Current Algebra and Ward Identities: Three- and Four-Point Functions

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We present a technique which allows us to display explicitly all the information which current algebra yields about n-point functions of vector and axial-vector currents. We write all three- and four-point functions in terms of a few primitive functions which are not determined by current algebra. Any approximation to these functions (subject to a single constraint) when used in our formulation yields three- and four-point functions guaranteed to satisfy all constraints imposed by current algebra and partially conserved axial-vector current. We give a very simple model where the primitive functions are given by as smooth functions of the momenta as possible and apply this model to the π^+-π^0 electromagnetic mass difference, A_1 decay, and π-π and π-ρ scattering.

I. INTRODUCTION

MANY results have been derived[1] from the assumptions of a local chiral $SU(2)\otimes SU(2)$ algebra[2] of vector and axial-vector current densities together with the partial conservation hypothesis[3] relating the divergence of the axial-vector current to the π-meson field. In this paper we shall present a technique for explicitly displaying all of the information that current algebra gives us about three- and four-point functions of the currents.[4-6] Making the additional assumption that the ρ, A_1, and π mesons dominate the 1^-, 1^+, and 0^- channels created by the currents from the vacuum, we obtain the current-algebra constraints imposed on the strong interactions of these mesons with each other as well as their radiative decays, etc.

Our method is to exploit the Ward identities obtained by taking divergences of the n-point functions composed of the isotopic vector currents $V_\mu(x)$, $A_\mu(x)$, and $\partial_\mu A_\mu(x)$. An n-point function which contains l factors of $\partial_\mu A_\mu(x)$ will be called "degree l"; degree-zero n-point functions will be called primitive. Since

$$GA_\mu(x)G^{-1}=-A_\mu(x),$$

where G is the G-parity operator, any n-point function must contain an even number of factors of $A_\mu(x)$ and/or $\partial_\mu A_\mu(x)$, so that we can label an n-point function with the triad $(n,2m,l)$ where $n-2m$ is the number of factors of $V_\mu(x)$ and, of course, $l<2m<n$. For example,

$$\langle 0|T(\partial_\mu A_\mu(x)A_\nu(y)V_\lambda(z)V_\sigma(0))|0\rangle=(4,2,1).$$

We may take the divergence of an n-point function either with respect to a vector line or an axial-vector line. In the first case, the Ward identity relates the longitudinal part of the n-point function to a sum of $(n-1)$-point functions. We shall call this a vector-constraint condition on $(n,2m,l)$. In the second case, we relate the longitudinal part of $(n,2m,l)$ to $(n,2m,l+1)$ as well as to $(n-1)$-point functions. Thus the axial-vector Ward identities allow us to write all nonprimitive n-point functions in terms of longitudinal components of primitive r-point functions for $r\leqslant n$. These primitive r-point functions are completely arbitrary from the point of view of current algebra except for their vector-constraint conditions. It will turn out that a sum rule following from the equality of vector and axial-vector Schwinger terms derived by Weinberg[7] will insure that all nonprimitive n-point functions will then automatically satisfy their vector-constraint equations.

The job of current algebra is finished when we have explicitly written the nonprimitive functions in terms of the primitive ones and specified the vector constraints. Any approximation scheme for the primitive functions will then satisfy current algebra. One particularly simple choice is to let the primitive functions have the smoothest dependence on the momenta that is possible.[8] To be more precise, the Ward identities imply a type of tree-diagram form for the n-point function

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[1] B. Renner, Rutherford Laboratory Report No. RHEL/R126, 1966 (unpublished).
[2] M. Gell-Mann, Physics 1, 63 (1964).
[3] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).
[4] We mean the chiral $SU(2)\otimes SU(2)$ algebra generated by the time components of the currents together with the usual transformation properties of the space components under this algebra. We do not consider, in this paper, the constraints imposed by commutators of the space components with each other which provide further (more model-dependent) information.
[5] Our results for three-point functions are contained implicitly in H. J. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967).
[6] In principle we could study n-point functions with these techniques although the algebra appears to become prohibitively complicated.
[7] S. Weinberg, Phys. Rev. Letters 18, 507 (1967). As is well known, this sum rule in no way depends on the assumptions of a conserved axial-vector current or zero-pion mass.
[8] This approximation was made throughout Ref. 5.

with combinations of primitive functions at the vertices. We make the smoothness assumption for the n-point vertex in the tree expansion of the n-point function.[9] With this approximation we find that our results are equivalent to those obtained from phenomenological Lagrangian techniques,[10] showing where the correspondence between the two methods lies.

II. COMMUTATION RELATIONS AND DIAGONALIZED n-POINT FUNCTIONS

The equal-time commutators which define our model are (a, b, c run from 1 to 3)[11]:

$$[V_0^a(x),V_\mu^b(y)]\delta(x^0-y^0) = i\epsilon^{abc}V_\mu^c(x)\delta(x-y)+\cdots, \quad (1a)$$

$$[V_0^a(x),A_\mu^b(y)]\delta(x^0-y^0) = i\epsilon^{abc}A_\mu^c(x)\delta(x-y)+\cdots, \quad (1b)$$

$$[A_0^a(x),A_\mu^b(y)]\delta(x^0-y^0) = i\epsilon^{abc}V_\mu^c(x)\delta(x-y)+\cdots, \quad (1c)$$

$$[V_0^a(x),\partial_\mu A_\mu^b(y)]\delta(x^0-y^0) = i\epsilon^{abc}\partial_\mu A_\mu^c(x)\delta(x-y), \quad (2a)$$

$$[A_0^a(x),\partial_\mu A_\mu^b(y)]\delta(x^0-y^0) = \sigma^{ba}(x)\delta(x-y). \quad (2b)$$

Equation (2b) defines the σ field. Conservation of the vector current, together with the locality assumption of Eq. (2b), yields

$$\sigma^{ab}(x)=\sigma^{ba}(x),$$

and using the Jacobi identity, we have

$$[V_0^a(x),\sigma^{bc}(y)]\delta(x^0-y^0) = [i\epsilon^{abe}\sigma^{ec}(x)+i\epsilon^{ace}\sigma^{be}(x)]\delta(x-y). \quad (3a)$$

We make the assumption that the equal-time commutator of σ with A_0^a is local and define

$$[A_0^a(x),\sigma^{bc}(y)]\delta(x^0-y^0)=\sigma^{bca}(x)\delta(x-y). \quad (3b)$$

Using the Jacobi identity in Eq. (3b), we find[12]

$$\sigma^{abc}(x)-\sigma^{acb}(x)=\delta^{ab}\partial_\mu A_\mu^c(x)-\delta^{ac}\partial_\mu A_\mu^b(x). \quad (4)$$

Of course, the commutation relations involving $\sigma(x)$ are true in the σ model[3] where, further,

$$\sigma^{ab}(x)=\delta^{ab}\sigma(x).$$

We shall, however, allow $\sigma^{ab}(x)$ to have an $I=2$ part as well as an $I=0$ part.

We shall use the following conventions to define the Fourier transform of the n-point function $(n,2m,l)$: (1) For $n=2$ we use the symbol Δ, for $n=3$ the symbol V, and for $n=4$ the symbol T. (2) Each function carries a superscript $n-2m$. (3) Axial currents are placed to the left of vector currents and divergences of axial currents are placed to the left of axial currents. (4) Fourier transforms are taken with all lines incoming. Of course, an n-point function carries n isotopic spin indices, is a function of $n-1$ momenta, and has $n-l$ vector indices. For example,

$$\Delta^{(2)}(q)_{\mu\nu}{}^{ab}=\int d^4x\, e^{-iq\cdot x}\langle 0|T(V_\mu^a(x)V_\nu^b(0))|0\rangle,$$

$$\Delta^{(0)}(q)_\nu{}^{ab}=\int d^4x\, e^{-iq\cdot x} \times\langle 0|T(\partial_\mu A_\mu^a(x)A_\nu^b(0))|0\rangle,$$

$$V^{(1)}(q_1,q_2)_{\nu\lambda}{}^{abc}=\int d^4x d^4y\, e^{-iq_1\cdot x-iq_2\cdot y} \times\langle 0|T(\partial_\mu A_\mu^a(x)A_\nu^b(y)V_\lambda^c(0))|0\rangle,$$

$$T^{(0)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd}=\int d^4xd^4yd^4z\, e^{-iq_1\cdot x-iq_2\cdot y-iq_3\cdot z} \times\langle 0|T(\partial_\mu A_\mu^a(x)\partial_\nu A_\nu^b(y)A_\lambda^c(z)A_\sigma^d(0))|0\rangle.$$

