pair of particles. Symbolically, K can be related to V by the operator equation[29]

$$K=V+VGGK\,, \tag{7.10}$$

where G is the free-particle boson Green's function *in the presence of the Bose condensation.*[30] Under conditions of a strong Bose condensation ($N_0\gg 1$), G can be written, to a good approximation,

$$iG(\mathbf{r},t)=iG_0(\mathbf{r},t)+N_0e^{-i\mu t}\,, \tag{7.11}$$

where G_0 is the empty-space free-particle Green's function. The second term takes account of the enhanced scattering into, or out of, the $\mathbf{k}=0$ single-particle state resulting from the Bose condensation. Because of the presence of N_0 in (7.11), the scattering matrix K will be a function of density ρ and temperature T (since N_0 changes with ρ and T). One can take the potential V_p of this paper to be an effective scattering matrix, but then one must assume this effective V_p to change with temperature in some manner not considered in the earlier equations of this section.

The second oversimplification results from neglecting lifetime and renormalization effects, probably very important in He^4. The hyperspin formulation is inherently incapable of treating such effects. Exactly the same situation is true with the isospin formulation of superconductivity,[13] the latter being incapable of treating lifetime and renormalization effects in a strong-coupling superconductor. Schrieffer and his coworkers[31] have achieved considerable success in treating such effects by means of Nambu's formulation of superconductivity[32] in terms of matrix Green's functions. An analogous sort of treatment appropriate for the many-boson problem is undoubtedly necessary for an accurate calculation of superfluid helium at finite temperatures.

[29] K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1117 (1957).
[30] S. T. Beliaev, Zh. Eksperim. i Teor. Fiz. **34**, 417 (1958) [English transl.: Soviet Phys.—JETP **7**, 289 (1958)].

[31] For a detailed discussion, see J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, Inc., New York, 1964), Chap. 7.
[32] Y. Nambu, Phys. Rev. **117**, 648 (1960).

PHYSICAL REVIEW VOLUME 170, NUMBER 1 5 JUNE 1968

Low-Temperature Expansion of the Transport Coefficients and Specific Heat of Fermi Liquids*†

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The low-temperature expansion of the properties of Fermi liquids is discussed within the framework of Landau's theory. It is shown that the leading corrections to the low-temperature asymptotic forms come mainly from the scattering of quasiparticles with small energy and momentum transfer, and that they are proportional to T^3 for the inverse mean free times for thermal conductivity and spin diffusion, and to $T^3\ln T$ for the specific heat. The energy and damping of a quasiparticle are calculated from the same point of view. The special case of a nearly ferromagnetic Fermi liquid is considered explicitly, and comparison is made with results already obtained from a Green's-function approach, from the random-phase approximation, and from the concept of "paramagnons."

I. INTRODUCTION

It has been known for some time[1] that at low temperatures the properties of liquid He^3 approach their asymptotic Fermi-liquid behavior rather slowly, and recently it was suggested[2,3] that this is a consequence of the fact that the exchange interaction is almost strong enough to make the liquid ferromagnetic. The dominant processes were imagined to be those in which a particle emitted or absorbed a persistent spin fluctuation or "paramagnon."

In particular, it was found that at temperature T there were contributions proportional to $T^3\ln T$ in the specific heat[2] and to T^3 in the inverse mean free times for thermal conductivity and spin diffusion.[3] These terms appeared with large coefficients and seemed to be of the right order of magnitude to account for the observed properties of liquid He^3.

* Work performed under the auspices of U. S. Atomic Energy Commission.
† Part of this work was included in lectures at the University of Sussex, England, in the Spring of 1967 and in a talk at the Gordon Conference on "Dynamics of Quantum Solids and Liquids," Crystal Mountain, Washington, July 1967.
[1] J. C. Wheatley, in *Quantum Fluids*, edited by D. F. Brewer (North-Holland Publishing Co., Amsterdam, 1966).
[2] S. Doniach and S. Engelsberg, Phys. Rev. Letters **17**, 750 (1966); S. Doniach, S. Engelsberg, and M. J. Rice, in *Proceedings of the Tenth International Conference on Low-Temperature Physics, Moscow, 1966* (VINITI, Moscow, 1967); N. F. Berk and J. R. Schrieffer, Phys. Rev. Letters **17**, 443 (1966); in *Proceedings of the Tenth International Conference on Low-Temperature Physics, Moscow, 1966* (VINITI, Moscow, 1967).
[3] M. J. Rice, Phys. Rev. **159**, 153 (1967); **162**, 189 (1967).

The first calculations[2,3] were carried out in the random-phase approximation, but, in this paper, the problem will be considered by means of Landau's theory of a Fermi liquid.[4,5] The Boltzmann equation will be used to obtain the transport lifetimes and quasiparticle damping. This approach clearly displays the origin of the terms proportional to T^3 and shows that they do not require proximity to a ferromagnetic phase transition, but, rather, they are a general feature of Fermi systems and are a consequence of quasiparticle scattering with small energy and momentum transfer. The coefficients may be obtained in terms of the forward-scattering amplitude which determines the equilibrium properties, and they are particularly large when the system is strongly coupled or nearly ferromagnetic. In the latter case they have the same form as those obtained by Rice,[3] with different numerical factors.

The quasiparticle energy and the specific heat will be deduced from the expression for the damping. The result is very similar to that obtained by Amit, Kane, and Wagner,[6] using a Green's-function method. The difference appears to come from scattering with large momentum transfer, which is outside the scope of the methods used here. Also, in either approach the coefficient of the $T^3 \ln T$ term in the specific heat is incomplete, since it depends in part upon the energy and temperature dependence of the forward-scattering amplitude which determines the equilibrium properties, and this is not known.

In an almost ferromagnetic liquid, the important feature is that quasiparticle scattering with small energy and momentum transfer is analogous to critical scattering of light or neutrons from a system which is close to a second-order, or continuous-phase-transition point.[7] The essential difference is that, in a transport process, the scattered particle is identical to the scatterer, and the symmetry of the scattering amplitudes implies that the backward scattering is critical also.

