Nucleon Magnetic Moments in a Bootstrap Model

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The bootstrap calculations of baryon magnetic moments by Dashen and Frautschi and by Barton and Dare are examined in an SU_2 model. The pion contribution to the nucleon magnetic moment, which Dashen and Frautschi neglected, is calculated using their method, and is shown to reduce to Barton and Dare's estimate in a nonrelativistic limit. In the SU_2 model, Dashen and Frautschi's method gives for the nucleon isoscalar and vector magnetic moments: $\mu_S=0$ and $0<\mu_V<5.7(e/2M)$, where M is the nucleon mass.

I. INTRODUCTION

TREATING the baryons as meson-baryon bound states, expressions for the baryon magnetic moments have been obtained by Dashen and Frautschi[1] using a dispersion technique, and by Barton and Dare[2] using a nonrelativistic model; Petras[3] has also used a model similar to that of Barton and Dare. Abarbanel, Callan, and Sharp[4] have evaluated the nucleon magnetic moments using Dashen and Frautschi's method. It is the purpose of this paper to compare the two methods. The essentials of the calculation can be seen if the discussion is confined to the calculation of nucleon magnetic moments in an SU_2 model.

The nucleon magnetic moment will have contributions from that of the constituent nucleon and from the pion. Consequently, the expression for the nucleon magnetic moment will be of the form

$$\mu=\alpha\mu+\beta e\,,$$

where e is the charge of the pion. Since the pion is an isovector, β will vanish in the expression for the isoscalar magnetic moment μ_S, so μ_S can only be zero in this simple model. The isovector magnetic moment μ_V, on the other hand, will depend in magnitude largely on the pion contribution, because in fact $\alpha=\frac{1}{9}$, as is shown below. Dashen and Frautschi (DF) do not calculate the pion term, or the corresponding meson contributions in the SU_3 case. Barton and Dare (BD) calculate the π contribution. Their estimate is not intended to be realistic, but more sophisticated calculations should give the BD result in an appropriate nonrelativistic limit.

The two approaches are examined in Secs. II and III, and in Sec. IV the π contribution is calculated by the DF method. It is shown that it does agree with the BD estimate in a nonrelativistic, weak-coupling limit. Lack of information about the πN scattering D function makes it difficult to obtain a realistic estimate of the π contribution. However, it can be shown to be smaller than the nonrelativistic term in magnitude, and of the same sign. An upper limit can be calculated for the π contribution, which gives

$$\mu_V<5.7e/2M \quad (\text{experimentally } \mu_V=4.7e/2M)\,,$$

where M is the nucleon mass. This shows that, with a reasonable D function, the pion term may be the dominant contribution to the nucleon magnetic moment.

II. BARTON AND DARE'S APPROACH

BD examine a simple nonrelativistic model in which a nucleon consists of a physical nucleon orbited by a physical pion in a $P_{1/2}$, isospin $T=\frac{1}{2}$ state. Contributions to the magnetic moment are shown in Fig. 1 and come from that of the constituent nucleon and the magnetic effect of the orbital motion of a charged pion.

Let $|N\pi\rangle$ represent a state with pion orbital angular momentum $L=1$ and total angular momentum $J=\frac{1}{2}$. Proton and neutron states are bound states in the $T=\frac{1}{2}$ πN channel and can be written

$$|p\rangle=-(\sqrt{\tfrac{1}{3}})|p\pi^0\rangle+(\sqrt{\tfrac{2}{3}})|n\pi^+\rangle\,,$$
$$|n\rangle=-(\sqrt{\tfrac{2}{3}})|p\pi^-\rangle+(\sqrt{\tfrac{1}{3}})|n\pi^0\rangle. \qquad (2.1)$$

The nonrelativistic magnetic moment operator in the static limit (in which the nucleon is infinitely heavy) is simply the sum of two terms, one relating to the nucleon

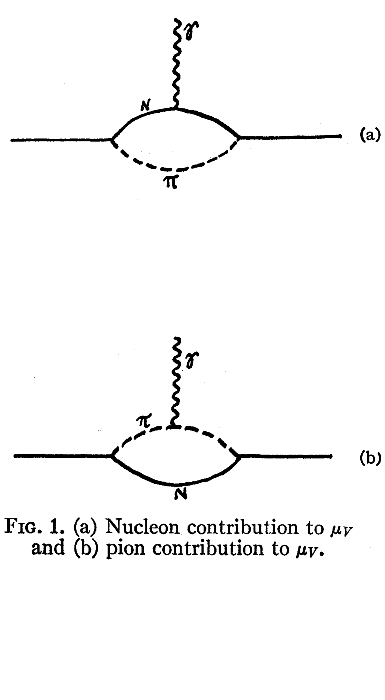

FIG. 1. (a) Nucleon contribution to μ_V and (b) pion contribution to μ_V.