Spectral representations for the two-point functions can be easily written down[13] and have been studied by Weinberg.[7] We shall follow Ref. 7 in using pion dominance for the 0^- spectral function of the axial-vector current.[14] We have

$$\Delta^{(2)}(q)_{\mu\nu}{}^{ab}=-i\delta^{ab}[\Delta^V(q)_{\mu\nu}-g_{\mu0}g_{\nu0}C_V], \quad (5a)$$

$$\Delta^{(0)}(q)_{\mu\nu}{}^{ab}=-i\delta^{ab}[\Delta^A(q)_{\mu\nu}+q_\mu q_\nu F_\pi^2/(m_\pi^2-q^2) -g_{\mu0}g_{\nu0}(C_A-F_\pi^2)], \quad (5b)$$

$$\Delta^{(0)}(q)_\nu{}^{ab}=\delta^{ab}q_\nu F_\pi^2m_\pi^2/(m_\pi^2-q^2), \quad (5c)$$

$$\Delta^{(0)}(q)^{ab}=-i\delta^{ab}F_\pi^2m_\pi^4/(m_\pi^2-q^2), \quad (5d)$$

where

$$\Delta^{V,A}(q)_{\mu\nu}=\int dm^2(g_{\mu\nu}-q_\mu q_\nu/m^2) \times\rho_{V,A}(m^2)/(m^2-q^2), \quad (6)$$

$$C_{V,A}=\int dm^2\rho_{V,A}(m^2)/m^2, \quad (7)$$

$$q_\mu\Delta^{V,A}(q)_{\mu\nu}=C_{V,A}q_\nu. \quad (8)$$

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[9] For three-point functions this is the three-point function itself. For four-point functions it is frequently called the contact or seagull term. Even this statement is not precisely correct. When axial currents are involved, we perform a diagonalization procedure which defines n-point functions which receive contributions from *either* the 0^- or 1^+ channel. We make the smoothness assumption for the diagonalized primitive contact term.

[10] J. Schwinger, Phys. Letters **24B**, 473 (1967); L. S. Brown, Phys. Rev. **163**, 1802 (1967); S. Weinberg, Phys. Rev. Letters **18**, 188 (1967); Phys. Rev. **166**, 1568 (1968); J. Wess and B. Zumino, *ibid.* **163**, 1727 (1967); B. W. Lee and H. T. Nieh, *ibid.* **166**, 1507 (1968); W. A. Bardeen and B. W. Lee, Canadian Summer Institute Lectures, 1967 (to be published).

[11] We have not written down the Schwinger terms. We make the assumption that they have no $I=1$ part which implies Weinberg's sum rule [Ref. 7, Eq. (1)]. We find that by imposing this sum rule, we may consistently ignore the Schwinger terms together with the noncovariant part of the T product in the Ward identities.

[12] N. Khuri, Phys. Rev. **153**, 1477 (1967).

[13] K. A. Johnson, Nucl. Phys. **25**, 431 (1961).

[14] Our metric is $- - - +$ so our results look slightly different from those of Refs. 5 and 7. We have also used a different normalization for the currents as expressed in Eq. (1).

For ρ and A_1 dominance of the 1^- and 1^+ spectral functions, we have

$$\rho_{V,A}(q^2)=-g_{\rho,A_1}{}^2\delta(m_{\rho,A_1}{}^2-q^2). \quad (9)$$

Weinberg's sum rule,[7] expressing the equality of the vector and axial-vector Schwinger terms, is

$$F_\pi{}^2=C_A-C_V. \quad (10)$$

The n-point functions as we have defined them are not the most convenient ones for presenting our results. The reason is that if q is a momentum associated with an axial-vector current, then the n-point function, considered as a function of q^2, receives contributions both from 0^- and 1^+ states. We will define a new set of diagonalized n-point functions (denoted by a bar over them) such that, on the mass shell, a channel associated with an axial current and an index ν receives contributions only from 1^+ states, while those without such an index receive contributions only from 0^- states. Of course, off the mass shell the 1^+ channels receive contributions from the (unphysical) 0^- part of the 1^+ particle in a way consistent with the propagator functions defined in Eq. (6).

For example,

$$T^{(2)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd}\equiv\bar{T}^{(2)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd}, \quad (11a)$$

$$T^{(2)}(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd}\equiv\bar{T}^{(2)}(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd} -iq_{2\nu}m_\pi{}^{-2}\bar{T}^{(2)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd}, \quad (11b)$$

$$\begin{aligned}T^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}\equiv{}&\bar{T}^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}\\ &-iq_{1\mu}m_\pi{}^{-2}\bar{T}^{(2)}(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd}\\ &-iq_{2\nu}m_\pi{}^{-2}\bar{T}^{(2)}(q_2,q_1,q_3)_{\mu\lambda\sigma}{}^{bacd}+(-iq_{1\nu}m_\pi{}^{-2})\\ &\times(-iq_{2\nu}m_\pi{}^{-2})\bar{T}^{(2)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd}. \quad (11c)\end{aligned}$$

Mnemonically, we replace each factor of $A_\mu{}^a(x)$ appearing in an n-point function by $A_\mu{}^a(x)\to\bar{A}_\mu{}^a(x) -iq_\mu m_\pi{}^{-2}\partial_\mu A_\mu(x)$ and write the resultant functions with bars over them according to the convention already given.[15] The barred functions in each line satisfy our diagonalization criteria in virtue of pion dominance of the (physical) 0^- channel. Of course, if the vector current were not conserved, it too would be diagonalized.

Finally, having defined diagonalized n-point functions whose structures in the $q_i{}^2$ match those of the propagators, we are led to define reduced, diagonalized n-point functions which explicitly display this structure. For three-point functions, these will be denoted Γ, while for four-point functions, we call them M. For example, we define

$$\begin{aligned}\bar{T}^{(0)}(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd}&\\ \equiv\Delta_\pi(q_1)\Delta^A(q_2)_{\nu\nu'}\Delta^A(q_3)_{\lambda\lambda'}\Delta^A(q_4)_{\sigma\sigma'}&\\ \times M^{(0)}(q_1,q_2,q_3)_{\nu'\lambda'\sigma'}{}^{abcd}, &\quad (12a)\end{aligned}$$

$$\begin{aligned}\bar{T}^{(2)}(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd}&\\ \equiv\Delta_\pi(q_1)\Delta^A(q_2)_{\nu\nu'}\Delta^V(q_3)_{\lambda\lambda'}\Delta^V(q_4)_{\sigma\sigma'}&\\ \times M^{(2)}(q_1,q_2,q_3)_{\nu'\lambda'\sigma'}{}^{abcd}, &\quad (12b)\end{aligned}$$

$$\begin{aligned}\bar{V}^{(1)}(q_1,q_2)_{\lambda\sigma}{}^{abc}&\\ \equiv\Delta_\pi(q_1)\Delta^A(q_2)_{\lambda\lambda'}\Delta^A(k)_{\sigma\sigma'}\Gamma^{(1)}(q_1,q_2)_{\lambda'\sigma'}{}^{abc}, &\quad (12c)\end{aligned}$$

where $\Delta^{A,V}(q)$ have been defined in Eq. (6),

$$\Delta_\pi(q)=F_\pi{}^2m_\pi{}^4/(m_\pi{}^2-q^2)$$

as in Eq. (5d),

$$q_4=-q_1-q_2-q_3,$$

and

$$k=-q_1-q_2.$$

All other reduced functions are defined analogously to Eqs. (12).

