The integral around the circle becomes

$$\lim_{\epsilon\to 0}\left[\tfrac{1}{6}efm\int_0^{2\pi}\frac{-\epsilon ie^{i\theta}d\theta}{(2m\epsilon e^{i\theta})^{3/2}}\right]$$

$$=\lim_{\epsilon\to 0}\left[-\tfrac{1}{12}ef\frac{1}{(2m\epsilon)^{1/2}}\int_0^{2\pi}ie^{-i\theta/2}d\theta\right]$$

$$=\lim_{\epsilon\to 0}\left[-\tfrac{1}{3}ef\frac{1}{(2m\epsilon)^{1/2}}\right].$$

The integral from $-\infty$ to $-m-\epsilon$ is

$$\lim_{\epsilon\to 0}\left\{\tfrac{1}{3}efm\left[\frac{\omega}{-m^2(\omega^2-m^2)^{1/2}}\right]_{-\infty}^{-m-\epsilon}\right\}$$

$$=\lim_{\epsilon\to 0}\left\{\frac{-ef}{3m}\left[\frac{-m}{(2m\epsilon)^{1/2}}-1\right]\right\}.$$

So with quadratic D, Eq. (A2) gives

$$ef/3m. \tag{A4}$$

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A Field Theory of Currents*†

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A new type of model field theory is constructed. Only currents appear as the coordinates. The canonical formalism is abandoned. The self-consistency, Lorentz covariance, energy-momentum conservation, etc., are checked.

I. INTRODUCTION

It is widely recognized among high-energy physicists that field variables may not be adequate to describe strong interactions. This implies among other things that we have to give up the canonical formalism. In other words, we cannot attribute to each particle a field which satisfies the canonical commutation relation.

Field theory itself, however, may not disappear. In fact, Gell-Mann repeatedly stresses[1] that currents which are measurable through electromagnetic, weak, and gravitational interactions will survive. We already partly understand the role of currents in the strong-interaction symmetry. But except for very limited cases we know nothing about the dynamical role of currents in strong interactions.

In this paper we try to investigate the possibility of constructing a field theory in terms of these observable currents.

We understand that the substitutes for the canonical commutation relations are the equal-time commutation relations among currents. Then, just as we construct a Hilbert space as a representation of the canonical commutation relations in the ordinary field theory, so we should find a representation of the equal-time commutation relations.[2] We may be able to find a physical representation independently of underlying field theory. Yet we have to be sure that we can actually construct this underlying field theory of currents.

First of all, what are the observable currents? Following Gell-Mann[1] we assume that they are the 8 vector currents $V_\mu{}^i(x)$, the 8 axial-vector currents $A_\mu{}^i(x)$, and the gravitational tensor $\theta_{\mu\nu}(x)$. No other observable currents have been implied so far by experiments. Suppose they are all independent dynamical variables. Then we have to write the equal-time commutation relations among all the components of these variables. This is obviously one possibility. But a more attractive theory would be when $\theta_{\mu\nu}(x)$ is expressible in terms of $V_\mu{}^i(x)$ and $A_\mu{}^i(x)$. We will only consider such a model in this paper.

We do not intend to construct a theory which describes a realistic strong interaction. This is possible only after we know enough about the equal-time commutation relations. Our intention is rather to show that at least it is possible to construct a simple nontrivial model, which may or may not be some limiting case of the realistic theory. Anyway, we will choose a relatively simple set of current commutators whenever we do not have experimental information.

What do we mean when we ask if a model is possible? Though we give up the canonical commutation relations we retain almost all the other axioms of quantum field theory. Thus when we construct a model we have to

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† The content of this paper is almost the same as its original version except that more details are explained to avoid misunderstanding.
[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[2] R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966); M. Gell-Mann, D. Horn, and J. Weyers (to be published); K. Bardakci, M. B. Halpern, and G. Segrè, Phys. Rev. **168**, 1728 (1968).

check if these axioms are indeed satisfied. We will show later than Lorentz covariance gives a very restrictive condition and it almost uniquely determines the form of $\theta_{\mu\nu}(x)$ in terms of the currents. If we require other "axioms" like energy-momentum conservation, the Heisenberg equation of motion, etc., our natural feeling would be that such a model is impossible except for trivial cases (linear theory) or canonical vector-meson theories in which we replace vector fields by currents. The latter case was investigated recently by Lee, Weinberg, and Zumino[3] in a paper in which they take the well-known Yang-Mills theory[4] and identify currents with gauge fields. Obviously we do not want this kind of theory here, because in the Yang-Mills theory we have the canonical commutation relations underlying the equal-time commutation relations. Moreover, we have to include explicitly in the Lagrangian all the canonical fields which correspond to hadrons other than vectors.