[1] R. Dashen and S. Frautschi, Phys. Rev. 143, 1171 (1966).
[2] G. Barton and D. Dare, Phys. Rev. 150, 1220 (1967).
[3] M. Petras, Nucl. Phys. 62, 526 (1965).
[4] H. D. I. Abarbanel, C. G. Callan, Jr., and D. H. Sharp, Phys. Rev. 143, 1225 (1966).

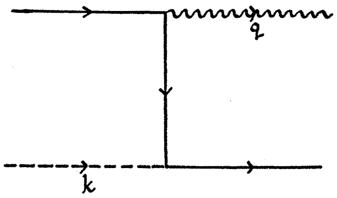

FIG. 2. Nucleon-exchange diagram.

only, and one to the pion only:

$$\boldsymbol{\mu}=\tfrac{1}{2}(\mu_S+\tau_3\mu_V)\boldsymbol{\sigma}+(e/2m)T_3\mathbf{L}, \tag{2.2}$$

where τ_3 and T_3 are the third components of the isospin operators for nucleon and pion, respectively, and $\boldsymbol{\sigma}$ and $\mathbf{L}$ are nucleon spin and orbital angular momentum. The isoscalar and vector magnetic moments are $\mu_S=\mu_p+\mu_n$ and $\mu_V=\mu_p-\mu_n$; m is the mass of the pion. The expectation values of the third component of $\boldsymbol{\mu}$ between proton and neutron states given by (2.1), are taken. If the nucleon states have spin up, represented by $|N\uparrow\rangle$, the left-hand sides will be $\langle p\uparrow|\mu_3|p\uparrow\rangle=\mu_p$ and $\langle n\uparrow|\mu_3|n\uparrow\rangle=\mu_n$.

Since the pion is in a $P_{1/2}$ state it will either have $L_3=+1$ with nucleon spin down, or $L_3=0$ with nucleon spin up. Consequently, the equation for the proton magnetic moment will be

$$\begin{aligned}\mu_p=&\tfrac{1}{3}[\tfrac{1}{3}\langle p\uparrow\pi^0|\mu_3|p\uparrow\pi^0\rangle+\tfrac{2}{3}\langle p\downarrow\pi^0|\mu_3|p\downarrow\pi^0\rangle]\\&+\tfrac{2}{3}[\tfrac{1}{3}\langle n\uparrow\pi^+|\mu_3|n\uparrow\pi^+\rangle+\tfrac{2}{3}\langle n\downarrow\pi^+|\mu_3|n\downarrow\pi^+\rangle]\\=&-(1/9)\mu_p-(2/9)\mu_n+(2/9)e/m.\end{aligned}$$

Similarly,

$$\mu_n=(2/9)\mu_p-(1/9)\mu_n-(2/9)(e/m).$$

This gives

$$\mu_S=-\tfrac{1}{3}\mu_S,\quad \mu_V=(1/9)\mu_V+(4/9)e/m, \tag{2.3}$$

i.e.,

$$\mu_S=0$$

as expected on this model, and

$$\mu_V=e/2m\approx 6.7e/2M,$$

where M is the mass of the nucleon.

These values should be compared with the experimental ones:

$$\mu_S=0.88e/2M,$$
$$\mu_V=4.70e/2M.$$

III. DASHEN AND FRAUTSCHI'S APPROACH

DF obtain dispersion relations for the nucleon magnetic moments by considering the process

$$\pi+N\rightarrow\gamma+N. \tag{3.1}$$

The nucleon is a bound state in the $P_{1/2}$, $T=\frac{1}{2}$ πN channel. $A(W)$ is the amplitude for process (3.1) at center-of-mass energy W, and $D(W)$ the D function for πN elastic scattering, both in the $P_{1/2}$ channels. Since $A(W)$ has the same phase as the πN elastic scattering amplitude in the physical region below the inelastic threshold, DF write a dispersion relation for $D(W)A(W)$:

$$A(W)D(W)=\frac{1}{\pi}\int_L\frac{D(W')\,\mathrm{Im}A(W')}{W'-W}dW'+\frac{1}{\pi}\int_{\mathrm{inel}}\frac{\mathrm{Im}[D(W')A(W')]}{W'-W}dW', \tag{3.2}$$