Although we shall not be primarily concerned with them in this paper, we must also define two- and three-point functions which contain the fields $\sigma^{ab}(x)$ and $\sigma^{abc}(x)$ of Eqs. (2b) and (3b) since they appear in the Ward-identity equations for $T^{(2)}$ and $T^{(0)}$. The three-point functions we shall need are

$$\begin{aligned}V_\Sigma{}^{(2)}(q_1q_2)_{\lambda\sigma}{}^{abcd}&=\int d^4xd^4y\,e^{-iq_1\cdot x-iq_2\cdot y}\\ &\times\langle0|T(\sigma^{ab}(x)V_\lambda{}^c(y)V_\sigma{}^d(0))|0\rangle, \quad (13)\end{aligned}$$

$$\begin{aligned}V_\Sigma{}^{(0)}(q_1,q_2)_{\lambda\sigma}{}^{abcd}&=\int d^4xd^4y\,e^{-iq_1\cdot x-iq_2\cdot y}\\ &\times\langle0|T(\sigma^{ab}(x)A_\lambda{}^c(y)A_\sigma{}^d(0))|0\rangle, \quad (14a)\end{aligned}$$

$$\begin{aligned}V_\Sigma{}^{(0)}(q_1,q_2)_{\sigma}{}^{abcd}&=\int d^4xd^4y\,e^{-iq_1\cdot x-iq_2\cdot y}\\ &\times\langle0|(\sigma^{ab}(x)\partial_\lambda A_\lambda{}^c(y)A_\sigma{}^d(0))|0\rangle, \quad (14b)\end{aligned}$$

$$\begin{aligned}V_\Sigma{}^{(0)}(q_1,q_2)^{abcd}&=\int d^4xd^4y\,e^{-iq_1\cdot x-iq_2\cdot y}\\ &\times\langle0|T(\sigma^{ab}(x)\partial_\lambda A_\lambda{}^c(y)\partial_\sigma A_\sigma{}^d(0))|0\rangle. \quad (14c)\end{aligned}$$

The diagonalized $\bar{V}$ functions are defined by the same procedure discussed above, while the reduced diagonalized three-point functions will be called Γ_Σ and have the structure in $q_2{}^2$ and k^2 explicitly factored out.[16]

We also define the two-point functions

$$\Lambda(q)_\lambda{}^{abcd}=\int d^4x\,e^{-iq\cdot x}\langle0|T(\sigma^{abc}(x)A_\lambda{}^d(0))|0\rangle, \quad (15a)$$

$$\Lambda(q)^{abcd}=\int d^4x\,e^{-iq\cdot x}\langle0|T(\sigma^{abc}(x)\partial_\lambda A_\lambda{}^d(0))|0\rangle. \quad (15b)$$

[15] This procedure has also been discussed by R. Arnowitt, M. Friedman, and P. Nath, Phys. Rev. Letters **19**, 1085 (1967). See also B. W. Lee and H. T. Nieh (Ref. 10) for the diagonalization in a Lagrangian context.

[16] We could also factor out the structure in the variable q^2, using the σ propagator defined below; we do not do this here since our main purpose at this time is not the study of the structure of the σ terms.

These two-point functions can be studied by the same spectral techniques used to study the propagation in Eq. (5) with the result[17]

$$\Lambda(q)_\lambda{}^{abcd}=iq_\lambda m_\pi^{-2}\Lambda(q)^{abcd}. \tag{16}$$

Using Eqs. (4) and (5d), we find

$$\Lambda(q)^{abcd}-\Lambda(q)^{acbd}=-i(\delta^{ab}\delta^{cd}-\delta^{ac}\delta^{bd})\Delta_\pi(q). \tag{17}$$

Finally, the σ propagator

$$\Sigma(q)^{abcd}=\int d^4x\, e^{-iq\cdot x}\langle 0|T(\sigma^{ab}(x)\sigma^{cd}(0))|0\rangle. \tag{18}$$

III. WARD IDENTITIES

To derive the Ward-identity equations we use the fundamental identity

$$\begin{aligned}\frac{\partial}{\partial x_\mu}&\langle 0|T(j_\mu(x)j_1(x_1)j_2(x_2)\cdots j_n(x_n))|0\rangle\\ &=\langle 0|T(\partial_\mu j_\mu(x)j_1(x_1)j_2(x_2)\cdots j_n(x_n))|0\rangle\\ &\quad+\sum_{i=1}^{n}\langle 0|\delta(x^0-x_i^0)T([j_0(x),j_i(x_i)]j_1(x_1)\cdots\\ &\qquad\times j_{i-1}(x_{i-1})j_{i+1}(x_{i+1})\cdots j_n(x_n))|0\rangle. \end{aligned}\tag{19}$$

We shall carry through the steps of our derivation for the functions $T^{(2)}$ as an example in this section and shall present the results for the other four- and three-point functions in Sec. IV.

Applying Eq. (19) to the four-point functions $T^{(2)}$, we arrive at the following set of Ward identities:

$$\begin{aligned}q_4{}^\sigma T^{(2)}&(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}\\ &=\epsilon^{dae}V^{(1)}(q_1+q_4,\, q_2)_{\mu\nu\lambda}{}^{ebc}+\epsilon^{dbe}V^{(1)}(q_1,\, q_2+q_4)_{\mu\nu\lambda}{}^{aec}\\ &\quad+\epsilon^{dce}V^{(1)}(q_1,q_2)_{\mu\nu\lambda}{}^{abe}, \end{aligned}\tag{20a}$$

$$\begin{aligned}q_4{}^\sigma T^{(2)}&(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd}\\ &=\epsilon^{dae}V^{(1)}(q_1+q_4,\, q_2)_{\nu\lambda}{}^{ebc}+\epsilon^{dbe}V^{(1)}(q_1,\, q_2+q_4)_{\nu\lambda}{}^{aec}\\ &\quad+\epsilon^{dce}V^{(1)}(q_1,q_2)_{\nu\lambda}{}^{abe}, \end{aligned}\tag{20b}$$

$$\begin{aligned}q_4{}^\sigma T^{(2)}&(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd}\\ &=\epsilon^{dae}V^{(1)}(q_1+q_4,\, q_2)_{\lambda}{}^{ebc}+\epsilon^{dbe}V^{(1)}(q_1,\, q_2+q_4)_{\lambda}{}^{aec}\\ &\quad+\epsilon^{dce}V^{(1)}(q_1,q_2)_{\lambda}{}^{abe}, \end{aligned}\tag{20c}$$

$$\begin{aligned}q_2{}^\nu T^{(2)}&(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}\\ &=-iT^{(2)}(q_2,q_1,q_3)_{\mu\lambda\sigma}{}^{bacd}+\epsilon^{bae}V^{(3)}(q_1+q_2,\, q_3)_{\mu\lambda\sigma}{}^{ecd}\\ &\quad+\epsilon^{bce}V^{(1)}(q_1,\, q_2+q_3)_{\mu\lambda\sigma}{}^{aed}\\ &\quad+\epsilon^{bde}V^{(1)}(q_1,\, q_2+q_4)_{\mu\sigma\lambda}{}^{aec}, \end{aligned}\tag{21a}$$

$$\begin{aligned}q_2{}^\nu T^{(2)}&(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd}\\ &=-iT^{(2)}(q_2,q_1,q_3)_{\lambda\sigma}{}^{bacd}-iV_\Sigma{}^{(2)}(q_1+q_2,\, q_3)_{\lambda\sigma}{}^{abcd}\\ &\quad+\epsilon^{bce}V^{(1)}(q_1,\, q_2+q_3)_{\lambda\sigma}{}^{aed}\\ &\quad+\epsilon^{bde}V^{(1)}(q_1,\, q_2+q_4)_{\sigma\lambda}{}^{aec}. \end{aligned}\tag{21b}$$

As we have mentioned, we regard Eqs. (21) as equations which yield $T_{\mu\lambda\sigma}{}^{(2)}$ and $T_{\lambda\sigma}{}^{(2)}$ as functions of $T_{\mu\nu\lambda\sigma}{}^{(2)}$, while Eqs. (20) are the vector-constraint equations. Of course, to implement this program, we need analogous equations for the three-point functions. We shall not write these down here, the equations for the spin-diagonalized; reduced three-point functions will be given below.

At this point we may ask whether all the vector-constraint equations are independent. That is, if $T_{\mu\lambda\sigma}{}^{(2)}$ and $T_{\lambda\sigma}{}^{(2)}$ are given, through Eqs. (21), as functions of $T_{\mu\nu\lambda\sigma}{}^{(2)}$, which satisfies Eq. (20a), are Eqs. (20b) and (20c) automatically true? The answer is yes, if the Weinberg sum rule Eq. (10) is satisfied. We must, of course, use the three-point function Ward identities, as well as Eqs. (8), to obtain this result. This is essentially the method Weinberg used to derive this equation although he studied the three-point function equations directly.

