqs. (4.5), (3.32), (3.19), (3.20), and (3.21), one has

$$a_{00}=a_{00ei}=32\pi^2(2\pi\hbar)^{-3}n_ie^4Z^2kT \times[\ln(Z^{1/3}/\lambda_i)-C_{ei}], \quad (4.7)$$

$$a_{11ei}=(32/3)\pi^4(2\pi\hbar)^{-3}n_ie^4Z^2kT \times[\ln(Z^{1/3}/\lambda_i)-C_{ei}], \quad (4.8)$$

$$a_{33ei}=(4608/175)\pi^8(2\pi\hbar)^{-3}n_ie^4Z^2kT \times[\ln(Z^{1/3}/\lambda_i)-C_{ei}]. \quad (4.9)$$

Note also that, since P_j is orthogonal to P_k,

$$a_{jkei}=0 \quad \text{for} \quad j\neq k. \quad (4.10)$$

According to Eqs. (3.29), (3.30) and (4.7), using $n_i=n/Z$, we have

$$S_{11}'=\frac{1}{4\pi}\frac{p_F^3}{m^2e^4Z[\ln(Z^{1/3}/\lambda_i)-C_{ei}]}, \quad (4.11)$$

$$S_{12}'=\frac{\pi}{4}\frac{p_F(kT)^2}{me^4Z[\ln(Z^{1/3}/\lambda_i)-C_{ei}]}, \quad (4.12)$$

and the one-polynomial result for thermal conductivity, including ei but not ee collisions, is, according to Eqs. (3.31a) and (4.8),

$$\kappa_{ei}^{[1]}=\tfrac{1}{12}\pi\frac{p_F^3k^2T}{m^2e^4Z[\ln(Z^{1/3}/\lambda_i)-C_{ei}]}. \quad (4.13)$$

The logarithm terms in Eqs. (4.11)–(4.13) have been given by Hubbard.[15] However, since his work is limited to static shielding, he could not correctly calculate C_{ei}. Inclusion of dynamic shielding effects reduces C_{ei} by 0.2 to 0.5, thus reducing the conductivities by about 10 to 30%.

5. ELECTRON-ELECTRON COLLISIONS

In order to calculate the collision integral $\mathbf{I}_{ee}$ of Eq. (3.6b), we shall transform to more suitable variables of integration, energies E_1', E_2', and E_2, normalized momentum transfer q, and angle θ between $\mathbf{p}_1$ and $\mathbf{p}_1+\mathbf{p}_2$ (see Fig. 3).

Since $\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_1'+\mathbf{p}_2'$, the bases of the two triangles $\mathbf{p}_1$, $\mathbf{p}_2$, $(\mathbf{p}_1+\mathbf{p}_2)$ and $\mathbf{p}_1'$, $\mathbf{p}_2'$, $(\mathbf{p}_1'+\mathbf{p}_2')$ coincide. Let φ be the angle between the planes of the two triangles. Figure 3 shows the plane of $\mathbf{p}_1'$, $\mathbf{p}_2'$ rotated by angle φ about $(\mathbf{p}_1+\mathbf{p}_2)$, so that all the vectors are coplanar. In a highly degenerate gas, all four p's are close to p_F; thus the vector $\mathbf{h}$ defined in Fig. 3 is a small quantity of order $p_F\alpha^{-1}$, whose components h_r and h_z are of orders $p_F\alpha^{-2}$ and $p_F\alpha^{-1}$, respectively.

It can be shown exactly that

$$\mathbf{I}_{ee}(\mathbf{p}A)=16\pi(2\pi\hbar)^{-3}m(\mathbf{p}_1/p_1^3)\int_0^{2\pi}d\varphi\int du\, dE_1'dE_2'\sigma_{ee}uf_1^-f_2^-f_1'^+f_2'^+[(p_1^2-u^2)^{-1/2}+(p_2^2-u^2)^{-1/2}]$$
$$\times\{p_1^2A(E_1)+[(p_1^2-u^2)^{1/2}(p_2^2-u^2)^{1/2}-u^2]A(E_2)-[\tfrac{1}{2}p_1^2+\tfrac{1}{2}p_2^2-u(u+h_r)(1-\cos\varphi)-\tfrac{1}{2}h^2]A(E_1')$$
$$+[u(u+h_r)\cos\varphi-(p_1^2-u^2)^{1/2}(p_2^2-u^2)^{1/2}+h_z(p_1^2-u^2)^{1/2}]A(E_2')\}, \quad (5.1)$$