where $\int_L$ and $\int_{\mathrm{inel}}$ are, respectively, integrals round the left-hand singularities and the right-hand inelastic singularities. The magnetic moment may be obtained by taking the limit of (3.2) as $W\rightarrow M$. In Chew-Goldberger-Low-Nambu (CGLN)[5] notation, the appropriate amplitude for (3.1) is $M^{1-}/(kq)$, where M^{1-} is the magnetic dipole amplitude, and k and q are, respectively, the π and γ momenta. The πN elastic scattering amplitude has a pole at $W=M$, so that near this point D must be linear:

$$D(W)=W-M\quad\text{near pole.} \tag{3.3}$$

$A(W)$ has a direct channel pole at $W=M$, with residue $-\frac{1}{2}\mu f$, where f is the πNN coupling constant. Consequently, taking the limit of (3.2) as $W\rightarrow M$ will give an expression for the magnetic moment.

DF neglect the inelastic right-hand cut, and treat N exchange as the only contribution to $\mathrm{Im}A(W)$ on the left. This corresponds to evaluating the nucleon terms only, in the BD method. Taking the limit $W\rightarrow M$ and keeping only the left-hand cut, (3.1) becomes

$$-\tfrac{1}{2}\mu_i f=\frac{1}{\pi}\int_L\frac{D(W')\,\mathrm{Im}A_i(W')}{W'-M}dW',\quad i=S,V. \tag{3.4}$$

Nucleon-Exchange Contribution

The nucleon-exchange graph shown in Fig. 2 contributes to the left-hand singularities of A. It has short cuts centered on $W=M$ which degenerate to a pole at $W=M$ in the static limit. To distinguish between direct and cross-channel poles (only the latter contributes to the left-hand singularities) the cross-channel pole may be displaced slightly to the left to $W=M-\epsilon$, and ϵ allowed to tend to zero at the end of the calculation. In

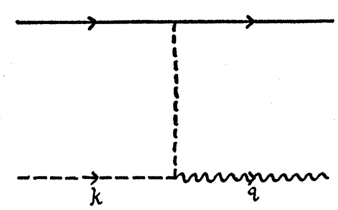

FIG. 3. Pion-exchange diagram.

[5] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1345 (1957).

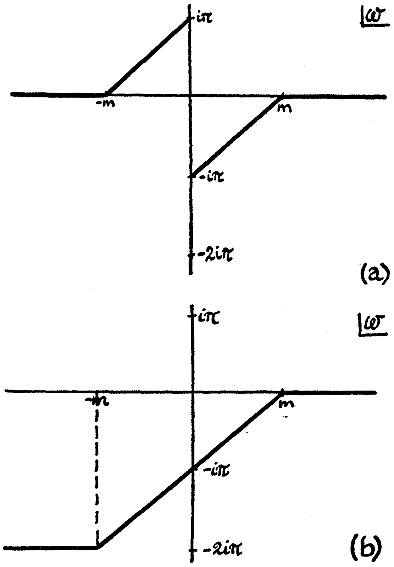

FIG. 4. (a) and (b), alternative choices for the phase of the logarithm.

the static limit the only left-hand singularity due to nucleon exchange is the pole at $W=M$, and at this point (3.3) can be used without introducing any approximation.

The nucleon contribution is then

$$\frac{1}{2\pi i}\int_{\circlearrowright}\frac{(W'-M)}{(W'-M)}\frac{R_i}{W'-M-\epsilon}dW'=-R_i.$$

$\circlearrowright$ indicates integration round the pole in a clockwise direction, and R_i is the residue at the cross-channel pole, given by

$$R_i=X_{ij}\tfrac{1}{2}\mu_i f.$$

X_{ij} is the crossing matrix element connecting direct and cross-channel residues. The crossing matrix elements are[6]

$$X_{SS}=-\tfrac{1}{3},$$
$$X_{VV}=\tfrac{1}{9},$$
$$X_{SV}=X_{VS}=0.$$

Therefore, the DF relations, neglecting π contributions, are

$$-\tfrac{1}{2}\mu_i f=-X_{ij}\tfrac{1}{2}\mu_i f \tag{3.5}$$

or

$$\mu_S=-\tfrac{1}{3}\mu_S,$$
$$\mu_V=\tfrac{1}{9}\mu_V. \tag{3.6}$$

Since there is no pion contribution to μ_S, the relation for μ_S agrees exactly with the BD relation, while the DF calculation reproduces the nucleon contribution to μ_V obtained by BD.

[6] P. Carruthers, *Introduction to Unitary Symmetry* (Interscience Publishers, Inc., New York, 1966), Chap. 7.