We do not want any canonical variables in our theory.

In Sec. II, we construct the model. Section III contains concluding remarks.

II. CONSTRUCTION OF MODEL

As is mentioned in the Introduction, we assume that the only independent dynamical variables are an octet of vector currents $V_\mu{}^i(x)$ $(i=1, 2, \cdots, 8)$ and an octet of axial-vector currents $A_\mu{}^i(x)$ $(i=1, 2, \cdots, 8)$. Among these dynamical variables we have to have a consistent set of equal-time commutation relations.

We start by postulating the time-time commutation relations

$$\begin{aligned}
[V_0{}^i(x),V_0{}^j(y)]_{x_0=y_0}&=if_{ijk}V_0{}^k(x)\delta^3(x-y)\,,\\
[V_0{}^i(x),A_0{}^j(y)]_{x_0=y_0}&=if_{ijk}A_0{}^k(x)\delta^3(x-y)\,, \qquad (1)\\
[A_0{}^i(x),A_0{}^j(y)]_{x_0=y_0}&=if_{ijk}V_0{}^k(x)\delta^3(x-y)\,.
\end{aligned}$$

These three commutation relations may presumably be correct. But we do not know what space-time and space-space commutators are. We assume, therefore, the following simple model which was first suggested in Ref. 3:

$$\begin{aligned}
[V_0{}^i(x),V_a{}^j(y)]&=if_{ijk}V_a{}^k(x)\delta^3(x-y)+iC\partial_a\delta^3(x-y)\,,\\
[V_0{}^i(x),A_a{}^j(y)]&=if_{ijk}A_a{}^k(x)\delta^3(x-y)\,,\\
[A_0{}^i(x),A_0{}^j(y)]&=if_{ijk}V_a{}^k(x)\delta^3(x-y)+iC\partial_a\delta^3(x-y)\,, \qquad (2)\\
[A_0{}^i(x),A_a{}^j(y)]&=if_{ijk}A_a{}^k(x)\delta^3(x-y)\,,\\
[V_a,V_b]&=[V_a,A_b]=[A_a,A_b]=0\,,
\end{aligned}$$

where C is a c-number constant and a runs from 1 to 3.

[3] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).

[4] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

As these commutators are suggested in the vector-meson theory, there is a danger of our theory being reduced to some canonical theory of vector mesons. We will show later, after constructing the model, that it is not a vector-meson theory.[5]

We note again that only $V_\mu{}^i(x)$ and $A_\mu{}^i(x)$ are the independent dynamical variables. The other dynamical variable $\theta_{\mu\nu}(x)$ should be a function of $V_\mu{}^i(x)$ and $A_\mu{}^i(x)$. But obviously it cannot be an arbitrary second-rank tensor because of the Schwinger condition[6]

$$[\theta_{00}(x),\theta_{00}(y)]=-i(\theta_{0a}(x)+\theta_{0a}(y))\partial_a\delta^3(x-y)\,. \qquad (3)$$

We have also to satisfy the energy-momentum conservation condition

$$\partial_\mu\theta_{\mu\nu}(x)=0\,. \qquad (4)$$

The condition (3) alone seems to be almost enough to fix the form of $\theta_{\mu\nu}(x)$ under the following two conditions: (a) $\theta_{\mu\nu}(x)$ is a polynomial of $V_\mu{}^i(x)$ and $A_\mu{}^i(x)$; (b) $\theta_{\mu\nu}(x)$ is a unitary singlet.