where $u\equiv p_1\sin\theta$ (see Fig. 3),

$$h_z=\frac{p_1'^2-p_1^2}{(p_1^2-u^2)^{1/2}+(p_2^2-u^2)^{1/2}}, \quad (5.2)$$

$$h_r=-u+[u^2-h_z^2-2(p_1^2-u^2)^{1/2}h_z -p_1^2-p_1'^2]^{1/2}, \quad (5.3)$$

and the momentum transfer is given by

$$P^2=2u(u+h_r)(1-\cos\varphi)+h^2. \quad (5.4)$$

We drop higher order in α^{-1}, i.e., we set p_1, p_2, p_1' and p_2' all equal to p_F, and $u=p_F\sin\theta$. We use the dimensionless variables

$$t\equiv(E_1-E_F)/kT, \quad x\equiv(E_1'-E_F)/kT, \quad y\equiv(E_2'-E_F)/kT, \quad (5.5)$$

$$q'\equiv\sin\theta\sin\tfrac{1}{2}\varphi, \quad (5.6)$$

insert the cross section σ_{ee} for direct ee scattering, Eqs. (1.3) and (1.6), and specialize to the first polynomial, $A=P_1(E_1)=t$. Equation (5.1) then becomes

$$\mathbf{I}_{ee}(\mathbf{p}t)=16\pi(2\pi\hbar)^{-3}m^3e^4(kT)^2(\mathbf{p}_1/\mathbf{p}_1^4)\int_{-\infty}^{\infty}dx\,dy \times f^-(t)f^-(x+y-t)f^+(x)f^+(y)M(x,y,t), \quad (5.7a)$$

where

$$M(x,y,t)\equiv(t-x)\int_0^1dq'\int_{\sin^{-1}q'}^{\pi/2}d\theta\sin^3\theta(\sin^2\theta-q'^2)^{-1/2} \times|q^2+\lambda^2L(q,V)|^{-2}+(x-y)\int_0^1dq'q'^2\int_{\sin^{-1}q'}^{\pi/2}d\theta\sin\theta \times(\sin^2\theta-q'^2)^{-1/2}|q^2+\lambda^2L(q,V)|^{-2}. \quad (5.7b)$$

The integrals over x and y have been allowed to run from $-\infty$ to ∞; this introduces an error only of order $e^{-\alpha}$.

According to Eqs. (5.4) and (5.6),

$$q^2=q'^2(1+h_r/p_F\sin\theta)+h^2/4p_F^2. \quad (5.8a)$$

Since h_r/p_F and $(h/p_F)^2$ are both of order α^{-2}, Abrikosov and Khalatnikov[8] identify q with q' in their general Fermi-liquid theory. But for a plasma, the dominant contribution to Eq. (5.7) comes from small-angle scattering with $q'<q\leqslant\lambda<1$. For such scattering, the crucial denominator $|q^2+\lambda^2L|^2$ in Eq. (5.7b) is approximately $|h^2/4p_F^2+\lambda^2L|^2$. Thus the term $h^2/4p_F^2$ may be dropped from q^2 *only* if $h/2p_F\ll\lambda$, i.e., $\alpha^{-1}\ll\lambda$ or $\gamma\gg1$, which is a much stronger requirement than the degeneracy condition $\alpha\gg1$. In view of the discussion of Sec. 2, we see that the Abrikosov-Khalatnikov calculation assumes that the dominant collisions are type I, which is true for a plasma only if $\gamma\gg1$. If we make no assumptions about the magnitude of γ, but drop terms of higher order in α^{-2} in Eqs. (5.8a), (5.2), and (5.3), we find

$$q^2\cong q'^2+\left(\frac{x-t}{4\alpha\cos\theta}\right)^2. \tag{5.8b}$$

Because of the complicated form of $L(q,V)$, it is necessary to make further approximations to perform the integrals of Eq. (5.7b). Since, according to Sec. 2, only small values of q' contribute significantly to $\mathbf{I}_{ee}$, one finds that the second integral of Eq. (5.7b) is smaller by order λ^2 or α^{-2}, whichever is larger, and will thus be neglected. The factor $\sin^3\theta$ ensures that there is very little contribution to the first integral of (5.7b) from small values of θ. One can see from Fig. 3 that this occurs because, when $\theta\cong0$, the two colliding particles have initially almost equal momenta, in magnitude and direction. We therefore make the complementary small approximations of setting the lower limit of integration over θ in Eq. (5.7b) to zero and replacing $(\sin^2\theta-q'^2)^{1/2}$ by $\sin\theta$. It is then easy to replace q' by q as a variable of integration, to obtain[23]