Inclusion of the pion-exchange diagram in $\mathrm{Im}A(W)$ on the left should correspond to BD's inclusion of pion effects.

IV. CALCULATION OF THE PION CONTRIBUTION BY DF METHOD

The pion contribution to Eq. (3.4) is

$$\frac{1}{2\pi i}\int_C\frac{D(W)T(W)}{W-M}dW, \tag{4.1}$$

where C is a contour enclosing the left-hand singularities of $T(W)$, the amplitude for the π-exchange diagram shown in Fig. 3, in the $L=1$, $J=\frac{1}{2}$, $T=\frac{1}{2}$ channel. In the static limit this is[5,7]

$$T(\omega)=\frac{ef}{3k^2}\left[1+\frac{m^2}{2\omega k}\ln\left(\frac{\omega-k}{\omega+k}\right)\right], \tag{4.2}$$

where $\omega=W-M=q$ in the static limit and $k^2=\omega^2-m^2$.

A. Singularities of $T(\omega)$ in the ω Plane

Since $k^2=\omega^2-m^2$, k has square-root branch points at $\omega=\pm m$. The cuts are taken from $+m$ to $+\infty$ and $-m$ to $-\infty$. Between $-m$ and $+m$, $k=+i|k|$. The logarithm is taken to have phase 0 above $\omega=m$. Below $\omega=m$, its argument becomes complex and its phase decreases to $-i\pi$ at $\omega=0$. Between $\omega=0$ and $\omega=-m$, the phase decreases by another $i\pi$ and below $\omega=-m$ the argument becomes real again. The logarithm can either be continued analytically across the imaginary ω axis, so that below $\omega=-m$ its phase is $-2\pi i$, or the phase below $\omega=-m$ can be made zero, in which case there is a discontinuity of $-2\pi i$ across the imaginary axis.

Consider the discontinuities of

$$\frac{1}{k^3}\ln\left(\frac{\omega-k}{\omega+k}\right).$$

Taking the logarithm to have a cut down the imaginary ω axis [Fig. 4(a)], this term also has a cut all down the imaginary axis, with discontinuity $-2\pi i/k^3$ from right to left across it. There are no other cuts.

Alternatively, the logarithm can be taken to have phase $-2\pi i$ along the negative real ω axis below $\omega=-m$ [Fig. 4(b)]. The term then has a cut along the negative real axis from $\omega=-m$ to $-\infty$, with discontinuity across it $-2\pi i(2/|k|^3)$. This is the only cut.

Apart from the cut due to the term involving the logarithm, the amplitude $T(\omega)$ has a pole at $\omega=0$. $D(\omega)$ has no left-hand singularities.

The singularities of $T(\omega)$ are shown in Fig. 5, with the contour of integration, C, to be used in evaluating (4.1).

[7] A. Donnachie and G. Shaw, Ann. Phys. (N. Y.) **37**, 333 (1966).

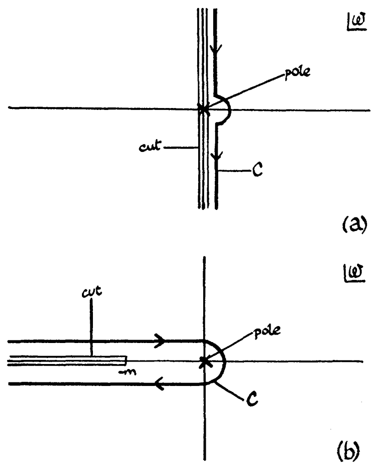

FIG. 5. (a) and (b), singularities of $T(\omega)$ and the contour of integration, C, corresponding to the choices of phase of the logarithm shown in Fig. 4.

As before, the left-hand singularities may be displaced slightly to the left so as to distinguish them from the direct-channel pole at $\omega=0$.

B. Evaluation of the π Contribution

The cut in $T(\omega)$ is chosen along the negative real ω axis and disc$T(\omega)$ is the discontinuity in $T(\omega)$ across this cut. Then the π contribution (4.1) is

$$\frac{1}{2\pi i}\int_{-\infty}^{-m}\frac{D(\omega)\,\mathrm{disc}T(\omega)}{\omega}d\omega+\frac{1}{2\pi i}\oint\frac{D(\omega)}{\omega}\frac{R}{\omega}d\omega, \quad (4.3)$$

where R is the residue of $T(\omega)$ at $\omega=0$. Near this pole $D(\omega)$ is given by (3.3) exactly, so the contribution of the pole term in (4.3) is (see Appendix A)

$$-R=-\frac{ef}{m}\frac{\pi}{6}. \quad (4.4)$$

To evaluate the cut contribution $D(\omega)$ must be known over the whole range of the cut. Now disc$T(\omega)\propto 1/k^3$, so $[\mathrm{disc}T(\omega)]/\omega$ decreases as ω^{-4} asymptotically. Consequently, a sufficiently good approximation should be to take a form for $D(\omega)$ which is suitable for the near part of the cut, and use it throughout the range.