In fact, the statement remains true for the n-point function. If the axial-vector Ward identities are used to define all nonprimitive ones, then, if the primitive n-point functions satisfy the vector constraints so do the nonprimitive ones, if Eq. (10) is satisfied. This shows that our assumption of neglecting Schwinger terms in the commutation relations (1) is self-consistent and may be made in a theory where the Schwinger terms have no $I=1$ part. We further see that of the entire set of Ward identities we need concern ourselves only with the axial-vector equations linking nonprimitive and primitive functions together with a single vector constraint on the primitive function.[18]

Substituting Eqs. (11) and (12) which define the diagonalized reduced functions M and Γ into Eqs. (20) and (21), we obtain

$$\begin{aligned}C_V q_4{}^\sigma M^{(2)}&(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}\\ &=\epsilon^{dce}\Delta^V(q_3)^{-1}{}_{\lambda\lambda'}\Delta^V(q_3+q_4)_{\lambda'\lambda''}\Gamma^{(3)}(q_1,q_2)_{\mu\nu\lambda''}{}^{abe}+\Big\{\epsilon^{dae}\Delta^A(q_1)^{-1}{}_{\mu\mu'}[\Delta^A(q_1+q_4)_{\mu'\mu''}\Gamma^{(1)}(q_1+q_4,\, q_2)_{\mu''\nu\lambda}{}^{ebc}\\ &\qquad-iq_{4\mu'}m_\pi^{-2}\Delta_\pi(q_1+q_4)\Gamma^{(1)}(q_1+q_4,q_2)_{\nu\lambda}{}^{ebc}]+\begin{Bmatrix}q_1\\ \mu\\ a\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_2\\ \nu\\ b\end{Bmatrix}\Big\}, \end{aligned}\tag{22}$$

[17] This result is analogous to the relationship expressed by Eqs. (5c) and (5d) between $\Delta^{(0)}(q)_\lambda{}^{ab}$ and $\Delta^{(0)}(q)^{ab}$ and is a result of our assumption of pion dominance of the 0^- channel.

[18] Of course, the symmetry properties of the n-point function may be used to demonstrate that there is only one vector and one axial-vector identity for each n-point function.

$$
\begin{aligned}
&C_A q_{2\nu} M^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}\\
&\quad= -iF_\pi{}^2 m_\pi{}^2 M^{(2)}(q_2,q_1,q_3)_{\mu\lambda\sigma}{}^{bacd} + C_A{}^{-1} q_{1\mu} m_\pi{}^{-2} \Gamma_\Sigma{}^{(2)}(q_1+q_2,\ q_3)_{\lambda\sigma}{}^{abcd}\\
&\quad+\epsilon^{bae}\Delta^A(q_1)^{-1}{}_{\mu\mu'}\Delta^V(q_1+q_2)_{\mu'\mu''}\Gamma^{(3)}(q_1+q_2,\ q_3)_{\mu\lambda\sigma}{}^{ecd} + \Bigg[\epsilon^{bce}\Delta^V(q_3)^{-1}{}_{\lambda\lambda'}[\Delta^A(q_2+q_3)_{\lambda'\lambda''}\Gamma^{(1)}(q_1,\ q_2+q_3)_{\mu\lambda''\sigma}{}^{aed}\\
&\qquad\qquad -i(q_2+q_3)_{\lambda'} m_\pi{}^{-2}\Delta_\pi(q_2+q_3)\Gamma^{(1)}(q_2+q_3,\ q_1)_{\mu\sigma}{}^{ead}] + \begin{Bmatrix} q_3\\ \lambda\\ c\end{Bmatrix} \leftrightarrow \begin{Bmatrix} q_4\\ \sigma\\ d\end{Bmatrix}\Bigg], \qquad (23a)
\end{aligned}
$$

$$
\begin{aligned}
&C_A q_{2\nu} M^{(2)}(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd}\\
&\quad= -iF_\pi{}^2 m_\pi{}^2 M^{(2)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd} - i\Delta_\pi(q_1)^{-1}\Gamma_\Sigma{}^{(2)}(q_1+q_2,\ q_3)_{\lambda\sigma}{}^{abcd}\\
&\quad+\Bigg[\epsilon^{bce}\Delta^V(q_3)^{-1}{}_{\lambda\lambda'}[\Delta^V(q_2+q_3)_{\lambda'\lambda''}\Gamma^{(1)}(q_1,q_2+q_3)_{\lambda''\sigma}{}^{aed}\\
&\qquad\qquad -i(q_2+q_3)_{\lambda'} m_\pi{}^{-2}\Delta_\pi(q_2+q_3)\Gamma^{(1)}(q_1,\ q_2+q_3)_\sigma{}^{aed}] + \begin{Bmatrix} q_3\\ \lambda\\ c\end{Bmatrix} \leftrightarrow \begin{Bmatrix} q_3\\ \sigma\\ d\end{Bmatrix}\Bigg]. \qquad (23b)
\end{aligned}
$$

The analogous equations for the three-point functions Γ are

$$C_V k^\sigma \Gamma^{(3)}(q_1,q_2)_{\nu\lambda\sigma} = \Delta^V(q_2)^{-1}{}_{\nu\lambda} - \Delta^V(q_1)^{-1}{}_{\nu\lambda}, \qquad (24)$$

$$C_V k^\sigma \Gamma^{(1)}(q_1,q_2)_{\nu\lambda\sigma} = \Delta^A(q_2)^{-1}{}_{\nu\lambda} - \Delta^A(q_1)^{-1}{}_{\nu\lambda}, \qquad (25)$$

$$
\begin{aligned}
&C_A q_2{}^\lambda \Gamma^{(1)}(q_1,q_2)_{\nu\lambda\sigma}\\
&\quad= iF_\pi{}^2 m_\pi \Gamma^{(1)}(q_2,q_1)_{\nu\sigma} + \Delta^A(q_1)^{-1}{}_{\nu\sigma} - \Delta^V(k)^{-1}{}_{\nu\sigma}, \qquad (26a)
\end{aligned}
$$

$$
\begin{aligned}
&C_A q_2{}^\lambda \Gamma^{(1)}(q_1,q_2)_{\lambda\sigma}\\
&\quad= -iF_\pi{}^2 m_\pi{}^2 \Gamma^{(1)}(q_1,q_2)_\sigma - iq_1{}^{\sigma'} m_\pi{}^{-2}\Delta^V(k)^{-1}{}_{\sigma\sigma'}. \qquad (26b)
\end{aligned}
$$

We have used isotopic spin invariance to write

$$\Gamma^{abc} = i\epsilon^{abc}\Gamma, \qquad (27)$$

defining the quantities Γ used above.

Before solving Eqs. (23) for the nonprimitive $M^{(2)}(q_i)$ we note that a certain structure is implied by them. The appearance of the propagators Δ^V, Δ^A, and Δ_π in these equations shows that the Ward identities imply the existence of ρ, A_1, and π poles in the four-point functions in the variables s, t, and u. More generally, since the four-point functions are constructed from currents which are generators of an algebra, we expect that the Ward identities will give us information about all channels which have the quantum numbers of these currents. The case $n=4$ is the first one for which this information is not merely in the external masses $q_i{}^2$. This is the origin of the tree structure for n-point functions which we find.

To utilize the tree structure implied by Eqs. (22) and (23), we shall define contact terms M_c by subtracting the 0^-, $1^\pm$ structure from the functions M. We do this for several reasons:

(1) We want to display, in an explicit way, the structure implied by the Ward identities.

(2) The functions M_c are simpler than the M since they do not contain π, ρ, and A_1 poles, for example.

(3) If we want to approximate any function by smooth functions in the momenta the only possible choice are the functions M_c since, at least as far as current algebra is concerned, they do not have to have any poles.

(4) The Ward identities for M_c are simpler than those for M.