At first sight, the calculation appears to be rather different from the paramagnon approach—in fact, it can be arranged to bring out either physical picture. The advantage of using Fermi-liquid theory, apart from the fact that it has been used very widely to understand the properties of liquid He^3, is that, instead of carrying out an approximate calculation on a model Hamiltonian with weak forces, one works with formally exact relationships involving quantities which have some connection with experiment. The other essential difference is that the particles which emit and absorb paramagnons are assumed to be the bare fermions,[2,3] whereas Fermi-liquid theory considers the scattering of quasiparticles. Thus every where we shall find that the full effective mass at the Fermi surface replaces the bare mass in the paramagnon results.[3] This is a feature of the work of Amit, Kane, and Wagner[6] also.

Expressions for the scattering amplitudes and the transport lifetimes will be given in Sec. II. Section III will be concerned with the low-temperature expansion of the transport coefficients, and Sec. IV with the quasiparticle damping and energy and the specific heat of the system.

II. SCATTERING RATES IN THE FERMI-LIQUID THEORY

In this section, some of the results of Landau's theory of a Fermi liquid will be discussed, in order to establish the notation and state the assumptions.

The transport coefficients are proportional to mean free times τ, and at sufficiently low temperatures they are determined by collisions of two quasiparticles. The momentum and energy will be denoted by $(\mathbf{p}_1,\epsilon_1)$ and $(\mathbf{p}_2,\epsilon_2)$ for the incoming quasiparticles and by $(\mathbf{p}_1',\epsilon_1')$ and $(\mathbf{p}_2',\epsilon_2')$ for the outgoing quasiparticles.

It has been shown[4,5] that at low temperatures

$$\tau^{-1}=\int_{-\infty}^{\infty} d\epsilon_2 \int_{-\infty}^{\infty} d\epsilon_1' \int_{-\infty}^{\infty} d\epsilon_2'\, \delta(\epsilon_1+\epsilon_2-\epsilon_1'-\epsilon_2') n_0(\epsilon_2) \times[1-n_0(\epsilon_1')][1-n_0(\epsilon_2')]W, \quad (1)$$

where $n_0(\epsilon)$ is the equilibrium Fermi function and W is a weighted average scattering rate which depends upon the transport process. For the thermal conductivity K, the viscosity η, and the spin diffusion D,

$$W_K=\frac{1}{12}\frac{m^{*3}}{\pi^5\hbar^6}\int_0^\pi d\phi\int_0^\pi d\theta\,\frac{\sin\theta}{\cos\frac{1}{2}\theta}(1-\cos\theta)w(\theta,\phi), \quad (2)$$

$$W_\eta=\frac{3}{32}\frac{m^{*3}}{\pi^5\hbar^6}\int_0^\pi d\phi\int_0^\pi d\theta\,\frac{\sin\theta}{\cos\frac{1}{2}\theta} \times(1-\cos\theta)^2\sin^2\phi\, w(\theta,\phi), \quad (3)$$

$$W_D=\frac{1}{8}\frac{m^{*3}}{\pi^5\hbar^6}\int_0^\pi d\phi\int_0^\pi d\theta\,\frac{\sin\theta}{\cos\frac{1}{2}\theta} \times(1-\cos\theta)(1-\cos\phi)w_D(\theta,\phi), \quad (4)$$

where m^* is the effective mass at the Fermi surface, θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$, and ϕ is the angle between the $(\mathbf{p}_1,\mathbf{p}_2)$ plane and the $(\mathbf{p}_1',\mathbf{p}_2')$ plane.

The scattering rates $2w$ and $2w_D$ may be expressed in terms of scattering amplitudes A_E and A_O, which are, respectively, even and odd under interchange of either the incoming particles on the outgoing particles. To lowest order in the temperature T, it is sufficient to calculate W with the incoming particles on the Fermi

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[7] T. Izuyama, D. J. Kim, and R. Kubo, J. Phys. Soc. Japan 18, 1025 (1963).

surface, and, indeed, this has been assumed in writing down Eqs. (2)–(4). In that case, A_E and A_O are, respectively, even and odd under change of sign of $\cos\phi$. Then, in Eq. (4),

$$2w_D=(2\pi/\hbar)|A_E+A_O|^2 \quad (5)$$

is the scattering rate for distinguishable particles, and, in Eqs. (2) and (3),

$$2w=(2\pi/\hbar)[3|A_O|^2+|A_E|^2] \quad (6)$$

is the scattering rate for indistinguishable particles with triplet and singlet spin weights. The alternative expression

$$2w=(2\pi/\hbar)[|A_E+A_O|^2+2|A_O|^2] \quad (7)$$

is often used. Here A_E+A_O and $2A_O$ are the scattering amplitudes for antiparallel spins and parallel spins, respectively. The two forms of w give the same results for τ_η and τ_K, since $A_EA_O^*$ and $A_E^*A_O$ are odd functions of $\cos\phi$ and so they do not contribute. However, since A_E and A_O are approximated by forward-scattering amplitudes, as is often done, the ϕ dependence is not treated correctly, and Eqs. (6) and (7) will lead to different values of τ_η and τ_K.

It is possible to obtain some information about A_O and A_E from the equilibruim properties. For this purpose, it is desirable to use the variables $\mathbf{p}_1$, $\mathbf{p}_2'$ and the momentum transfer $\mathbf{q}=\mathbf{p}_1'-\mathbf{p}_1=\mathbf{p}_2-\mathbf{p}_2'$ to specify the collision. For small $|\mathbf{q}|$, it turns out that $\mathbf{q}$ enters through the variable

$$s=m^*\omega/qp_F, \quad (8)$$

where p_F is the Fermi momentum and ω is the energy transfer

$$\omega=\epsilon_1'-\epsilon_1. \quad (9)$$

The spin-dependent scattering amplitude may then be written

$$\begin{aligned}A(1,2',s)&\equiv A(\mathbf{p}_1,\boldsymbol{\sigma}_1;\mathbf{p}_2',\boldsymbol{\sigma}_2;s)\\&=\tfrac{1}{2}(3A_O+A_E)+2(A_O-A_E)\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\end{aligned} \quad (10)$$

where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are spin operators for the scattering particles. It has been shown[4] that, for small $|\omega|$ and $|\mathbf{q}|$,

$$A(1,2',s)=\Phi(1,2')+\frac{N_0}{8\pi p_F^2}\mathrm{Tr}_{\sigma_3}\int d\mathbf{p}_3 \times\delta(p_3-p_F)\Phi(1,3)\frac{\hat{p}_3\cdot\hat{q}}{s-\hat{p}_3\cdot\hat{q}}A(3,2',s), \quad (11)$$

where N_0 is the density of states at the Fermi surface and is equal to $p_Fm^*/\pi^2\hbar^3$, and $\Phi(1,2')$ is the forward-scattering amplitude which determines the equilibrium properties. In Eq. (11) it is assumed that ω has a small positive imaginary part.