1. *Linear D*

As a first approximation $D(\omega)$ was taken as linear. The cut contribution is then

$$\frac{1}{2\pi i}\int_{-\infty}^{-m}\mathrm{disc}T(\omega)d\omega,$$

which is evaluated in Appendix A, and found to be $+(ef/m)\frac{1}{6}\pi)$. So using a linear D function the cut contribution just cancels that of the pole, so that the total pion contribution is zero, which implies $\mu_V=0$.

2. *Improved D Function*

If a more realistic D function is used, the cut contribution will, in general, no longer cancel the pole.

Shaw and Wong[8] have suggested forms for the πN D function in the $J=\frac{1}{2}$, $T=\frac{1}{2}$ channel, taking into account the Roper resonance[9] observed at pion lab energy 600 MeV. The form of D depends on whether this resonance is mainly due to forces in the πN channel, or mainly inelastic. Two of the forms for $D(\omega)$ suggested by Shaw and Wong are shown in Fig. 6, in the region $\omega<0$. Also shown is the linear form for $D(\omega)$, and the form

$$D(\omega)=-\omega^2/m. \quad (4.5)$$

It can be seen that Shaw and Wong's D functions lie between the linear form and form (4.5) for D in the region of interest. Consequently, use of (4.5) for D in evaluating the cut contribution should give a limit on this contribution.

Explicit calculation of the cut contribution using (4.5) gives

$$\frac{1}{2\pi i}\int_{\mathrm{cut}}\frac{-\omega^2}{m}\frac{efm^2}{6\omega}\left(-2\pi i\frac{2}{|k|^3}\right)\frac{d\omega}{\omega}=\frac{ef}{3m} \quad (4.6)$$

(see Appendix A).

As we have seen, use of a linear D function makes the pion contribution identically zero. Use of form (4.5) for D gives the total contribution

$$-(ef/3m)(\tfrac{1}{2}\pi-1). \quad (4.7)$$

Since a realistic D function would appear to lie somewhere between the D forms used, the π contribution is expected to lie between these values.

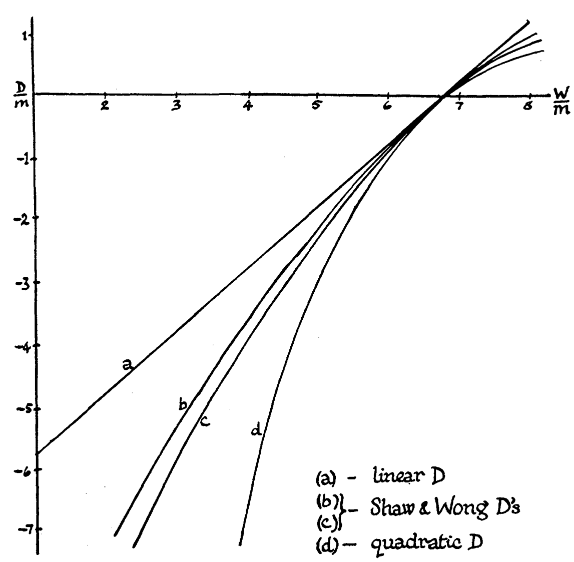

FIG. 6. D functions in the region of the left-hand cut.

[8] G. L. Shaw and D. Y. Wong, Phys. Rev. **147**, 1028 (1966).
[9] D. Roper, Phys. Rev. Letters **12**, 340 (1964).

The equations for the isovector magnetic moment in these two limiting cases then become

$$\mu_V=\tfrac{1}{9}\mu_V$$

and

$$\mu_V=\tfrac{1}{9}\mu_V+(e/3m)(\pi-2)\,,$$

giving

$$\mu_V=0$$

or

$$\mu_V\approx 5.7e/2M\,. \qquad (4.8)$$

So an upper bound for μ_V is obtained, and we expect

$$0<\mu_V<5.7e/2M\,.$$

3. *Nonrelativistic Limit of* π *Contribution*

It is of interest to see how the BD result arises from the DF treatment. The nonrelativistic limit corresponding to the BD model should be one in which the pion and nucleon are weakly bound. The nucleon is then a bound state in the $P_{1/2}$ πN channel as before, and the process $\pi+N\rightarrow N+\gamma$ becomes

$$\pi+N\rightarrow(\pi N)_B+\gamma\,, \qquad (4.9)$$

where $(\pi N)_B$ is the bound state with quantum numbers of the nucleon, and mass

$$m_B=M+m-\delta\,, \qquad (4.10)$$

where δ is the binding energy, which is small.