$$M=(t-x)\int_0^{\pi/2}d\theta\,\sin^2\theta\times\int_{|s|/4\alpha\cos\theta}^1\frac{dq\,q}{|q^2+\lambda^2L(q,V)|^2(q^2-s^2/16\alpha^2\cos^2\theta)^{1/2}}$$
$$=\tfrac{1}{4}\pi(t-x)\int_{|s|/4\alpha}^1\frac{dq(1-s^2/16\alpha^2q^2)}{|q^2+\lambda^2L(q,V)|^2}, \tag{5.9}$$

where $s\equiv x-t$.

If we use the identity

$$f^-(t)f^-(x+y-t)f^+(x)f^+(y)=f^-(t)f^+(t)\times[f^-(x)+(e^{x-t}-1)^{-1}][f^-(y)-f^-(x+y+t)] \tag{5.10}$$

to perform the integration over y exactly, Eqs. (5.7a) and (5.9) become

$$\mathbf{I}_{ee}(\mathbf{p}t)=-4\pi^2(2\pi\hbar)^{-3}m^3e^4(kT)^2f^-(t)f^+(t)(\mathbf{p}_1/p_F^4)\times\int_{-\infty}^{\infty}ds\,s^2[(e^{s+t}+1)^{-1}+(e^s-1)^{-1}]\times\int_{|s|/4\alpha}^1dq[1-(s/4\alpha q)^2]|q^2+\lambda^2L(q,V)|^{-2}. \tag{5.11}$$

When we use (5.11) in (3.32) and perform the integration over t exactly, we have

$$a_{11ee}=32\pi^3(2\pi\hbar)^{-6}m^2e^4(kT)^3p_F^{-1}\int_0^{\infty}ds\,s^4e^s(e^s-1)^{-2}\times\int_{s/4\alpha}^1dq\,(1-s^2/16\alpha^2q^2)|q^2+\lambda^2L(q,V)|^{-2}. \tag{5.12}$$

Double integrals similar to Eq. (5.12) frequently occur in plasma transport theory. Usually $L(q,V)$ is an insensitive fuction of small q, and one approximates $L(q,V)$ by $L(0,V)$. For degenerate electrons $L_r(q,V)$ depends logarithmically on q near $V=1$, $q=0$; however, the contribution of this region where L_r is sensitive to q is smaller than the remaining integral by order $\lambda^2\ln\lambda$ or α^{-2}, whichever is smaller, and is thus negligible. Hence it is consistent to approximate $L(q,V)$ by $L(0,V)$, even for degenerate electrons.

Equation (5.12) can now be rewritten as

$$a_{11ee}=256(2\pi\hbar)^{-6}\pi^3m^{-1}e^4p_F^5J, \tag{5.13}$$

where

$$J=\int_0^1dV\,V^2(1-V^2)\int_0^{\infty}ds\,s^5\frac{e^s}{(e^s-1)^2}\times\frac{1}{|s^2+\gamma^2V^2L(V)|^2} \tag{5.14a}$$

$$=(4\alpha)^{-3}\int_0^{\infty}ds\,s^4\frac{e^s}{(e^s-1)^2}\times\int_{s/4\alpha}^1\frac{dq(1-s^2/16\alpha^2q^2)}{|q^2+\lambda^2L(V)|^2}. \tag{5.14b}$$

When electron shielding is dominant, Eq. (5.14a) clearly depends only on the single parameter γ.