Conventionally, $\Phi(1,2')$ is written in the form

$$N_0\Phi(1,2')=F(\mathbf{p}_1\cdot\mathbf{p}_2')+Z(\mathbf{p}_1\cdot\mathbf{p}_2')\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2. \quad (12)$$

It is well known,[4] that the averages F_0 and Z_0 of F and Z over the Fermi surface are determined by the first sound velocity and spin susceptibility, respectively, and that the average of F weighted with $\mathbf{p}_1\cdot\mathbf{p}_2'$ over the Fermi surface is determined by m^*. This is the only empirical information available, but it may be supplemented by enforcing condition that A_O vanishes when $\mathbf{p}_1=\mathbf{p}_2$.

Equation (11) gives an exact description of screening of the equilibruim scattering amplitudes for small $|\epsilon_1'-\epsilon_1|$ and q. The paramagnon theory[2,3] uses the random-phase approximation for this purpose. The existing theory does not make any statement about larger values of $|\epsilon_1'-\epsilon_1|$ or q except that A_O and A_E may be determined for $\phi\sim\pi$ (which is the same as $q\sim 2p_F\sin\frac{1}{2}\theta$) from their symmetry under change of sign of $\cos\phi$.

In order to consider a nearly ferromagnetic Fermi liquid, explicit results will be quoted for the special case in which $F(\mathbf{p}_1\cdot\mathbf{p}_2')$ is zero and Z is a constant and equal to Z_0. Then Eq. (11) may be solved to give

$$2N_0A_O=-\tfrac{2}{3}N_0A_E=\tfrac{1}{4}Z_0/\{1+\tfrac{1}{4}Z_0[1+\alpha(s)]\}, \quad (13)$$

where

$$\alpha(s)=\tfrac{1}{2}s\ln[(s-1)/(s+1)]. \quad (14)$$

In a ferromagnetic Fermi liquid, $1+\frac{1}{4}Z_0<0$. In a nearly ferromagnetic Fermi liquid, $1+\frac{1}{4}Z_0$ is positive but very small, and the right-hand side of Eq. (13) becomes large as $s\rightarrow 0$. Note that Eq. (13) cannot be accurate, since A_O should vanish when $\mathbf{p}_1=\mathbf{p}_2$. It could be a good approximation, if this condition were satisfied and then, as θ increased, F and Z changed rapidly to the values assumed in Eq. (13). There is no reason to believe that this is the case in liquid He^3, and Eq. (13) should be regarded as an example for which the integrals may be evaluated explicitly.

Finally, the relation between q and ϕ will be required. It will be sufficient to take both particles on the Fermi surface, and then

$$q^2=2p_F^2\sin^2\tfrac{1}{2}\theta\left\{1-\left[1-\left(\frac{\epsilon_1'-\epsilon_1}{E_F\sin\theta}\right)^2\right]^{1/2}\cos\phi\right\}, \quad (15)$$

where E_F is the Fermi energy $p_F^2/2m^*$. For small ϕ and $|\epsilon_1'-\epsilon_1|$, Eq. (15) may be rewritten

$$s^2=\frac{(\epsilon_1'-\epsilon_1)^2}{4\phi^2E_F^2\sin^2\frac{1}{2}\theta+(\epsilon_1'-\epsilon_1)^2\sec^2\frac{1}{2}\theta}, \quad (16)$$

where q is related to s by Eq. (8).

III. LOW-TEMPERATURE EXPANSION OF TRANSPORT COEFFICIENTS

The thermal conductivity will be considered first, and we shall calculate W_K^0, which is the contribution to W_K in Eq. (2) from the region $0\leqq\phi\leqq\phi_0$, where ϕ_0 is small enough for Eqs. (11) and (16) to be valid. It will turn out that the quantities of interest will be independent of ϕ_0.

Since $n_0(\epsilon)$ is a function of ϵ/T, it follows from Eq. (1) that $\hbar/\tau$ may be obtained as a series in powers of T by writing W_K as a series in powers of $(\epsilon_1'-\epsilon_1)$. For this purpose, it is not possible to expand the integrand in Eq. (2), because, according to Eqs. (6), (10), (11), and (16), w depends upon ϵ_1 and ϵ_1' through s and so is a function of $(\epsilon_1'-\epsilon_1)^2/\phi^2$. The first term of the expansion is a constant and gives no trouble, but in all higher orders the ϕ integral in Eq. (2) diverges at the lower limit. The divergence is a consequence of the expansion, and it may be avoided by changing variables to $\bar{\phi}=\phi/|\epsilon_1'-\epsilon_1|$. Then, from Eq. (2),

$$W_K{}^0=\frac{1}{12}\frac{m^{*3}}{\pi^5\hbar^6}|\epsilon_1'-\epsilon_1|\int_0^\pi d\theta \times\frac{\sin\theta}{\cos\frac{1}{2}\theta}(1-\cos\theta)\int_0^{\phi_0/|\epsilon_1'-\epsilon_1|} d\bar{\phi}\, w(\theta,\phi). \quad (17)$$

Now, to obtain the lowest order in $|\epsilon_1'-\epsilon_1|$, the upper limit of the $\bar{\phi}$ integral may be extended to infinity, after integrating once by parts to ensure convergence:

$$W_K{}^0\approx\frac{1}{12}\frac{m^{*3}}{\pi^5\hbar^6}\int_0^\pi d\theta\frac{\sin\theta}{\cos\frac{1}{2}\theta}(1-\cos\theta) \times\left[\phi_0 w(\theta,\phi_0)-|\epsilon_1'-\epsilon_1|\int_0^\infty d\bar{\phi}\,\bar{\phi}\frac{\partial w}{\partial\bar{\phi}}\right]. \quad (18)$$