The pion-exchange diagram for process (4.9) is shown in Fig. 7. The amplitude for this process is

$$T_{\rm nr}(\omega)=\frac{ef}{3qk^2}\left[\omega+\frac{m^2}{2k}\ln\left(\frac{\omega-k}{\omega+k}\right)\right]. \qquad (4.11)$$

By the "nonrelativistic (nr) limit" we mean $k\ll\omega$, i.e., $\omega\approx m+k^2/2m$. The logarithmic term in $T_{\rm nr}(\omega)$ becomes, expanding the logarithm in powers of k/ω,

$$\frac{efm^2}{3qk^2}\frac{1}{2k}\left(-\frac{2k}{\omega}-\frac{2k^3}{\omega^3}\cdots\right)=\frac{efm^2}{3qk^2}\left(-\frac{1}{\omega}-\frac{k^2}{3\omega^3}\cdots\right).$$

Then this term is a polynomial in k^2. Since the argument of the logarithm is real in the nonrelativistic limit, the phase is zero, and $T_{\rm nr}(\omega)$ can be expressed as a polynomial in k^2. Consequently, there is no left-hand cut in the nonrelativistic limit, and the only remaining left-hand singularity is a pole at $q=0$.

The center-of-mass energy is

$$W=M+\omega=m_B+q\,.$$

Therefore, at the pole $q=0$,

$$\omega=m_B-M=m-\delta$$

from (4.10). At the pole the pion momentum is k_0, given by

$$k_0^2/2m=-\delta\,.$$

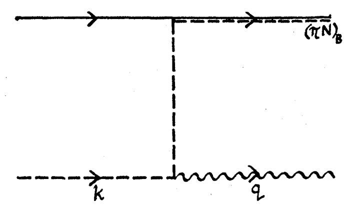

FIG. 7. Nonrelativistic pion-exchange diagram.

Since in the nonrelativistic limit we must have $k_0/m\ll 1$, we must also have $\delta/m\ll 1$. The pole $q=0$ is less than m away from $\omega=0$, so that a linear form for D is still a reasonably good approximation. Then the nonrelativistic π contribution is

$$\frac{1}{2\pi i}\int_{\circlearrowleft}\frac{R}{\omega-(m-\delta)}d\omega\,,$$

where R is the residue of $T_{\rm nr}(\omega)$ at the pole:

$$R=\lim_{q\to 0}\frac{ef}{3k^2}\left[\omega+\frac{m^2}{2k}\ln\left(\frac{\omega-k}{\omega+k}\right)\right].$$

Expanding the logarithm in powers of k/ω gives

$$\begin{aligned}R&=\frac{ef}{3k_0^2}\left[\omega_0+\frac{m^2}{2k_0}\left(-\frac{2k_0}{\omega_0}-\frac{2k_0^3}{\omega_0^3}\cdots\right)\right]\\&=(ef/3k_0^2\omega_0)[\omega_0^2-m^2-m^2k_0^2/3\omega_0^2]+O(k_0/\omega_0)\\&\approx(ef/3\omega_0)[1-\tfrac{1}{3}]\\&=(2/9)ef/m\,.\end{aligned}$$

Therefore, the π contribution in this limit is $-(2/9)ef/m$, which gives for the magnetic moment

$$\mu_V=(1/9)\mu_V+(4/9)e/m\,,$$

and this is exactly the result of BD [Eq. (2.3)].

V. CONCLUSION

In a suitable nonrelativistic limit, the π contribution to the nucleon magnetic moments calculated by the DF method, agrees with the BD estimate. The calculation indicates that μ_V lies between the limits 0 and $5.7e/2M$, so that π exchange could be the dominant contribution.

The SU_2 calculations simply illustrate the main features of the two methods. In a more sophisticated relativistic calculation, such as that of Abarbanel, Callan, and Sharp,[4] other exchange mechanisms must be included; in particular the exchange of an isoscalar particle such as the ω is necessary for the prediction of a nonzero isoscalar magnetic moment. A realistic calculation should treat simultaneously the magnetic moments of all the baryons in the SU_3 octet and decuplet.