We have

$$
\begin{aligned}
&M^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}\\
&\quad\equiv M_c{}^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd} + i\Gamma^{(3)}(q_3,\ q_1+q_2)_{\lambda\alpha\sigma}{}^{ced}\Delta^V(q_1+q_2)_{\alpha\alpha'}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\alpha'}{}^{abe}\\
&\quad+\Bigg[i\Gamma^{(1)}(q_1+q_3,\ q_2)_{\nu\sigma}{}^{ebd}\Delta_\pi(q_1+q_3)\Gamma^{(1)}(q_2+q_4,\ q_1)_{\mu\lambda}{}^{eac}\\
&\qquad\qquad + i\Gamma^{(1)}(q_2,\ q_1+q_3)_{\nu\alpha\sigma}{}^{bed}\Delta^A(q_1+q_3)_{\alpha\alpha'}\Gamma^{(1)}(q_1,\ q_2+q_4)_{\mu\alpha'\lambda}{}^{aec} + \begin{Bmatrix} q_1\\ \mu\\ a\end{Bmatrix} \leftrightarrow \begin{Bmatrix} q_2\\ \nu\\ b\end{Bmatrix}\Bigg]. \qquad (28)
\end{aligned}
$$

The definitions of $M_c{}^{(2)}(q_1,q_2,q_3)_{\nu\lambda\sigma}$ and $M_c{}^{(3)}(q_1,q_2,q_3)_{\lambda\sigma}$ are obtained from the above by simply deleting the indices μ and/or ν from Eq. (28). Using Eqs. (22)–(27), we obtain the following simple equations for the contact terms:

$$C_V q_4{}^\sigma M_c{}^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd} = \epsilon^{aed}\Gamma^{(1)}(q_1+q_4,\ q_2)_{\mu\nu\lambda}{}^{ebc} + \epsilon^{bed}\Gamma^{(1)}(q_1,q_2+\ q_4)_{\mu\nu\lambda}{}^{aec} + \epsilon^{ced}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\lambda}{}^{abe}, \qquad (29)$$

$$
\begin{aligned}
C_A q_2{}^\nu M_c{}^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd} &= -iF_\pi{}^2 m_\pi{}^2 M_c{}^{(2)}(q_2,q_1,q_3)_{\mu\nu\sigma}{}^{bacd} + C_A{}^{-1} q_{1\mu} m_\pi{}^{-2}\Gamma_\Sigma{}^{(2)}(q_1+q_2,\ q_3)_{\lambda\sigma}{}^{abcd}\\
&\quad+\epsilon^{bae}\Gamma^{(3)}(q_1+q_2,\ q_3)_{\mu\lambda\sigma}{}^{ecd} + \epsilon^{bce}\Gamma^{(1)}(q_1,\ q_2+q_3)_{\mu\lambda\sigma}{}^{aed} + \epsilon^{bde}\Gamma^{(1)}(q_1,\ q_2+q_4)_{\mu\sigma\lambda}{}^{aec}, \qquad (30a)
\end{aligned}
$$

$$
\begin{aligned}
&C_A q_2{}^\nu M_c{}^{(2)}(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd}\\
&\quad= -iF_\pi{}^2 m_\pi{}^2 M_c{}^{(2)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd} + (1/i)[(m_\pi{}^2-q_1{}^2)/F_\pi{}^2 m_\pi{}^4]\Gamma_\Sigma{}^{(2)}(q_1+q_2,\ q_3)_{\lambda\sigma}{}^{abcd}\\
&\qquad -i\epsilon^{abe} q_1{}^\alpha m_\pi{}^{-2}\Gamma^{(3)}(q_3,\ q_1+q_2)_{\lambda\alpha\sigma}{}^{ced} + \epsilon^{bce}\Gamma^{(1)}(q_1,\ q_2+q_3)_{\lambda\sigma}{}^{aed} + \epsilon^{bde}\Gamma^{(1)}(q_1,\ q_2+q_4)_{\sigma\lambda}{}^{aec}. \qquad (30b)
\end{aligned}
$$

A comparison of Eqs. (22) and (23) with Eqs. (29) and (30) reveals the gain in simplicity achieved by writing equations for the contact terms M_c. We also see from Eqs. (29) and (30) that the definition used in Eq. (28) successfully subtracts out all the structure induced in the 0^-, $1^\pm$ channels by the Ward identities.

IV. RESULTS FOR THREE- AND FOUR-POINT FUNCTIONS

We are now in a position to write all nonprimitive three- and four-point functions in terms of the primitive ones. For the three-point functions and the four-point functions $M^{(2)}$ we have written all the relevant Ward identities in the previous section; here we shall also present the final results for the other four-point functions. We shall display the structure of the type revealed in Eq. (28) explicitly, and shall use the primitive contact terms to contain the information not provided by current algebra. These contact terms are arbitrary except for the vector constraints.

A. Three-Point Functions

$$m_\pi^2F_\pi^2\Gamma^{(1)}(q_1,q_2)_{\nu\lambda} = -iC_Aq_{1\mu}\Gamma^{(1)}(q_2,q_1)_{\nu\mu\lambda} + i\Delta^A(q_2)^{-1}{}_{\nu\lambda} - i\Delta^V(k)^{-1}{}_{\nu\lambda}, \quad (31a)$$

$$m_\pi^4F_\pi^4\Gamma^{(1)}(q_1,q_2)_\lambda = -C_A^2q_{1\mu}q_{2\nu}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\lambda} + \tfrac{1}{2}(q_1-q_2)_\lambda + (C_A-\tfrac{1}{2}C_V)(q_2-q_1)_\alpha\Delta^V(k)^{-1}{}_{\alpha\lambda}. \quad (31b)$$

Vector Constraints

$$C_Vk^\sigma\Gamma^{(1)}(q_1,q_2)_{\nu\lambda\sigma} = \Delta^A(q_2)^{-1}{}_{\nu\lambda} - \Delta^A(q_1)^{-1}{}_{\nu\lambda}, \quad (32a)$$

$$C_Vk^\sigma\Gamma^{(3)}(q_1,q_2)_{\nu\lambda\sigma} = \Delta^V(q_2)^{-1}{}_{\nu\lambda} - \Delta^V(q_1)^{-1}{}_{\nu\lambda}. \quad (32b)$$

B. Four-Point Functions

Primitive Function: $(n-2m=4)$

$$M^{(4)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd} \equiv M_c{}^{(4)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd} + i\epsilon^{ade}\epsilon^{bce}\Gamma^{(3)}(q_1,\, q_2+q_3)_{\mu\alpha\sigma}\Delta^V(q_2+q_3)_{\alpha\alpha'}\Gamma^{(3)}(q_2,q_3)_{\nu\lambda\alpha'} + \begin{Bmatrix} q_1 \\ \mu \\ a \end{Bmatrix} \leftrightarrow \begin{Bmatrix} q_2 \\ \nu \\ b \end{Bmatrix} + \begin{Bmatrix} q_1 \\ \mu \\ a \end{Bmatrix} \leftrightarrow \begin{Bmatrix} q_3 \\ \lambda \\ c \end{Bmatrix}. \quad (33)$$

Vector Constraint: $(n-2m=4)$

$$C_Vq_4{}^\sigma M_c{}^{(4)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd} = i\epsilon^{aed}\epsilon^{bce}\Gamma^{(3)}(q_2,q_3)_{\nu\lambda\mu} + i\epsilon^{bed}\epsilon^{ace}\Gamma^{(3)}(q_1,q_3)_{\mu\lambda\nu} + i\epsilon^{ced}\epsilon^{abe}\Gamma^{(3)}(q_1,q_2)_{\mu\nu\lambda}. \quad (34)$$

Primitive Function: $(n-2m=2)$

$$\begin{aligned} M^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd} \equiv{} & M_c{}^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd} + i\epsilon^{abe}\epsilon^{cde}\Gamma^{(3)}(q_3,\, q_1+q_2)_{\lambda\alpha\sigma}\Delta^V(q_1+q_2)_{\alpha\alpha'}\Gamma^{(3)}(q_1,q_2)_{\mu\nu\alpha'} \\ & + \Big\{ i\epsilon^{ade}\epsilon^{ebc}[\Gamma^{(1)}(q_1,\, q_2+q_3)_{\mu\alpha\sigma}\Delta^A(q_2+q_3)_{\alpha\alpha'}\Gamma^{(1)}(q_1+q_4,\, q_2)_{\alpha'\nu\lambda} \\ & - \Gamma^{(1)}(q_2+q_3,\, q_1)_{\mu\sigma}\Delta_\pi(q_2+q_3)\Gamma^{(1)}(q_1+q_4,\, q_2)_{\nu\lambda}] + \begin{Bmatrix} q_1 \\ \mu \\ a \end{Bmatrix} \leftrightarrow \begin{Bmatrix} q_2 \\ \nu \\ b \end{Bmatrix} \Big\}. \end{aligned} \quad (35)$$