Here $w(\theta,\phi_0)$ is to be evaluated with $\epsilon_1'-\epsilon_1=0$. If Eq. (18) is substituted into Eq. (1) and ϵ_1 is set equal to the chemical potential μ, τ_K^{-1} has the form

$$\tau_K^{-1}=a_K T^2+b_K T^3. \quad (19)$$

In view of the symmetry about the Fermi surface, $\hbar/\tau_K$ is expected to be an even function of T, and in fact, in Eq. (19), T^3 should be written $|T|^3$. The appearance of an odd power of T stems from the factor $|\epsilon_1'-\epsilon_1|$ in Eq. (18), which, in turn, is a consequence of the divergence at $\phi=0$ of the expansion of $w(\theta,\phi)$ in powers of $(\epsilon_1'-\epsilon_1)$.

The T^2 term in Eq. (19) is the familiar result of Fermi-liquid theory, although only a small part of the coefficient a_K has been calculated, since only the region $0\leqq\phi\leqq\phi_0$ has been included. On the other hand, b_K is independent of ϕ_0 and the contribution from the neighborhood of $\phi=0$ is given exactly. Since $w(\theta,\phi)$ is unchanged if ϕ is replaced by $\pi-\phi$, there is an equal contribution from $\phi\sim\pi$, and, from Eqs. (1) and (18), the total coefficient of T^3 is given by

$$b_K'=-\frac{7\zeta(3)k_B{}^3m^{*3}}{12\pi^5\hbar^6}\int_0^\pi d\theta \times\frac{\sin\theta}{\cos\frac{1}{2}\theta}(1-\cos\theta)\int_0^\infty d\bar{\phi}\,\bar{\phi}\frac{\partial w}{\partial\bar{\phi}}, \quad (20)$$

where k_B is Boltzmann's constant and $\zeta(3)$ is Riemann's zeta function of argument 3. In Sec. IV, it will be seen that there may be an additional contribution from the neighborhood of $q=2p_F$, which cannot be calculated from the theory presented here.

We now turn to the calculation of the other transport coefficients. In the case of the viscosity, there is a factor $\sin^2\phi$ in the integrand of Eq. (3), and this removes the divergence from the first-order term in the expansion of $w(\theta,\phi)$ in powers of $(\epsilon_1'-\epsilon_1)^2/\phi^2$, but not from higher orders. The net result is that $\hbar/\tau_\eta$ is of the form $aT^2+cT^4+dT^5$. The value of c depends upon scattering for all ϕ, but d may be calculated in the same way as b_K' was. The expression for d will not be quoted, since it is not easy to determine it from experiment.

In the same way, the factor $(1-\cos\phi)$ which occurs in Eq. (4) removes the contribution to the T^3 term in τ_D^{-1} for the neighborhood of $\phi=0$, but not for $\phi\approx\pi$. Using Eq. (5) and the symmetry of A_E and A_O, the coefficient of the T^3 term in τ_D^{-1} is

$$b_D'=-\frac{7\zeta(3)k_B{}^3m^{*3}}{8\pi^5\hbar^6}\int_0^\pi d\theta \times\frac{\sin\theta}{\cos\frac{1}{2}\theta}(1-\cos\theta)\int_0^\infty d\bar{\phi}\,\bar{\phi}\frac{\partial w_D}{\partial\bar{\phi}}, \quad (21)$$

where

$$2w_D=(2\pi/\hbar)|A_E-A_O|^2. \quad (22)$$

Once again, there may be an additional contribution from the neighborhood of $q=2p_F$. Equations (19)–(21) constitute the main result, that the most important corrections to the low-temperature Fermi-liquid-theory expressions for the inverse mean free times for thermal conductivity and spin diffusion are proportional to T^3 and the coefficients may be calculated from the forward-scattering amplitudes which determine the equilibrium properties (with the proviso about the region $q\sim 2p_F$). The significance is that there may be experimentally detectable departures from the low-temperature asymptotic form $\tau\sim T^{-2}$, even when T is much less than the Fermi temperature $T_F=E_F/k_B$.

Note that it has not been assumed that the system is near a ferromagnetic phase-transition point, but, if it is, b_K' and b_D' may be evaluated explicitly, and they are very large in magnitude. In general, the integrals in Eqs. (20) and (21) must be carried out numerically, but, if A_O and A_E are given by Eq. (13) and $(1-\frac{1}{4}Z_0)\ll 1$, Eq. (20) becomes

$$b_K'\approx-\frac{7\zeta(3)\pi^3}{32}\frac{k_B}{\hbar T_F{}^2}\left|\frac{\frac{1}{4}Z_0}{1+\frac{1}{4}Z_0}\right|^3, \quad (23)$$

and from Eqs. (21) and (22), $b_D'=2b_K'$. Both the numerical coefficient and the factor $|1+\frac{1}{4}Z_0|^{-3}$ make b_K' and b_D' large, and the expansion of τ^{-1} in powers of T may not converge very rapidly. In fact, the whole theory may not be valid if the system is too close to a ferromagnetic instability. The scattering of a quasiparticle is analogous to critical scattering of light or neutrons from a system near a second-order phase-transition point, except that identity of the particles leads to critical scattering in the backward direction, as well as at small angles. The method used here is equivalent to the Ornstein-Zernicke theory, which is known to be inadequate very near the transition.[8]

Equation (23) is very similar to the result first obtained by Rice[3] from the paramagnon theory, although, for several reasons, his numerical coefficient was different. He used $2w_D$ instead of $2w$ as the scattering rate for K and omitted the region $\phi\sim 0$. An approximate spectral function affected the numerical coefficients and removed the T^4 term in η. Also, Rice used a variation principle to solve the Boltzmann equation, and this leads to an additional, energy-dependent, factor in the integrand in Eq. (1). The difference is a consequence of approximations made in both approaches. As a result, the T^2 term was multiplied by 12/5 in τ_k^{-1} and by $\frac{4}{3}$ in τ_D^{-1} and τ_η^{-1}. The same factors have been found by Baym and Ebner.[9] A calculation based upon a more accurate solution of the Boltzmann equation is in progress and so the "variational" values of b_K' and b_D' will not be quoted here. This question does not change the origin of the T^3 terms except that there will be a small T^3 term in the viscosity.