ACKNOWLEDGMENTS

I would like to thank Dr. J. E. Paton for suggesting the problem, and both him and Dr. I. J. R. Aitchison for helpful discussions.

APPENDIX

A. Residue at Pole

$$T(\omega)=\frac{ef}{3k^2}\left[1+\frac{m^2}{2\omega k}\ln\left(\frac{\omega-k}{\omega+k}\right)\right].$$

Near the pole $\omega=0$ the logarithm has phase $-i\pi$, and can be expanded in a power series in ω/k:

$$\ln\left(\frac{\omega-k}{\omega+k}\right)=-i\pi+\mathcal{P}\ln\left(\frac{1-\omega/k}{1+\omega/k}\right)$$

$$\approx -i\pi-2\left[\frac{\omega}{k}+\frac{1}{3}\left(\frac{\omega}{k}\right)^3+\cdots\right].$$

Therefore,

$$R=\lim_{\omega\to 0}\frac{ef}{3k^2}\left[\omega+\frac{m^2}{2k}\left(-i\pi-\frac{2\omega}{k}\cdots\right)\right]$$

$$=\tfrac{1}{6}ef(-i\pi m^2)\frac{1}{k^3|_{\omega=0}}.$$

But $k^2=\omega^2-m^2$. Therefore,

$$R=\frac{ef}{m}\frac{\pi}{6}.$$

Taking the cut along the negative real axis, the pole contribution is

$$\frac{1}{2\pi i}\int_{\circlearrowleft}\frac{R}{\omega}d\omega=-\frac{ef}{m}\frac{\pi}{6}. \tag{A1}$$

B. Evaluation of the Cut Contribution

The left-hand cut of the π-exchange amplitude is taken along the negative real ω axis from $-\mu$ to $-\infty$.

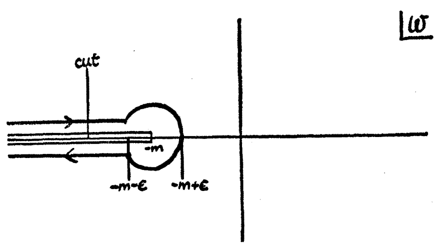

FIG. 8. Contour of integration around the left-hand cut.

The discontinuity across this cut is $-(4\pi i/|k|^3)efm^2/6\omega$. Since $k^2=\omega^2-m^2$, this discontinuity appears to diverge at the upper end of the cut because of the pole $k^2=0$. Consequently, in evaluation of the integral around the left-hand cut, which is finite, the contour must be distorted to avoid the pole, as shown in Fig. 8.

The cut contribution to the π term then becomes

$$\lim_{\epsilon\to 0}\left[\frac{1}{2\pi i}\int_{-\infty}^{-\mu-\epsilon}\frac{\mathrm{disc}T(\omega)D(\omega)}{\omega}d\omega\right.$$

$$\left.+\frac{1}{2\pi i}\int\frac{\frac{1}{2}\,\mathrm{disc}T(\omega)D(\omega)}{\omega}d\omega\right], \tag{A2}$$

where the second integral is around the circumference of the circle with radius ϵ and center $\omega=-m$.

1. Linear D

Using a linear D function, Eq. (A2) becomes

$$\lim_{\epsilon\to 0}\left[-\tfrac{1}{3}efm^2\left(\int_{-\infty}^{-\mu-\epsilon}\frac{d\omega}{\omega(\omega^2-m^2)^{3/2}}+\frac{1}{2}\int\frac{d\omega}{\omega(\omega^2-m^2)^{3/2}}\right)\right].$$

On the circle $\omega=-m-\epsilon e^{i\theta}$, so that the integral around the circle becomes

$$\lim_{\epsilon\to 0}\left[-\tfrac{1}{6}efm^2\int_0^{2\pi}\frac{-\epsilon ie^{i\theta}d\theta}{-m(2m\epsilon e^{i\theta})^{3/2}}\right]$$

$$=\lim_{\epsilon\to 0}\left[-\tfrac{1}{6}efm^2\frac{1}{2m^2}\int_0^{2\pi}\frac{e^{-i\theta/2}id\theta}{(2m\epsilon)^{1/2}}\right]$$

$$=\lim_{\epsilon\to 0}\left(-\tfrac{1}{3}ef\frac{1}{(2m\epsilon)^{1/2}}\right).$$

The integral from $-\infty$ to $-m-\epsilon$ is

$$\lim_{\epsilon\to 0}\left[-\tfrac{1}{3}efm^2\left(-\frac{\pi}{2m^3}-\frac{1}{m^2(2m\epsilon)^{1/2}}\right)\right].$$