Since analytic evaluation of J is complicated and is only possible in limiting cases, we first calculate J with static (electron) shielding, i.e., using $L(0,0)=1$, so that

$$J_{\text{stat}}=(4\alpha)^{-3}\int_0^{\infty}ds\,s^4e^s(e^s-1)^{-2}D(s), \tag{5.15}$$

[23] The small spurious region of integration with $q>1$, $\theta\cong\frac{1}{2}\pi$ has been neglected. This region, whose contribution to the integral is order λ^2, arises from the failure of (5.8b) near $\theta=\frac{1}{2}\pi$.

where

$$D(s)=\int_{s/4\alpha}^{1} dq(1-s^2/16\alpha^2q^2)(q^2+\lambda^2)^{-2}$$
$$=\tfrac{1}{2}\lambda^{-3}\{\tan^{-1}(\gamma/s)+3(s/\gamma)^2 \times[\tan^{-1}(\gamma/s)-(\gamma/s)]\}+O(1+s^2/\gamma^2) \quad (5.16a)$$
$$\rightarrow \tfrac{1}{4}\pi\lambda^{-3}, \quad \gamma\gg s, \quad (5.16b)$$
$$\rightarrow (2/15)(4\alpha/s)^3, \quad \gamma\ll s. \quad (5.16c)$$

The two different limiting forms (5.16b) and (5.16c) reflect the fact [discussed in Sec. 2, and in connection with Eq. (5.8)] that when $\gamma\gg1$ the dominant collisions are type I, with $q\gtrsim\lambda$, but when $\gamma\ll1$ they are type II with $q\gtrsim\frac{1}{4}s\alpha^{-1}$.

The integral

$$J_{\rm stat}=\tfrac{1}{2}\gamma^{-3}\int_0^\infty ds\, s^4e^s(e^s-1)^{-2} \times\{\tan^{-1}(\gamma/s)+3(s/\gamma)^2[\tan^{-1}(\gamma/s)-\gamma/s]\} \quad (5.17)$$

can be performed analytically in each of the two limits $\gamma\gg1$ and $\gamma\ll1$, using (5.16b) and (5.16c), and for $s^2\leqslant\gamma\ll1$, using the approximation

$$e^s(e^s-1)^{-2}\cong s^{-2}(1+\text{order}\tfrac{1}{12}s^2), \quad s<1. \quad (5.18)$$

Neglecting order γ^{-1}, we find

$$J_{\rm stat}\cong\pi^5/15\gamma^3, \quad \gamma\gg1; \quad (5.19)$$

neglecting order γ, we find

$$J_{\rm stat}\cong(2/15)[\ln(1/\gamma)+31/30], \quad \gamma\ll1. \quad (5.20)$$

For the complete range of γ, $J_{\rm stat}$ has been calculated numerically and tabulated in Table II. We see that (5.20) is an excellent approximation when $\gamma<1$, but that (5.19) is quantitatively accurate only for very large values of γ.

Returning to the evaluation of J with the correct dynamic shielding, we shall distinguish between the nonresonant region $V>V_0$, where there is no ion plasma oscillation, and the resonant region $V\leqslant V_0$, by letting

$$J=J_{\rm res}+J_{\rm np}, \quad (5.21a)$$

where

$$J_{\rm res}=\int_0^{V_0} dV\, H, \quad (5.21b)$$

$$J_{\rm nr}=\int_{V_0}^{1} dV\, H, \quad (5.21c)$$

$$H\equiv V^2(1-V^2)\int_0^\infty ds\, s^6e^s(e^s-1)^{-2} \times|s^2+\gamma^2V^2L(V)|^{-2}. \quad (5.21d)$$

We consider first the nonresonant region, which turns out to be far more important.

Since the ionic part $L_i^{(i)}$ of L_i is negligible in $J_{\rm nr}$, it is evident that if we use the approximation (1.10), $L(V)$ is independent of α. Thus $J_{\rm nr}$ depends primarily on γ, slightly on A/Z (through $L_r^{(i)}$), but not on α. We again consider first the two limiting cases $\gamma\ll1$ and $\gamma\gg1$, where analytic methods are applicable.

For $\gamma\ll1$, the physical situation is much like that of a nondegenerate plasma. We break up the integration over s in Eq. (5.21) into the two ranges $s^2<\gamma$, where we use (5.18), and $s^2>\gamma$, where

$$|s^2+\gamma^2V^2L|^{-2}\cong s^{-4}(1+\text{order }\gamma).$$

Neglecting higher order in γ, Eq. (5.21) then becomes

$$J_{\rm nr}\cong(2/15)[\ln(1/\gamma)+C_{ee}], \quad \gamma\ll1 \quad (5.22)$$

where

$$C_{ee}\equiv(23/15)-(15/8)\int_0^1 dV\, V^2(1-V^2) \times[\ln(L_r^2+L_i^2)+2(L_r/L_i)\cot^{-1}(L_r/L_i)]=1.30. \quad (5.23)$$