Vector Constraint: $(n-2m=2)$

$$C_Vq_4{}^\sigma M_c{}^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd} = i\epsilon^{aed}\epsilon^{ebc}\Gamma^{(1)}(q_1+q_4,\, q_2)_{\mu\nu\lambda} + i\epsilon^{bed}\epsilon^{aec}\Gamma^{(1)}(q_1,\, q_2+q_4)_{\mu\nu\lambda} + i\epsilon^{ced}\epsilon^{abe}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\lambda}. \quad (36)$$

Nonprimitive Functions: $(n-2m=2)$

$$\begin{aligned} m_\pi^2F_\pi^2M^{(2)}&(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd} \\ ={}& iC_Aq_{1\mu}M_c{}^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd} - iC_A{}^{-1}q_{2\nu}m_\pi{}^{-2}\Gamma_\Sigma{}^{(2)}(q_1+q_2,\, q_3)_{\lambda\sigma}{}^{abcd} + \epsilon^{ced}\epsilon^{abe}\Gamma^{(3)}(q_3,\, q_1+q_2)_{\lambda\alpha\sigma}\Delta^V(q_1+q_2)_{\alpha\alpha'} \\ & \times[-C_Aq_{1\mu}\Gamma^{(1)}(q_2,q_1)_{\nu\mu\alpha'} + \Delta^A(q_2)^{-1}{}_{\nu\alpha'}] + \Big\{ \epsilon^{bed}\epsilon^{aec}[\Gamma^{(1)}(q_2,\, q_1+q_3)_{\nu\alpha\sigma}\Delta^A(q_1+q_3)_{\alpha\alpha'} \\ & \times[-C_Aq_{1\mu}\Gamma^{(1)}(q_2+q_4,\, q_1)_{\alpha'\mu\lambda} - \Delta^V(q_3)^{-1}{}_{\alpha'\lambda}] - [-C_A{}^2(q_1+q_3)_\mu\Gamma^{(1)}(q_2,\, q_1+q_3)_{\nu\mu\sigma} + \Delta^A(q_2)^{-1}{}_{\nu\sigma} - \Delta^V(q_4)^{-1}{}_{\nu\sigma}] \\ & \times(F_\pi{}^{-2}/[m_\pi{}^2-(q_1+q_3)^2])[-C_A{}^2q_{1\mu}(q_2+q_4)_\alpha\Gamma^{(1)}(q_1,\, q_2+q_4)_{\mu\alpha\lambda} + \tfrac{1}{2}(q_1-q_2-q_4)_\lambda \\ & + (C_A-\tfrac{1}{2}C_V)(q_2+q_4-q_1)_\alpha\Delta^V(q_3)^{-1}{}_{\alpha\lambda})]] + \begin{Bmatrix} q_1 \\ \mu \\ a \end{Bmatrix} \leftrightarrow \begin{Bmatrix} q_2 \\ \nu \\ b \end{Bmatrix} \Big\}, \end{aligned} \quad (37)$$

$$m_\pi{}^4F_\pi{}^4M^{(2)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd}$$
$$=-C_A{}^2q_{1\mu}q_{2\nu}M_c{}^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}+(m_\pi{}^2-q_1{}^2-q_2{}^2)m_\pi{}^{-2}\Gamma_\Sigma{}^{(2)}(q_1+q_2,\,q_3)_{\lambda\sigma}{}^{abcd}$$
$$+\tfrac{1}{2}i(\epsilon^{ade}\epsilon^{bec}+\epsilon^{bed}\epsilon^{ace})[\Delta^V(q_3)^{-1}{}_{\sigma\lambda}+\Delta^V(q_4)^{-1}{}_{\sigma\lambda}]+i\epsilon^{abe}\epsilon^{cde}\Gamma^{(3)}(q_3,\,q_1+q_2)_{\lambda\alpha\sigma}\Delta^V(q_1+q_2)_{\alpha\alpha'}$$
$$\times[-C_A{}^2q_{1\mu}q_{2\nu}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\alpha'}+\tfrac{1}{2}(q_1-q_2)_{\alpha'}]+\Bigg\{i\epsilon^{bed}\epsilon^{aec}[[C_Aq_{2\nu}\Gamma^{(1)}(q_1+q_3,\,q_2)_{\alpha\nu\sigma}+\Delta^V(q_4)^{-1}{}_{\alpha\sigma}]\Delta^A(q_1+q_3)_{\alpha\alpha'}$$
$$\times[C_Aq_{1\mu}\Gamma^{(1)}(q_2+q_4,\,q_1)_{\alpha'\mu\lambda}+\Delta^V(q_3)^{-1}{}_{\alpha'\lambda}]-[-C_A{}^2q_{2\nu}(q_1+q_3)_\alpha\Gamma^{(1)}(q_2,\,q_1+q_3)_{\nu\alpha\sigma}+\tfrac{1}{2}(q_2-q_1-q_3)_\sigma$$
$$+(C_A-\tfrac{1}{2}C_V)(q_1+q_3-q_2)_\alpha\Delta^V(q_4)^{-1}{}_{\alpha\sigma}](F_\pi{}^{-2}/[m_\pi{}^2-(q_1+q_3)^2])$$
$$\times[-C_A{}^2q_{1\mu}(q_2+q_4)_\alpha\Gamma^{(1)}(q_1,\,q_2+q_4)_{\mu\alpha\lambda}+\tfrac{1}{2}(q_1-q_2-q_4)_\lambda$$
$$+(C_A-\tfrac{1}{2}C_V)(q_2+q_4-q_1)_\alpha\Delta^V(q_3)^{-1}{}_{\alpha\lambda}]]+\begin{Bmatrix}q_1\\ \mu\\ a\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_2\\ \nu\\ b\end{Bmatrix}\Bigg\}. \quad (38)$$

Primitive Function: $(n=2m=0)$

$$M^{(0)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}\equiv M_c{}^{(0)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}-i\epsilon^{ade}\epsilon^{bce}\Gamma^{(1)}(q_1,q_4)_{\mu\sigma\alpha}\Delta^V(q_2+q_3)_{\alpha\alpha'}\Gamma^{(1)}(q_2,q_3)_{\nu\lambda\alpha'}$$
$$+\begin{Bmatrix}q_1\\ \mu\\ a\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_2\\ \nu\\ b\end{Bmatrix}+\begin{Bmatrix}q_1\\ \mu\\ a\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_3\\ \lambda\\ c\end{Bmatrix}. \quad (39)$$

Nonprimitive Functions: $(n-2m=0)$

$$m_\pi{}^2F_\pi{}^2M^{(0)}(q_1,q_2,q_3)_{\nu\lambda\sigma}{}^{abcd}$$
$$=iC_Aq_{1\mu}M_c{}^{(0)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}+\Bigg\{-iC_A{}^{-1}q_{2\nu}m_\pi{}^{-2}\Gamma_\Sigma{}^{(0)}(q_1+q_2,\,q_3)_{\lambda\sigma}{}^{abcd}$$
$$+\epsilon^{abe}\epsilon^{cde}\Gamma^{(1)}(q_3,q_4)_{\lambda\sigma\alpha}\Delta^V(q_1+q_2)_{\alpha\alpha'}[-C_Aq_{1\mu}\Gamma^{(1)}(q_2,q_1)_{\nu\mu\alpha'}+\Delta^A(q_2)^{-1}{}_{\nu\alpha'}]+\begin{Bmatrix}q_2\\ \nu\\ b\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_3\\ \lambda\\ c\end{Bmatrix}+\begin{Bmatrix}q_2\\ \nu\\ b\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_4\\ \sigma\\ d\end{Bmatrix}\Bigg\}, \quad (40)$$