There is one other significant difference. In the paramagnon theory, it is insisted that T_F in Eq. (23) should be calculated with the bare mass m of the He^3 atoms. From the point of view of Fermi-liquid theory, we can see no reason why this should be so, and the method used in the present paper indicates that the effective mass m^* should be used.

We have made no attempt to determine a_K and a_D or to calculate the transport coefficients at higher temperatures, since this cannot be done without making further assumptions about the behavior of the scattering amplitudes when $T\neq 0$ and $|\omega|$ and $|\mathbf{q}|$ are not small. Rice[3] used a weak effective interaction and the random-phase approximation for this purpose. In the special case of a δ-function interaction, this amounts to using the $s=0=q$ scattering amplitudes for estimating a_K and a_D. As mentioned above, there is an ambiguity here for a_K, since either Eq. (6) or Eq. (7) could be used.

Liquid He^3 has an exchange integral of the appropriate sign and is not too close to a ferromagnetic instability, so the theory should apply. Experimentally, the thermal conductivity is most accurately known,[10] and at very low temperatures $(\tau_K T^2)^{-1}$ does seem to have a term which is linear in T, and has a large coefficient. It is not reasonable to compare Eq. (23) directly with experiments, since $\frac{1}{4}Z_0(1+\frac{1}{4}Z_0)^{-1}$, which varies from -2 to -2.6 as the pressure is increased from 0.28 to 27 atm, is scarcely large enough for the approximation to Eq. (20) to be valid. It is possible to make use of the other limited experimental information about $F(\mathbf{p}_1,\mathbf{p}_2')$ and $Z(\mathbf{p}_1,\mathbf{p}_2')$ and to evaluate the integrals in Eqs. (20) and (21) numerically. This has not been done at present because the doubt about solution of the Boltzmann equation, mentioned above, is particularly serious for the thermal conductivity, which happens to be the transport coefficient which is known best from experiment.[10] Nevertheless, Eq. (20) does give the right order of magnitude for b_K, but, since He^3 is a strongly coupled Fermi liquid, the existence of the effect and the precise magnitude of the coefficient cannot be attributed solely to the proximity of liquid He^3 to a ferromagnetic instability.

IV. DAMPING AND ENERGY OF A QUASIPARTICLE

The behavior of the scattering amplitudes for small $|\epsilon_1'-\epsilon_1|$ and q also gives rise to a $T^3\ln T$ term in the low-temperature specific heat,[2] and Amit, Kane, and Wagner[6] have used a Green's-function method to calculate its coefficient. In this section, the problem will be considered from the point of view of Fermi-liquid theory. The damping of a quasiparticle will be obtained from the Boltzmann equation, and this will lead to an expression for the quasiparticle energy and hence the specific heat.

Suppose that the system is in equilibrium, and then, at time $t=0$, a quasiparticle with momentum $\mathbf{p}_1$ is added. At temperatures which are low enough for two-body collisions to dominate, the relaxation time of this nonequilibrium distribution is described by the Boltzmann equation[4,11]

$$\frac{\partial n_1}{\partial t}=-\frac{2}{(2\pi\hbar)^6}\int d\mathbf{p}_2 d\mathbf{p}_1' d\mathbf{p}_2' \times\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')\delta(\epsilon_1+\epsilon_2-\epsilon_1'-\epsilon_2')w \times[n_1n_2(1-n_1')(1-n_2')-(1-n_1)(1-n_2)n_1'n_2'], \quad (24)$$

where w is the symmetrized scattering rate of Eq. (6) and the variables in the two-body collision are the same as those defined in Sec. II. The notation n_1 and n_1' is used for the nonequilibrium Fermi function for momentum $\mathbf{p}_1$ and $\mathbf{p}_1'$, respectively. To order $1/N$ (where N is the number of particles), for $\mathbf{p}\neq\mathbf{p}_1'$, n may be replaced by the equilibrium function n_0 and ϵ may be taken to be the equilibrium energy. Then, using the properties of n_0, Eq. (24) may be rewritten

$$\partial n_1/\partial t=-(n_1-n_0(\epsilon_1))/\tau_1, \quad (25)$$

[8] See, for example, *Critical Phenomena*, edited by M. S. Green and J. V. Sengers, Natl. Bur. Std. (U. S.) Misc. Publ. No. **273** (1965).

[9] G. Baym and C. Ebner (private communication).

[10] J. C. Wheatley, Phys. Rev. **165**, 304 (1968).

[11] P. Morel and P. Nozieres, Phys. Rev. **126**, 1909 (1962).

where

$$\frac{1}{\tau_1}=\frac{2}{(2\pi\hbar)^6}\int d\mathbf{p}_2 d\mathbf{p}_1' d\mathbf{p}_2' \times\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')\delta(\epsilon_1+\epsilon_2-\epsilon_1'-\epsilon_2')w \times[n_0(\epsilon_2)(1-n_0(\epsilon_1'))(1-n_0(\epsilon_2')) +(1-n_0(\epsilon_2))n_0(\epsilon_1')n_0(\epsilon_2')]. \quad (26)$$

Now it is possible to remove the momentum δ function by integration over $\mathbf{p}_2'$ and then to introduce the variables θ and ϕ used in Secs. II and III, to find that τ_1 is given by Eq. (1), with[11]

$$W=\frac{1}{8}\frac{m^{*3}}{\pi^5\hbar^6}\int_0^\pi d\phi\int_0^\pi d\theta\frac{\sin\theta}{\cos\frac{1}{2}\theta}w(\theta,\phi), \quad (27)$$

and the temperature dependence of τ_1 is similar to that of τ_K, discussed in Sec. III. However, the integrations will be carried out in a rather different way, in order to obtain the real part of the quasiparticle energy and to make a comparison with the results obtained by Amit, Kane, and Wagner.[6]