Effects of dynamic shielding first appear in the subdominant term C_{ee}, which differs slightly from the static shielding value 31/30. To understand this, note that for $\gamma\ll1$, Eqs. (5.21c) and (5.21d) are roughly of the form

$$J_{\rm nr}\propto\int_0^1 ds\, s^3|s^2+\gamma^2L|^{-2}\sim\int_0^{\alpha^{-1}} dq\, q^3 \times|q^2+\lambda^2L|^{-2}\sim-\ln(\gamma|L|), \quad (5.24)$$

so that dynamic-shielding effects appear only in the argument of a logarithm. This form is typical for plasma collision integrals, except for the small upper limit α^{-1} on q, due to the exclusion principle (see Sec. 2).

When $\gamma\gg1$, similar manipulations indicate that the static shielding result, Eq. (5.19), is correct[24] to order γ^{-1}, because the exclusion principle essentially prevents energy transfer (i.e., $V\ll1$ is the dominant range of integration). A crude mathematical form of $J_{\rm nr}$, analogous to Eq. (5.24), is

$$\int_0^1 ds\, |s^2+\lambda^2L|^{-2}\sim\lambda^{-3}. \quad (5.25)$$

In the intermediate region $\gamma\sim1$, $J_{\rm nr}$ takes roughly the form

$$\int_0^1 ds\, s^n|s^2+\lambda^2L|^{-2}\sim(\lambda|L|)^{n-3},$$

where $0<n<3$. Since L appears as a multiplicative factor rather than in the argument of a logarithm, dynamic-electron-shielding effects are not negligible and cannot be separated out. It is therefore necessary to

[24] Small, nonresonant effects of *ion* dynamic shielding occur for $V\cong V_0$. These effects increase with γ, but for physically significant values of γ (up to 100) never alter the thermal conductivity by more than about 1%.

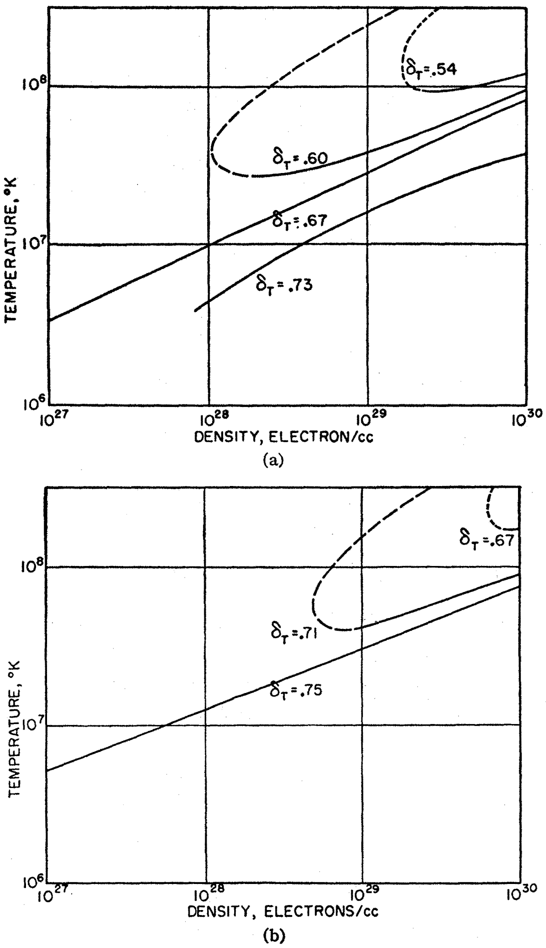

FIG. 4. The reduction in thermal conductivity due to *ee* collisions is given by $\delta_T \equiv \kappa/\kappa_{ei}$. Curves of specified values of δ_T are plotted in the temperature-density plane for (a) hydrogen and (b) helium plasmas. The dashed portions of the curves, where the extreme degeneracy assumption is suspect, are probably not physically realistic. It is believed that a correct treatment of partial degeneracy will show that δ_T always decreases with temperature and increases with density.

perform the integrations numerically. The numerical results are given in Table II. In this calculation, the exact expression for $L^{(e)}(0,V)$ has been used, but $L_i^{(i)}$ has been neglected and the asymptotic forms (1.10) and (1.12) have been used for $L_r^{(i)}$ and V_0. Inaccuracies due to these approximations in ion shielding are very small.