$$m_\pi{}^4F_\pi{}^4M^{(0)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd}$$
$$=-C_A{}^2q_{1\mu}q_{2\nu}M_c{}^{(0)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}-(m_\pi{}^2-q_1{}^2-q_2{}^2)m_\pi{}^{-2}\Gamma_\Sigma{}^{(0)}(q_1+q_2,\,q_3)_{\lambda\sigma}{}^{abcd}$$
$$+i\epsilon^{abe}\epsilon^{cde}\Gamma^{(1)}(q_3,q_4)_{\lambda\sigma\alpha}\Delta^V(q_1+q_2)_{\alpha\alpha'}[C_A{}^2q_{1\mu}q_{2\nu}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\alpha'}-\tfrac{1}{2}(q_1-q_2)_{\alpha'}]$$
$$+\Bigg\{q_{3\lambda}m_\pi{}^{-2}[q_{1\mu}\Gamma_\Sigma{}^{(0)}(q_2+q_3,\,q_4)_{\sigma\mu}{}^{bcda}+q_{2\nu}\Gamma_\Sigma{}^{(0)}(q_1+q_3,\,q_2)_{\nu\sigma}{}^{acbd}]-C_A{}^2m_\pi{}^{-4}q_{3\lambda}q_{4\sigma}\Sigma(q_1+q_3)^{acbd}$$
$$+i\epsilon^{ade}\epsilon^{bce}\{[-C_Aq_{1\mu}\Gamma^{(1)}(q_4,q_1)_{\sigma\mu\alpha}+\Delta^A(q_4)^{-1}{}_{\sigma\alpha}]\Delta^V(q_2+q_3)_{\alpha\alpha'}$$
$$\times[-C_Aq_{2\nu}\Gamma^{(1)}(q_3,q_2)_{\lambda\nu\alpha'}+\Delta^A(q_3)^{-1}{}_{\lambda\alpha'}]+\tfrac{1}{2}i\Delta^A(q_3)^{-1}{}_{\sigma\lambda}\}+\begin{Bmatrix}q_3\\ \lambda\\ c\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_4\\ \sigma\\ d\end{Bmatrix}\Bigg\}, \quad (41)$$

$$m_\pi{}^6F_\pi{}^6M^{(0)}(q_1,q_2,q_3)_\sigma{}^{abcd}=-iC_A{}^3q_{1\mu}q_{2\mu}q_{3\lambda}M_c{}^{(0)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}$$
$$+\Bigg\{-iC_A(m_\pi{}^2-q_1{}^2-q_2{}^2)m_\pi{}^{-2}q_{3\lambda}\Gamma_\Sigma{}^{(0)}(q_1+q_2,q_4)_{\sigma\lambda}{}^{abcd}+iC_Aq_{4\sigma}m_\pi{}^{-2}q_{1\mu}q_{2\nu}\Gamma_\Sigma{}^{(0)}(q_3+q_4,q_1)_{\mu\nu}{}^{cdab}$$
$$+iC_A{}^{-1}q_{4\sigma}m_\pi{}^{-2}(m_\pi{}^2-q_1{}^2-q_2{}^2)m_\pi{}^{-2}\Sigma(q_1+q_2)^{abcd}+\epsilon^{abe}\epsilon^{cde}\{[-C_Aq_{3\lambda}\Gamma^{(1)}(q_4,q_3)_{\sigma\lambda\alpha}+\Delta^A(q_4)^{-1}{}_{\sigma\alpha}]\Delta^V(q_1+q_2)_{\alpha\alpha'}$$
$$\times[-C_A{}^2{}_1q_\mu q_{2\nu}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\alpha'}+\tfrac{1}{2}(q_1-q_2)_{\alpha'}]+\tfrac{1}{6}(q_2-q_1)_\sigma+(\tfrac{1}{2}C_A-\tfrac{1}{3}C_V)(q_2-q_1)_\alpha\Delta^A(q_4)^{-1}{}_{\alpha\sigma}\}$$
$$+\begin{Bmatrix}q_2\\ \\ b\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_3\\ \\ c\end{Bmatrix}+\begin{Bmatrix}q_1\\ \\ a\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_3\\ \\ c\end{Bmatrix}\Bigg\}+iC_A{}^{-1}q_{4\sigma}F_\pi{}^2\,\tfrac{1}{3}(L^{cdab}+L^{bdca}+L^{adbc}). \quad (42)$$

We have defined

$$L^{abcd}=\Delta_\pi(q)^{-1}\Lambda^{abcd}(q)\,,$$

and note that L^{abcd} is independent of q because of the assumption of pion dominance.

$$m_\pi{}^8F_\pi{}^8M^{(0)}(q_1,q_2,q_3)^{abcd}=C_A{}^4q_{1\mu}q_{2\nu}q_{3\lambda}q_{4\sigma}M_c{}^{(0)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}$$
$$+\Bigg\{C_A{}^2[(m_\pi{}^2-q_1{}^2-q_2{}^2)m_\pi{}^{-2}q_{3\lambda}q_{4\sigma}\Gamma_\Sigma{}^{(0)}(q_1+q_2,q_3)_{\lambda\sigma}{}^{abcd}+(m_\pi{}^2-q_3{}^2-q_4{}^2)m_\pi{}^{-2}q_{1\mu}q_{2\nu}\Gamma_\Sigma{}^{(0)}(q_3+q_4,q_1)_{\mu\nu}{}^{cdab}]$$
$$+(m_\pi{}^2-q_1{}^2-q_2{}^2)(m_\pi{}^2-q_3{}^2-q_4{}^2)m_\pi{}^{-4}\Sigma(q_1+q_2)^{abcd}-i\epsilon^{abe}\epsilon^{cde}\{[-C_A{}^2q_{3\lambda}q_{4\sigma}\Gamma^{(1)}(q_3,q_4)_{\lambda\sigma\alpha}+\tfrac{1}{2}(q_3-q_4)_\alpha]$$
$$\times\Delta^V(q_1+q_2)_{\alpha\alpha'}[-C_A{}^2q_{1\mu}q_{2\nu}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\alpha'}+\tfrac{1}{2}(q_1-q_2)_{\alpha'}]+[\tfrac{1}{6}F_\pi{}^2+\tfrac{1}{2}(C_A-\tfrac{1}{2}C_V)](q_3-q_4)\cdot(q_2-q_1)\}$$
$$+\begin{Bmatrix}q_2\\ \\ b\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_3\\ \\ c\end{Bmatrix}+\begin{Bmatrix}q_1\\ \\ a\end{Bmatrix}\leftrightarrow\begin{Bmatrix}q_3\\ \\ c\end{Bmatrix}\Bigg\}+[m_\pi{}^2-\tfrac{1}{3}(q_1{}^2+q_2{}^2+q_3{}^2+q_4{}^2)]F_\pi{}^2(L^{abcd}+L^{acbd}+L^{adbc}). \quad (43)$$

Equations (31) through (43) are as far as current algebra alone can take us in a study of the three- and four-point functions. The soft-pion results previously obtained in the literature for meson interactions are found from these equations in the limits where various of the q_i are allowed to go to zero. In this approximation the contribution of the unknown primitive functions vanishes and we are left with an exact current-algebra prediction (although at an unphysical point where the pion mass vanishes). Our equations, on the other hand, give information on the mass shell, at the price of having to make some estimate of the primitive functions.

A possible approximation for the primitive functions is to assume they are given by as smooth functions of the momenta as possible. This approximation has been previously used and discussed in Ref. 5. It is more or less reasonable depending on the question being asked. If one only wishes to use the function at moderate energies and momenta, then it is reasonable to expect that the approximation works fairly well. On the other hand, it presumably becomes less good at higher energies. We have verified for a few specific cases that this approximation yields three- and four-point functions identical to those given by effective Lagrangians.[10,19]

In the following we give these approximations for the primitive functions appearing in Eqs. (31)–(45). They have, of course, been chosen to be consistent with the vector constraints. It has been previously observed[5] that the Ward identities for three-point functions imply that if the primitive three-point function is as slowly varying as possible, then the two-point functions have the form of free propagators. With our notation,

$$\Delta^V(q)_{\mu\nu}=[C_V m_\rho^2/(m_\rho^2-q^2)](g_{\mu\nu}-q_\mu q_\nu/m_\rho^2)\,, \tag{44a}$$

$$\Delta^A(q)_{\mu\nu}=[C_A m_{A_1}^2/(m_{A_1}^2-q^2)](g_{\mu\nu}-q_\mu q_\nu/m_{A_1}^2)\,, \tag{44b}$$

$$\Delta^V(q)^{-1}{}_{\mu\nu}=C_V^{-1}m_\rho^{-2}[(m_\rho^2-q^2)g_{\mu\nu}+q_\mu q_\nu]\,, \tag{45a}$$

$$\Delta^A(q)^{-1}{}_{\mu\nu}=C_A^{-1}m_{A_1}^{-2}[(m_{A_1}^2-q^2)q_{\mu\nu}+q_\mu q_\nu]\,. \tag{45b}$$