So with linear D, Eq. (A2) gives

$$+\frac{ef}{m}\frac{\pi}{6}. \tag{A3}$$

2. $D=-\omega^2/m$

Using $D=-\omega^2/m$, Eq. (A2) becomes

$$\lim_{\epsilon\to 0}\left[+\tfrac{1}{3}efm\left(\int_{-\infty}^{-m-\epsilon}\frac{d\omega}{(\omega^2-m^2)^{3/2}}+\frac{1}{2}\int_{\text{circle}}\frac{d\omega}{(\omega^2-m^2)^{3/2}}\right)\right].$$

The integral around the circle becomes

$$\lim_{\epsilon\to 0}\left[\tfrac{1}{6}efm\int_0^{2\pi}\frac{-\epsilon ie^{i\theta}d\theta}{(2m\epsilon e^{i\theta})^{3/2}}\right]$$

$$=\lim_{\epsilon\to 0}\left[-\tfrac{1}{12}ef\frac{1}{(2m\epsilon)^{1/2}}\int_0^{2\pi}ie^{-i\theta/2}d\theta\right]$$

$$=\lim_{\epsilon\to 0}\left[-\tfrac{1}{3}ef\frac{1}{(2m\epsilon)^{1/2}}\right].$$

The integral from $-\infty$ to $-m-\epsilon$ is

$$\lim_{\epsilon\to 0}\left\{\tfrac{1}{3}efm\left[\frac{\omega}{-m^2(\omega^2-m^2)^{1/2}}\right]_{-\infty}^{-m-\epsilon}\right\}$$

$$=\lim_{\epsilon\to 0}\left\{\frac{-ef}{3m}\left[\frac{-m}{(2m\epsilon)^{1/2}}-1\right]\right\}.$$

So with quadratic D, Eq. (A2) gives

$$ef/3m. \tag{A4}$$

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A Field Theory of Currents*†

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A new type of model field theory is constructed. Only currents appear as the coordinates. The canonical formalism is abandoned. The self-consistency, Lorentz covariance, energy-momentum conservation, etc., are checked.

I. INTRODUCTION

It is widely recognized among high-energy physicists that field variables may not be adequate to describe strong interactions. This implies among other things that we have to give up the canonical formalism. In other words, we cannot attribute to each particle a field which satisfies the canonical commutation relation.

Field theory itself, however, may not disappear. In fact, Gell-Mann repeatedly stresses[1] that currents which are measurable through electromagnetic, weak, and gravitational interactions will survive. We already partly understand the role of currents in the strong-interaction symmetry. But except for very limited cases we know nothing about the dynamical role of currents in strong interactions.

In this paper we try to investigate the possibility of constructing a field theory in terms of these observable currents.

We understand that the substitutes for the canonical commutation relations are the equal-time commutation relations among currents. Then, just as we construct a Hilbert space as a representation of the canonical commutation relations in the ordinary field theory, so we should find a representation of the equal-time commutation relations.[2] We may be able to find a physical representation independently of underlying field theory. Yet we have to be sure that we can actually construct this underlying field theory of currents.

First of all, what are the observable currents? Following Gell-Mann[1] we assume that they are the 8 vector currents $V_\mu{}^i(x)$, the 8 axial-vector currents $A_\mu{}^i(x)$, and the gravitational tensor $\theta_{\mu\nu}(x)$. No other observable currents have been implied so far by experiments. Suppose they are all independent dynamical variables. Then we have to write the equal-time commutation relations among all the components of these variables. This is obviously one possibility. But a more attractive theory would be when $\theta_{\mu\nu}(x)$ is expressible in terms of $V_\mu{}^i(x)$ and $A_\mu{}^i(x)$. We will only consider such a model in this paper.

We do not intend to construct a theory which describes a realistic strong interaction. This is possible only after we know enough about the equal-time commutation relations. Our intention is rather to show that at least it is possible to construct a simple nontrivial model, which may or may not be some limiting case of the realistic theory. Anyway, we will choose a relatively simple set of current commutators whenever we do not have experimental information.

What do we mean when we ask if a model is possible? Though we give up the canonical commutation relations we retain almost all the other axioms of quantum field theory. Thus when we construct a model we have to

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[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[2] R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966); M. Gell-Mann, D. Horn, and J. Weyers (to be published); K. Bardakci, M. B. Halpern, and G. Segrè, Phys. Rev. **168**, 1728 (1968).