We now consider the integral $J_{\rm res}$ over the range of V where scattering is dominated by the ion plasma oscillation. The contribution to the double integral (5.21c) and (5.21d) from this resonance at $s^2=-\gamma^2V^2L_r$ is[25]

$$J_{\rm res}=\frac{1}{2}\int_0^{V_0} dV\ V^2(1-V^2)\gamma^2V^2L_re^{\gamma V|L_r|^{1/2}} \times(e^{\gamma V|L_r|^{1/2}}-1)^{-2}(L_r/L_i)\cot^{-1}(L_r/L_i). \quad (5.26a)$$

[25] The contribution from nonresonant *ee* scattering in the region $0\leqslant V<V_0$ is quite negligible. Making the approximations used to derive Eq. (5.28) ,we may estimate this contribution as $(3^{1/3}/27)\times(m_{ep}Z/A)^{3/2}[\frac{1}{3}\ln(3A)/m_{ep}Z\gamma^2)-\frac{1}{10}]$.

TABLE II. The nonresonant *ee* collision integral, as a function of γ. The exact integral $J_{\rm nr}$ is given for $A/Z=1$, 2. $J_{\rm stat}$ is the value obtained with the approximation of static shielding. $J_{\rm asymp}$ is the appropriate asymptotic expression for $J_{\rm nr}$: for $\gamma>1$, $J_{\rm asymp}=\pi^5/15\gamma^3$; for $\gamma\leqslant1$, $J_{\rm asymp}=(2/15)[\ln(1/\gamma)+1.30]$.

γ	$J_{\rm nr}(A/Z=1)$	$J_{\rm nr}(A/Z=2)$	$J_{\rm stat}$	$J_{\rm asymp}$
100	1.91×10^{-5}	1.85×10^{-5}	1.81×10^{-5}	2.04×10^{-5}
80	3.54×10^{-5}	3.47×10^{-5}	3.43×10^{-5}	3.98×10^{-5}
60	7.83×10^{-5}	7.72×10^{-5}	7.73×10^{-5}	9.45×10^{-5}
40	2.36×10^{-4}	2.34×10^{-4}	2.37×10^{-4}	3.19×10^{-4}
20	1.43×10^{-3}	1.42×10^{-3}	1.43×10^{-3}	2.55×10^{-3}
10	7.21×10^{-3}	7.20×10^{-3}	6.97×10^{-3}	2.04×10^{-2}
5	2.76×10^{-2}	2.76×10^{-2}	2.51×10^{-2}	1.63×10^{-1}
2	9.55×10^{-2}	9.55×10^{-2}	8.32×10^{-2}	
1	1.73×10^{-1}	1.73×10^{-1}	1.53×10^{-1}	1.73×10^{-1}
0.8	$2.0\ \times10^{-1}$	$2.0\ \times10^{-1}$	1.78×10^{-1}	2.03×10^{-1}
0.6	$2.4\ \times10^{-1}$	$2.4\ \times10^{-1}$	2.13×10^{-1}	2.41×10^{-1}
0.4	$3.0\ \times10^{-1}$	$3.0\ \times10^{-1}$	2.62×10^{-1}	2.96×10^{-1}

Numerical results for this integral are given in Table III, as a function of αZ, A/Z, for $\gamma=0$. Since $\gamma^2V^2|L_r|\ll1$, the dependence of (5.26) on γ is negligible (less than 3% for $\gamma\leqslant50$).

The following approximations permit qualitative analytic evaluation of (5.26). The asymptotic forms (1.10) and (1.12), and the approximation (5.18), are employed. The integration is restricted to the region of weakly damped ion plasma waves, very roughly defined by

$$4m_{\rm ep}/\alpha A<V^2<m_{\rm ep}Z/3A\,,$$

where $m_{\rm ep}$ is the ratio of electron to proton mass. In this region $L_r/L_i\ll-1$, so that $\cot^{-1}(L_r/L_i)\cong\pi$. We then find

$$J_{\rm res}\approx\tfrac{1}{6}m_{\rm ep}(Z/A)[\ln(\tfrac{1}{12}\alpha Z)-1]. \quad (5.26b)$$

This order-of-magnitude estimate of $J_{\rm res}$ is independent of γ and is much smaller than $J_{\rm nr}$ (because $m_{\rm ep}=1/1836$), for γ up to 50. At larger values of γ, the assumptions of nondegenerate and weakly coupled ions begin to fail.