Suppose $\theta_{\mu\nu}(x)$ contains more than a quadratic term. Then the fourth-order term in $\theta_{\mu\nu}(x)$ will give a sixth-order term to the left-hand side of (3) unless we have a nice cancellation among the terms which some from different fourth-order terms in $\theta_{\mu\nu}(x)$. We note the non-Schwinger parts of the current commutators do not contribute because of the condition (b). Unless there is a cancellation we have to add sixth-order terms to $\theta_{\mu\nu}(x)$. Then again because of (3) we have to add tenth-order terms to $\theta_{\mu\nu}(x)$. Finally we will end up with an infinite series which contradicts the condition (a). $\theta_{\mu\nu}(x)$ should therefore be a quadratic polynomial. We cannot exclude the possibility of cancellations but, as our purpose is to make a simple model, we are satisfied with the above discussion which looks plausible and we set

$$\theta_{\mu\nu}(x)=a_V(V_\mu{}^i(x)V_\nu{}^i(x)+V_\nu{}^i(x)V_\mu{}^i(x))+b_Vg_{\mu\nu}(V_\rho{}^i(x)V_\rho{}^i(x))+(\text{similar axial-vector terms})\,, \qquad (5)$$

where a_V, b_V are arbitrary coefficients to be determined by the condition (3). We take the metric $(1, 1, 1, -1)$.

Substituting (5) into (3) and calculating the left-hand side using the current commutators (1) and (2), we get as a unique solution

$$a_V=-b_V=-1/2C\,, \qquad (6)$$

where C has already appeared in (2). We also get

$$a_A=-b_A=-1/2C\,, \qquad (7)$$

where a_A, b_A are the corresponding coefficients for the axial-vector terms in (5).

[5] This important observation was made by K. Bardakci, Y. Frishman, and M. B. Halpern, Phys. Rev. **170**, 1353 (1968).

[6] J. Schwinger, Phys. Rev. **130**, 406 (1963); **130**, 800 (1963).

Our $\theta_{\mu\nu}(x)$ therefore has the following form:

$$\theta_{\mu\nu}(x)=-(1/2C)[V_\mu^i(x)V_\nu^i(x)+V_\nu^i(x)V_\mu^i(x)-g_{\mu\nu}(V_\rho^i(x)V_\rho^i(x))]-(1/2C)[A_\mu^i(x)A_\nu^i(x)+A_\nu^i(x)A_\mu^i(x)-g_{\mu\nu}(A_\rho^i(x)A_\rho^i(x))], \quad (8)$$

where the sum should be taken over repeated indices.

Thus we can satisfy at least (3). But obviously this is far from sufficient to claim that we have a consistent model. Moreover, as we have already fixed the form of $\theta_{\mu\nu}(x)$, we no longer have any arbitrary parameter to be determined by other conditions. We simply have to satisfy them.

Heisenberg Equation of Motion

Obviously we want our theory to be quantum mechanical, which means that we have to satisfy the Heisenberg equation of motion

$$[P_\mu,T_\rho^i(x)]=-i\partial_\mu T_\rho^i(x), \quad (9)$$

where $T_\rho^i(x)$ is either a vector or axial-vector current, and

$$P_\mu=-\int\theta_{0\mu}(x)d^3x,$$

$\theta_{0\mu}$ being defined in (8).

Now we have to calculate the left-hand side of (9) using the definition of P_μ [Eq. (8)] and the equal-time current commutators (1) and (2).

First, by a direct calculation it is easy to see that

$$[\theta_{00}(x),V_0^j(y)]_{x_0=y_0}=iV_a^j(x)\partial_a\delta^3(x-y), \quad (a=1,2,3). \quad (10)$$

Substituting (10) into (9), we get

$$\partial_\mu V_\mu^i(x)=0. \quad (11)$$

Similarly, we get

$$\partial_\mu A_\mu^i(x)=0. \quad (12)$$

Next we note that

$$[\theta_{a0}(x),V_0^j(y)]_{x_0=y_0}=iV_0^j(x)\partial_a^x\delta^3(x-y). \quad (13)$$

Substituting (13) into (9), we get an identity

$$\partial V_0^j/\partial x_a=\partial V_0^j/\partial x_a.$$

Nonlinear equations come from the following commutators:

$$[\theta_{00}(x),V_b^j(y)]_{x_0=y_0}=(i/2C)f_{ijk}(V_0^iV_b^k+V_b^kV_0^i+A_0^iA_b^k+A_b^kA_0^i)\delta^3(x-y)+iV_0^j(x)\partial_b\delta^3(x-y), \quad (14)$$

$$[\theta_{a0}(x),V_b^j(y)]_{x_0=y_0}=(i/2C)f_{ijk}(V_a^iV_b^k+V_b^kV_a^i+A_a^iA_b^k+A_b^kA_a^i)\delta^3(x-y)+iV_a^j(x)\partial_b\delta^3(x-y). \quad (15)$$