First, the momentum δ function is removed by integration over $\mathbf{p}_2$, and then the $\mathbf{p}_1'$ integration may be expressed in bipolar coordinates:

$$\int d\mathbf{p}_1'=\frac{2\pi}{p_1}\int_0^\infty p_1'dp_1'\int_{|p_1'-p_1|}^{p_1'+p_1} q\,dq, \quad (28)$$

where $\mathbf{q}$ is the momentum transfer, so that

$$\mathbf{p}_1'=\mathbf{p}_1+\mathbf{q}\,,\qquad \mathbf{p}_2=\mathbf{p}_2'+\mathbf{q}. \quad (29)$$

The Fermi functions in Eq. (26) imply that the dominant contribution to the integral comes for p_1' near p_F, and so, in Eq. (28), $p_1'dp_1'$ may be replaced by $m^*d\epsilon_1'$ and $|p_1'-p_1|$ by $m^*|\epsilon_1'-\epsilon_1|/p_F$. Since we are interested in small momentum transfers, the upper limit of integration will be set equal to q_0, which is a cutoff corresponding to ϕ_0 in Sec. III.

Now, using energy conservation, $\epsilon_1+\epsilon_2=\epsilon_1'+\epsilon_2'$, and Eq. (29) for small q,

$$n_0(\epsilon_2)(1-n_0(\epsilon_2'))=(n_0(\epsilon_2')-n_0(\epsilon_2))(e^{\beta(\epsilon_1'-\epsilon_1)}-1)^{-1} \approx -\hat{p}_2'\cdot\mathbf{q}\frac{\partial n_0(\epsilon_2')}{\partial p_2'}(e^{\beta(\epsilon_1'-\epsilon_1)}-1)^{-1}. \quad (30)$$

Making this substitution and the corresponding one for $n_0(\epsilon_2')(1-n_0(\epsilon_2))$ in Eq. (26), using Eq. (28), and replacing $\epsilon_2'-\epsilon_2$ by $-\mathbf{q}\cdot\mathbf{p}_2'/m^*$, the contribution to $1/\tau_1$ from the region $q\leqq q_0$ is given by

$$\frac{\hbar}{\tau_1^0}=-\frac{2}{(2\pi\hbar)^5}\frac{m^{*2}}{p_1'p_F}\int_0^\infty d\epsilon_1' \times\left[n_0(\epsilon_1')+\frac{1}{e^{\beta(\epsilon_1'-\epsilon_1)}-1}\right]\int_{(m^*/p_F)|\epsilon_1'-\epsilon_1|}^{q_0} qdq\int d\mathbf{p}_2' \times\frac{\mathbf{p}_2'\cdot\mathbf{q}}{m^*}\delta\left(\epsilon_1'-\epsilon_1-\frac{\mathbf{q}\cdot\mathbf{p}_2'}{m^*}\right)\delta(p_2'-p_1)w. \quad (31)$$

Now

$$\operatorname{Im}\frac{1}{s-\hat{p}_3\cdot\hat{q}}A=A^*\operatorname{Im}\frac{1}{s-\hat{p}_3\cdot\hat{q}}+\frac{1}{s-\hat{p}_3\cdot\hat{q}}\operatorname{Im}A, \quad (32)$$

and, if ImΦ is neglected, since it contributes $O(T^4)$ to $\hbar/\tau$, the imaginary part of Eq. (11) is

$$\operatorname{Im}A(1,2',s)=-\frac{N_0}{8p_F^2}\operatorname{Tr}_{\sigma_3}\int d\mathbf{p}_3 \times\delta(p_3-p_F)\Phi(1,3)\frac{\mathbf{p}_3\cdot\mathbf{q}}{m^*}\delta\left(\epsilon_1'-\epsilon_1-\frac{\mathbf{p}_3\cdot\mathbf{q}}{m^*}\right)A^*(3,2',s) +\frac{N_0}{8p_F^2}\operatorname{Tr}_{\sigma_3}\int d\mathbf{p}_3\,\delta(p_3-p_F)\Phi(1,3)\frac{\hat{p}_3\cdot\hat{q}}{s-\hat{p}_3\cdot\hat{q}} \times\operatorname{Im}A(3,2',s). \quad (33)$$

Then, comparing Eq. (33) with Eq. (11),

$$\operatorname{Im}A(1,2',s)=-\frac{N_0}{8p_F^2}\operatorname{Tr}_{\sigma_3}\int d\mathbf{p}_3\,\delta(p_3-p_F)\frac{\mathbf{p}_3\cdot\mathbf{q}}{m^*} \times\delta\left(\epsilon_1'-\epsilon_1-\frac{\mathbf{p}_3\cdot\mathbf{q}}{m^*}\right)A(1,3,q)A^*(3,2',s), \quad (34)$$

since each side of Eq. (34) satisfies Eq. (33).

From $\Phi(2',1)=\Phi(1,2')$ and Eq. (11) it follows that $A(1,2',s)=A(2',1,s)$, and then, using Eqs. (6), (10), and (34), Eq. (31) may be rewritten

$$\frac{\hbar}{\tau_1}=\frac{N_0}{4p_F^2}\int_0^\infty d\epsilon_1'\left[n_0(\epsilon_1')+\frac{1}{e^{\beta(\epsilon_1'-\epsilon_1)}-1}\right] \times\int_{(m^*/p_F)|\epsilon_1'-\epsilon_1|}^{q_0} qdq\operatorname{Im}a(1,s), \quad (35)$$

where

$$a(1,s)=3A_O(1,1,s)-A_E(1,1,s). \quad (36)$$

Note that there is no additional contribution analogous to that from the region $\phi\sim\pi$ in Sec. III. This corresponds to the neighborhood of $\mathbf{p}_2'=\mathbf{p}_1$, and it has already been taken into account by carrying out the $\mathbf{p}_2'$ integral with the aid of Eq. (11), which is exact.