The one-polynomial thermal conductivity $\kappa^{[1]}$, including both *ei* and *ee* collisions, is given by

$$1/\kappa^{[1]}=(1/\kappa_{ei}^{[1]})+(108me^4/\pi^3p_Fk^3T^2) \times(J_{\rm nr}+J_{\rm res}), \quad (5.27)$$

TABLE III. The resonant part of the *ee* collision integral, $J_{\rm res}$ is given as a function of αZ, for $A/Z=1$, 2, and $\gamma=0$. The dependence of $J_{\rm res}$ on γ is negligible.

αZ	$A/Z=1$	$A/Z=2$
1000	2.07×10^{-4}	1.02×10^{-4}
500	1.55×10^{-4}	7.62×10^{-5}
200	9.09×10^{-5}	4.43×10^{-5}
100	4.95×10^{-5}	2.36×10^{-5}
50	1.86×10^{-5}	8.32×10^{-6}
20	1.91×10^{-6}	7.04×10^{-7}
10	3.56×10^{-7}	1.27×10^{-7}

where κ_{ei}, J_{nr}, and J_{res} are given by Eqs. (4.13) and (5.21), and numerical values are given in Tables I–III. The quantity $\delta_T=\kappa/\kappa_{ei}$ indicates the relative reduction in thermal conductivity due to inclusion of *ee* collisions. In Fig. 4, values of δ_T are plotted for hydrogen and helium plasmas, over the region of the temperature-density plane where electrons are degnerate and nonrelativistic and ions are weakly coupled. In general, *ee* collisions reduce κ by about 25 to 50% over a wide range of temperatures of about 10^8 °K and densities up to 10^{30} electrons/cm^3, appropriate to red-giant stellar cores. The dashed portions of the curves, where the assumption of extreme degeneracy may be suspect, are not believed to be physically realistic. A complete study of the partially degenerate regime, now in progress, is expected to show that δ_T always decreases with temperature and increases with density.

It is well known[14] that for a nondegenerate plasma, the one-polynomial approximation to transport coefficients is incorrect by a factor of order 2, but that the two-polynomial results are accurate to order 1%. In the case of a highly degenerate electron gas, using the Lorentzian approximation for *ei* collisions, we have seen that $a_{ijei}=0$ for $i\neq j$, to lowest order[21] in α^{-1}, and that consequently one polynomial gives exact results for S_{11}' and S_{12}', to lowest order in α^{-2}. Furthermore, $a_{ijee}=0$ to lowest order[21] in α^{-1}, if $i+j$ is odd, since $\mathbf{I}_{ee}(\mathbf{p}P_j)$ is odd or even in t according to the parity of j. Thus, to lowest order in α^{-2}, the two- and three-polynomial expressions for thermal conductivity, Eqs. (3.31b) and (3.31c), reduce to

$$\kappa^{[2]}=\kappa^{[1]},\qquad \kappa^{[3]}=\kappa^{[1]}(1-a_{13}^2/a_{11}a_{33}). \tag{5.28}$$

The quantities in Eq. (5.28) that have not already been calculated are a_{13ee} and a_{33ee}. These calculations straightforwardly follow that of a_{11ee} (quite laboriously in the case of a_{33ee}). The results are

$$a_{13}=2(2\pi\hbar)^{-6}\pi^3m^5(kT)^6e^4p_F^{-7}\int_0^1 dV\ V^2(1-V^2)\times\int_0^\infty ds\ s^5(s^2-\tfrac{4}{5}\pi^2)e^s(e^s-1)^{-2}\times|s^2+\gamma^2V^2L|^{-2}, \tag{5.29}$$

$$a_{33ee}=\tfrac{2}{5}\pi^3(2\pi\hbar)^{-6}m^5(kT)^6e^4p_F^{-7}\int_0^1 dV\ V^2\times\int_0^\infty ds\ s^5e^s(e^s-1)^{-2}|s^2+\gamma^2V^2L|^{-2}\times[(31s^4+5\pi^2s^2+52\pi^4)-V^2(25s^4-5\pi^2s^2-12\pi^4)]. \tag{5.30}$$

It is not surprising that, upon evaluating these expressions numerically or, in limiting cases, analytically, we find the correction (5.28) to be less than 1%. The one-polynomial approximation is extremely accurate for degenerate electrons because the usual two-polynomial correction vanishes to lowest order in α^{-2}.

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