Substituting (14) and (15) into (9), we get

$$\partial_\mu V_\nu^i-\partial_\nu V_\mu^i=(1/2C)f_{ijk}(V_\mu^jV_\nu^k+V_\nu^kV_\mu^j+A_\mu^jA_\nu^k+A_\nu^kA_\mu^j), \quad (\mu,\nu=0,1,2,3). \quad (16)$$

Similarly, we have

$$\partial_\mu A_\nu^i-\partial_\nu A_\mu^i=(1/2C)f_{ijk}(A_\mu^jV_\nu^k+V_\nu^kA_\mu^j+V_\mu^jA_\nu^k+A_\nu^kV_\mu^j). \quad (17)$$

Thus to satisfy Eq. (9) we need to have Eqs. (11), (12), (16), and (17). It is easy to see that we can drop the axial part in (8) without violating the condition (3). In this case (when axial currents do not come into the theory) Eq. (16) can be written

$$F_{\mu\nu}^i\equiv D_\mu^{ij}V_\nu^j-D_\nu^{ij}V_\mu^j=0, \quad (18)$$

where

$$D_\mu^{ij}=\delta_{ij}\partial_\mu-(1/2C)f_{ikj}V_\mu^k.$$

This is the covaraint derivative in the Yang-Mills theory. The Yang-Mills field equation[4] is

$$\partial_\mu F_{\mu\nu}^i=0. \quad (19)$$

There seems to be a rather profound difference between (18) and (19) which we will discuss later.

Energy-Momentum Conservation

That $\partial_\mu\theta_{\mu0}=0$ is clear because of the Schwinger commutation relation (3). To show that

$$\partial_\mu\theta_{\mu a}=0, \quad (a=1,2,3) \quad (20)$$

we calculate the commutator

$$[\theta_{a0}(x),\theta_{00}(y)]_{x_0=y_0}. \quad (21)$$

Rewriting $\theta_{a0}(x)$ in terms of $V_\mu^i(x)$ and $A_\mu^i(x)$ and using (10), (14), and similar commutators for the axial currents, we get

$$[\theta_{a0}(x),\theta_{00}(y)]=-(i/2C)\{[V_0^i(x)V_0^i(y)+V_0^i(y)V_0^i(x)]\times\partial_a^x\delta^3(x-y)+[V_a^i(x)V_b^i(y)+V_b^i(y)V_a^i(x)]\times\partial_b^x\delta^3(x-y)\}+(V\to A). \quad (22)$$

Using (9), we get

$$[\partial\theta_{a0}(x)/\partial x_0]=(1/2C)\{\partial_a(V_0^i(x)V_0^i(x))+V_a^i(x)\partial_bV_b^j(x)+(\partial_bV_b^i(x))V_a^i(x)\}+(V\to A). \quad (23)$$

In deriving this we have to keep in mind that in (22) the quadratic form of currents on the right-hand side has the different arguments x and y. Then, making use of (16) appropriately in (23), we arrive at (20).

Lorentz Covariance

The only reasonable way to define the Lorentz generators is through

$$M_{\mu\nu}=-\int[x_\nu\theta_{0\mu}(x)-x_\mu\theta_{0\nu}(x)]d^3x. \quad (24)$$

First of all, we have to show that $V_\mu^i(x)$ and $A_\nu^i(x)$ have actually the transformation property of 4-vectors.

This is again done by a direct application of the current commutators. For example

$$[M_{0a},V_b{}^j(y)]$$
$$=-\int d^3x\{x_0[\theta_{a0}(x),V_b{}^j(y)] -x_a[\theta_{0b}(x),V_b{}^j(y)]\} \quad (25)$$
$$=-\int d^3x\{x_0[-iV_a{}^j(x)\partial_b\delta(x-y) +i\partial_aV_b{}^j-i\partial_bV_a{}^j]-x_a[-iV_0{}^j(x)\partial_b{}^x\delta(x-y) +i\partial_0V_b{}^j-i\partial_bV_0{}^j]\}. \quad (26)$$

In deriving this we first calculate the two commutators in (25) using (14) and (15). Then we rewrite it using (16). From (26) we immediately get

$$[M_{0a},V_b{}^j(x)]=i(x_a\partial_0-x_0\partial_a)V_b{}^j+ig_{ab}V_0{}^j. \quad (27)$$

We repeat similar calculations and show that

$$[M_{\mu\nu},V_\rho{}^i(x)]=-i(x_\mu\partial_\nu-x_\nu\partial_\mu)V_\rho{}^i(x) +i(g_{\nu\rho}V_\mu{}^i-g_{\mu\rho}V_\nu{}^i), \quad (28)$$

which is precisely the condition for $V_\mu{}^i(x)$ to be a 4-vector. The same equation is true also for $A_\mu{}^i(x)$.