In order to find $\hbar/\tau_1$ for small values of T and $|\epsilon_1-\mu|$, it is necessary to expand the q integral in Eq. (35) in powers of $|\epsilon_1'-\epsilon_1|$. The problem is similar to that encountered with the transport lifetimes, and it is necessary to change variables from q to s, integrate once by parts, and use the fact that $\mathrm{Im}a(1,s)$ is an odd function of s and is proportional to s for small s. Then keeping lowest orders in $|\epsilon_1'-\epsilon_1|$ in the integrand,

$$\frac{\hbar}{\tau_1}=-\int_0^\infty d\epsilon_1'\left[n_0(\epsilon_1')+\frac{1}{e^{\beta(\epsilon_1'-\epsilon_1)}-1}\right]\times(\epsilon_1'-\epsilon_1)(b+c|\epsilon_1'-\epsilon_1|), \quad (37)$$

where

$$b=\frac{N_0q_0}{3p_FE_F}\left(\frac{\mathrm{Im}a(1,s)}{s}\right)_{s=0}, \quad (38)$$

$$c=\frac{N_0}{16E_F^2}\left[\mathrm{Im}a(1,1)-\int_0^1\frac{ds}{s}\frac{\partial}{\partial s}\frac{\mathrm{Im}a(1,s)}{s}\right]. \quad (39)$$

When $T=0$, Eq. (37) gives

$$\hbar/\tau_1=-\tfrac{1}{2}b(\epsilon_1-\mu)^2-\tfrac{1}{3}c|\epsilon_1-\mu|^3. \quad (40)$$

Equation (40) is analogous to Eq. (20), except that τ_1 has been evaluated at $T=0$ instead of $\epsilon_1=\mu$. The doubts which were expressed in Sec. III about the solution of the Boltzmann equation for transport processes do not apply to the calculation of the quasiparticle lifetime, and the contribution from the region $q\sim0$ to the coefficient of the cubic term in Eq. (40) is given exactly.

The integral in Eq. (39) must be evaluated numerically in general, but, for the special assumption of Eq. (13), if $1+\frac{1}{4}Z_0$ is small,

$$c=-(3\pi^3/128E_F^2)|\tfrac{1}{4}Z_0/(1+\tfrac{1}{4}Z_0)|^3, \quad (41)$$

and is very large. This result could have been obtained directly from Eqs. (1), (13), (14), and (27), and it is the analog of Eq. (23).

The transformation from Eq. (31) to Eq. (35) is the replacement of a transition rate by the imaginary part of a forward-scattering amplitude, and it is, therefore, a form of optical theorem. Since $\hbar/\tau_1$ is the imaginary part of the quasiparticle energy, the real part should be obtained by substituting $\mathrm{Re}a(1,s)$ for $\mathrm{Im}a(1,s)$ in Eq. (35). This statement is plausible, but it cannot be proved from what has been done here (for example, by means of a dispersion relation), since Eq. (35) has been derived for small ϵ_1 only. Also, for the equilibrium properties at finite temperatures, it is possible to define a quasiparticle energy which is purely real and does not coincide with the complex energy of an added quasiparticle.[12] However, the imaginary part of the quasiparticle energy does not affect the $T^3\ln T$ term in the specific heat, and so, for present purposes, there is no essential difference.

The real part of $a(1,s)$ is an even function of s, and it is not difficult to see that, as a result, the leading correction to the quasiparticle energy has a logarithmic term which may be obtained from the second order in the expansion of $a(1,s)$ in powers of s. If

$$a(1,s)=a(1,0)+a's^2, \quad (42)$$

the correction to the quasiparticle energy is

$$\Delta\epsilon_1=-\frac{N_0a'}{16E_F^2}\int_0^\infty d\epsilon_1'\left[n_0(\epsilon_1')+\frac{1}{e^{\beta(\epsilon_1'-\epsilon_1)}-1}\right]\times(\epsilon_1'-\epsilon_1)^2\ln\left|\frac{(\epsilon_1'-\epsilon_1)m^*}{p_Fq_0}\right|. \quad (43)$$

In order to obtain the specific heat, $\Delta\epsilon_1$ should be evaluated for $T\neq0$, and keeping the terms which lead to $T^3\ln T$ in the specific heat,

$$\Delta\epsilon_1=\tfrac{1}{16}N_0a'\left[\frac{(\epsilon_1-\mu)^2}{T_F^2}+\pi^2\frac{T^2}{T_F^2}\right]\times(\epsilon_1-\mu)\ln\left|\frac{(\epsilon_1-\mu)m^*}{p_Fq_0}\right|. \quad (44)$$

The total single-particle energy is then

$$E_1=\epsilon_1+\Delta\epsilon_1. \quad (45)$$

In obtaining Eq. (44), the energy and temperature dependence of $F(\mathbf{p}_1,\mathbf{p}_2')$ has been omitted. Its significance has been emphasized[13] in connection with the interpretation of the specific heat of liquid He^3, and the requirement of Galilean invariance,[4] when E_1 has the form given by Eqs. (44) and (45), shows that it is reflected in $a(1,0)$ in Eq. (42) in such a way that the coefficients of $(\epsilon_1-\mu)^3$ and $T^2(\epsilon_1-\mu)$ will be changed. The effect cannot be calculated from the theory presented here, since Φ is assumed to be given phenomenologically.

If Eq. (44) is assumed for the moment, the change in the specific heat may be evaluated from the entropy

$$S=-2\sum_{\mathbf{p}_1}[n_0\ln n_0+(1-n_0)\ln(1-n_0)], \quad (46)$$

where n_0 is a function of E_1.

Changing the sum to an integral, introducing E_1 as variable of integration, and extending the lower limit of integration to $-\infty$,

$$\frac{S}{N}=-\frac{3}{2T_F}\int_{-\infty}^\infty dE_1\left[1-\frac{N_0a'}{48}\frac{T^2}{T_F^2}\left(3\frac{E_1^2}{T^2}+\pi^2\right)\ln\left|\frac{m^*E_1}{p_Fq_0}\right|\right]\times[n_0\ln n_0+(1-n_0)\ln(1-n_0)]. \quad (47)$$

Then, keeping the lowest order in T and the term pro-

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[13] V. J. Emery, Ann. Phys. (N. Y.) **28**, 1 (1964).

portional to $\ln T$,

$$\frac{S}{N}=\frac{1}{2}\pi^2\frac{T}{T_F}\left[1-\frac{\pi^2}{20}N_0a'\frac{T^2}{T_F^2}\ln\left|\frac{k_BTm^*}{q_0p_F}\right|\right], \qquad (48)$$

from which the specific heat is given by

$$\frac{C_V}{N}=\frac{1}{2}\pi^2\frac{T}{T_F}\left[1-\frac{3\pi^2}{20}N_0a'\frac{T^2}{T_F^2}\ln\left|\frac{k_BTm^*}{q_0p_F}\right|\right]. \qquad (49)$$

A similar result has been obtained by Amit, Kane, and Wagner,[6] who calculated the self-energy rather than the single quasiparticle energy and also solved for the special form of $\Phi(1,2')$ for which

$$F(\mathbf{p}_1,\mathbf{p}_2')=F_0+F_1\hat{p}_1\cdot\hat{p}_2', \quad z(\mathbf{p}_1,\mathbf{p}_2')=Z_0+Z_1\hat{p}_1\cdot\hat{p}_2'. \qquad (50)$$

This was chosen because F_0, F_1, and Z_0 are known from experiment, and Z_1 may be chosen so that A_O vanishes for $\mathbf{p}_1=\mathbf{p}_2$ when $s=0=q$.

Solving Eq. (11) and using Eqs. (8) and (36), Eq. (47) gives for a'

$$N_0a'=-4(\phi_0+\phi_1), \qquad (51)$$

where

$$\phi_0=-\tfrac{1}{4}[\bar{F}_0^2(1+\bar{F}_1-\tfrac{1}{4}\pi^2\bar{F}_0)+\bar{F}_1^2(1-\tfrac{1}{16}\pi^2\bar{F}_1)] -\tfrac{3}{4}[\bar{Z}_0^2(1+\bar{Z}_1-\tfrac{1}{4}\pi^2\bar{Z}_0)+\bar{Z}_1^2(1-\tfrac{1}{16}\pi^2\bar{Z}_1)] \qquad (52)$$

and

$$\phi_1=\tfrac{1}{2}\bar{F}_0\bar{F}_1+\tfrac{3}{2}\bar{Z}_0\bar{Z}_1, \qquad (53)$$

with

$$\bar{F}_l=\frac{F_l}{1+F_l/(2l+1)}, \quad \bar{Z}_l=\frac{\frac{1}{4}Z_l}{1+\frac{1}{4}Z_l(2l+1)}. \qquad (54)$$

The real part of the self-energy $\Sigma(\epsilon,p_1)$ of a quasiparticle may be obtained in the present approach by replacing $\epsilon_1-\mu$ by ϵ and $\mathrm{Im}a(1,s)$ by $\mathrm{Re}a(1,s)$ in Eq. (35), and at $T=0$, assuming Eq. (50), the contribution to $\mathrm{Re}\Sigma(\epsilon,\mathbf{p}_1)$ is given by

$$\mathrm{Re}\delta\Sigma(\epsilon,\mathbf{p}_1)=\frac{1}{4E_F^2}[(\tfrac{1}{3}\phi_0-\tfrac{1}{6}\phi_1)\epsilon^3+\tfrac{1}{2}\phi_1\epsilon^2(\epsilon_1-\mu)] \times\ln\left|\frac{(\epsilon-\mu)m^*}{p_Fq_0}\right|. \qquad (55)$$

The energy ϵ_1 appears in this equation because $\mathbf{q}\cdot\mathbf{p}_1/m^*$ has been rewritten in terms of ϵ_1 and ϵ_1'.

Amit, Kane, and Wagner[6] obtained two expressions for the contribution to $\mathrm{Re}\Sigma(\epsilon,\mathbf{p}_1)$ from the neighborhood of $q=0$, and they disagreed with each other and with Eq. (55). It was pointed out, however, that there is a further contribution from the neighborhood of $q=2p_F$, and it was suggested that it be determined so that the two ways of finding $\mathrm{Re}\delta\Sigma(\epsilon,\mathbf{p}_1)$ agreed, The result may be obtained by adding $\frac{1}{4}(\bar{F}_0+\frac{1}{3}\bar{F}_1)\phi_1$ to ϕ_0 in Eq. (55), and there is an analogous change in S and C_V. This argument will be modified when the corrections to $\mathrm{Re}\delta\Sigma(\epsilon,\mathbf{p}_1)$ from the energy and temperature dependence of Φ are included. Also, the magnitude of the contribution from the neighborhood of $q=2p_F$ depends strongly upon the special assumption that $F(\mathbf{p}_1,\mathbf{p}_2')$ and $Z(\mathbf{p}_1,\mathbf{p}_2')$ are given by Eq. (50). Presumably, in a more accurate calculation there would be a residual discrepancy to be attributed to the neighborhood of $q\simeq 2p_F$, and it will affect the lifetimes too. To take this region into account would require an extension of the methods used here.

In the case of a nearly ferromagnetic Fermi liquid, $\bar{Z}_0$ is assumed to dominate and Eqs. (51)–(53) give approximately

$$N_0a'=-\tfrac{3}{4}\pi^2\bar{Z}_0^3, \qquad (56)$$

which is large and positive.

As mentioned above, Eq. (49) does not give the entire coefficient of $T^3\ln T$ in C_V. If Eq. (56) is substituted into Eq. (49) and Z_0 is calculated to first order in the interaction, this coefficient differs by a factor of 3 from that obtained by Brenig, Mikeska, and Riedel[14] and by Brinkman and Engelsberg,[14] using the random-phase approximation. The difference was attributed to a "bosonlike" contribution from the paramagnons,[14] but it may equally be interpreted as a consequence of the temperature dependence of $F(\mathbf{p}_1,\mathbf{p}_2')$. This function may consistently be calculated in the random-phase approximation by functional differentiation,[15] so that the formulas of Fermi-liquid theory apply.

The temperature and energy dependence of Φ does not change the lifetimes, to the order in which they have been calculated.

ACKNOWLEDGMENTS

I have enjoyed discussions with Dr. S. Doniach and Dr. M. J. Rice of Imperial College, London, and I am indebted to them for several comments. Part of this work was carried out during a visit to the University of Sussex, England, in the Spring of 1966, and I wish to thank Professor D. F. Brewer and Professor R. J. Blin-Stoyle for their hospitality.

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