Our next task is to show that P_μ and M_μ constitute the Poincaré algebra:

$$[P_\mu,P_\nu]=0, \quad (29)$$
$$[M_{\mu\nu},P_\rho]=i(g_{\nu\rho}P_\mu-g_{\mu\rho}P_\nu), \quad (30)$$
$$[M_{\mu\nu},M_{\rho\kappa}]=i(g_{\nu\rho}M_{\mu\kappa}+g_{\mu\kappa}M_{\nu\rho}-g_{\nu\kappa}M_{\mu\rho} -g_{\mu\rho}M_{\nu\kappa}). \quad (31)$$

The way we prove these equation is quite similar to what we have done above. We express everything in terms of $\theta_{\mu\nu}(x)$ and calculate commutators

$$[\theta_{\mu\nu}(x),\theta_{\rho\kappa}(y)]_{x_0=y_0}$$

using the original current commutators. We use the equations of motion (16) and (17) whenever necessary. After a few steps of simple manipulations we get Eqs. (29)–(31). Actually, the existence of the Poincaré algebra is obvious because of the Schwinger condition (3) and the energy-momentum conservation (4).

Positive Definiteness of Energy Spectrum

This is obvious from the expression

$$H=\frac{1}{2C}\int\{[V_0{}^i(x)V_0{}^i(x)+V_a{}^i(x)V_a{}^i(x)] +(V\to A)\}d^3x. \quad (32)$$

III. REMARKS AND CONCLUSIONS

It is very plausible that the model described above is self-consistent though there are many things still to be shown: microcausality, existence of S matrix, etc. As a first step to understanding our model, we show that it is not a canonical vector-meson theory.[5] Assume that there exists a canonical conjugate momentum $\pi_a(x)$ which satisfies

$$[\pi_a{}^i(x),V_b{}^j(y)]_{x_0=y_0}=i\delta_{ij}\delta_{ab}\delta^3(x-y). \quad (33)$$

Then consider the commutator of $\pi_a(x)$ with each side of the equation of motion (16) (taking only the space-component part). We immediately notice that there is a contradiction. This means that our model is not a canonical vector-meson theory; $\pi_a(x)$ cannot exist. This also means that we cannot diagonalize the $V_a{}^i(x)$ ($a=1,2,3$) though they commute with each other. The proof goes as follows. Suppose that it is possible to diagonalize $V_a{}^i(x)$; then we have a set of states $|v\rangle$ which satisfies

$$V_a{}^i(x)|v\rangle=v_a{}^i(x)|v\rangle. \quad (34)$$

$v_a{}^i(x)$ is a c-number function. Then in the functional space $\{|v\rangle\}$ we can represent $V_0{}^i(x)$ as

$$V_0{}^i(x)=-if_{ijk}v_a{}^j(x)\frac{\delta}{\delta v_a{}^k(x)}+iC\frac{\partial}{\partial x_a}\frac{\delta}{\delta v_a{}^i(x)}. \quad (35)$$

It is easy to check that they satisfy the required commutation relations. Obviously,

$$\left[i\frac{\delta}{\delta v_a{}^i(x)},v_b{}^j(y)\right]=i\delta_{ij}\delta_{ab}\delta(x-y). \quad (36)$$

This is nothing but a functional representation of a vector-meson theory. We have already shown that this is impossible.

If we look at our model from a different point of view, we can say that we are just trying to find a representation of the current algebra in a specific model. In particular, the angular condition of Dashen and Gell-Mann[2] is replaced by a set of equations of motion (12), (16), and (17). We have to find a solution to these equations which satisfies the current commutation relations.

It might well be that our model is some particular limit[5] of the Yang-Mills model,[4] though the limit must be such that the canonical commutation relation is no longer true. These problems will be discussed in a future publication.

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