The approximate primitive three- and four-point functions are

$$\begin{aligned}\Gamma^{(3)}(q_1,q_2)_{\nu\lambda\sigma}&\\ =C_V^{-2}m_\rho^{-2}[g_{\nu\lambda}(q_2-q_1)_\sigma&\\ &+g_{\lambda\sigma}(k-q_2)_\nu+g_{\sigma\nu}(q_1-k)_\lambda]\,,\end{aligned} \tag{46}$$

$$\begin{aligned}\Gamma^{(1)}(q_1,q_2)_{\nu\lambda\sigma}&\\ =C_V^{-1}C_A^{-1}m_A^{-2}[g_{\nu\lambda}(q_2-q_1)_\sigma+g_{\lambda\sigma}(k-q_2)_\nu&\\ &+g_{\sigma\nu}(q_1-k)_\lambda-\delta(g_{\nu\sigma}k_\lambda-g_{\lambda\sigma}k_\nu)]\,,\end{aligned} \tag{47}$$

$$\begin{aligned}M_c^{(4)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}&\\ =iC_V^{-3}m_\rho^{-2}[\epsilon^{abe}\epsilon^{dce}\tfrac{3}{2}(g_{\nu\lambda}g_{\mu\sigma}-g_{\mu\lambda}g_{\nu\sigma})&\\ +(\epsilon^{aed}\epsilon^{ebc}-\epsilon^{dbe}\epsilon^{aec})&\\ &\times(\tfrac{1}{2}g_{\nu\lambda}g_{\mu\sigma}+\tfrac{1}{2}g_{\mu\lambda}g_{\nu\sigma}-g_{\mu\nu}g_{\lambda\sigma})]\,,\end{aligned} \tag{48}$$

$$\begin{aligned}M_c^{(2)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}&\\ =iC_V^{-2}C_A^{-1}m_A^{-2}[\epsilon^{abe}\epsilon^{dce}(\tfrac{3}{2}+\delta)(g_{\nu\lambda}g_{\mu\sigma}-g_{\lambda\mu}g_{\nu\sigma})&\\ +(\epsilon^{aed}\epsilon^{ebc}-\epsilon^{dbe}\epsilon^{aec})&\\ &\times(\tfrac{1}{2}g_{\nu\lambda}g_{\mu\sigma}+\tfrac{1}{2}g_{\mu\lambda}g_{\nu\sigma}-g_{\mu\nu}g_{\lambda\sigma})]\,,\end{aligned} \tag{49}$$

$$\begin{aligned}M_c^{(0)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}&\\ =i\xi C_A^{-3}m_A^{-2}[\epsilon^{abe}\epsilon^{dce}\tfrac{3}{2}(g_{\nu\lambda}g_{\mu\sigma}-g_{\mu\lambda}g_{\nu\sigma})&\\ +(\epsilon^{aed}\epsilon^{ebc}-\epsilon^{dbe}\epsilon^{aec})&\\ &\times(\tfrac{1}{2}g_{\nu\lambda}g_{\mu\sigma}+\tfrac{1}{2}g_{\mu\lambda}g_{\nu\sigma}-g_{\mu\nu}g_{\lambda\sigma})]\,.\end{aligned} \tag{50}$$

The parameter δ, introduced in Eq. (47), represents the anomalous magnetic moment of the A_1 meson and is the same parameter used in Ref. 5. In fact, if Eqs. (46) and (47) are used in Eqs. (31), then the resultant three-point functions are identical (except for normalization) to those of Ref. 5. The parameter ξ in Eq. (50) appears because there is no vector constraint for n-point functions of axial currents alone, and so there is no equation to determine a scale for the primitive contact term. To be completely general, we could have written

$$\begin{aligned}M_c^{(0)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}=\delta^{ab}\delta^{cd}[2\xi_1 g_{\mu\nu}g_{\lambda\sigma}&\\ &-\xi_2(g_{\mu\lambda}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\lambda})]+\text{crossed terms},\end{aligned}$$

with ξ_1 and ξ_2 arbitrary. We have chosen

$$\xi_1=\xi_2=iC_A^{-3}m_{A_1}^{-2}$$

so that $M_c^{(0)}$ has the same form as $M_c^{(4)}$ where the vector constraint determines the form.

V. APPLICATIONS

We shall present in this section a few applications of Eqs. (31)–(43), most of which use the approximate contact terms. We shall not discuss better models for the contact terms or the σ terms here. We shall present more detailed analysis and applications elsewhere.

A. Sum Rules

We have already pointed out that the sum rule

$$F_\pi^2=C_A-C_V \tag{10}$$

is implied by our choice of Ward identities. There are two other interesting sum rules which appear in the literature whose derivation is not on completely secure ground. One, derived by Weinberg,[7,20] is

$$C_V m_\rho^2=C_A m_{A_1}^2\,, \tag{51}$$

while the other is[21]

$$F_\pi^2=\tfrac{1}{2}g_\rho^2/m_\rho^2\,. \tag{52}$$

Equation (52) can be rewritten, using Eq. (10), as

$$C_A-\tfrac{1}{2}C_V=0\,,$$

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[19] R. Perrin (private communication).

[20] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).

[21] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Fayyazuddin and Riazuddin, Phys. Rev. **147**, 1071 (1966).

with this combination appearing throughout our three- and four-point functions. Equation (31b), which represents the amplitude for ρ decay into two pions, is comprised of three terms: the first is an interaction induced by the A_1 meson, the second is the coupling of the pion isotopic spin current to the ρ, while the third is strictly an off-mass-shell effect and has $C_A-\frac{1}{2}C_V$ as a coefficient. Thus, Eq. (52) makes this last term vanish. Since the π-π amplitude, Eq. (41), has ρ mesons off their mass shell as intermediate states, this sum rule guarantees that no direct π-π interactions are induced from this term.[22]

We have considered various high-energy constraints on Eq. (31b); however, we have not been able to find a satisfactory reason for the acceptance of Eq. (51).

B. Pion Electromagnetic Mass Difference

We have previously reported[23] a calculation of the difference $m_{\pi^+}-m_{\pi^0}$, which is determined from a knowledge of $M^{(2)}(q_1,q_2,q_3)_{\lambda\sigma}$. In that calculation we used Eqs. (47) and (49) for want of a better approximation although we recognize that high-energy effects are not treated correctly. We found that the mass difference diverged logarithmically, Eq. (10) removing a possible quadratic divergence. Here we would like to note that in our previous calculation we also used Eqs. (50) and (51) although they are certainly not implied by our model. If we relax them, then we find that the coefficient of the divergent logarithm can be set equal to zero by using a sum rule which differs from Eq. (51) by terms of higher order in m_π^2/m_ρ^2. The first-order correction is

$$C_Am_A^2=C_Vm_\rho^2\left[1+\frac{1}{16}\frac{m_\pi^2}{m_\rho^2}(1+\delta^2)\right].$$

Using Eq. (51), we find

$$\tfrac{1}{2}m_A^2=m_\rho^2\left[1+\frac{1}{16}\frac{m_\pi^2}{m_\rho^2}(1+\delta^2)\right]$$

or, alternatively, we could accept the relation

$$\tfrac{1}{2}m_A^2=m_\rho^2$$

and correct Eq. (52). None of these corrections are large enough to be observable; however, the point remains that even such a simple model has the possibility of yielding finite electromagnetic mass shifts for on-mass-shell pions.

C. A_1 Decay

The matrix element for the decay of the A_1 into three pions is given by $M^{(0)}(q_1,q_2,q_3)_\sigma$ and contains both a vector-meson-dominance term and a direct $A_1\to 3\pi$ contact term. If we make a narrow-width approximation for the ρ, then the decay rate is given by the sum of $A_1\to\rho+\pi$ and the direct term. The first term is precisely that computed by Schnitzer and Weinberg in Ref. 5 from a knowledge of the three-point function. Approximating three-body phase space by taking the pion mass m_π to be zero, we find the correction to the A_1 width to be

$$\Gamma=18+21\xi+8\xi^2 \text{ MeV}.$$

Thus, for $\xi=1$, we find an additional 50 MeV which requires a lower value of δ than that given in Ref. 5 to fit the total width of about 130 MeV.

Of course, since we have an explicit matrix element for this decay, we can find more sensitive tests of our theory than merely finding the total width. In particular, with more detailed experimental information about A_1 decay we could certainly determine δ and ξ. It is encouraging, however, that even making the above approximations the direct three-π term (for reasonable values of ξ) does not overwhelm the $\rho\pi$ part so that the A_1 width can be fitted satisfactorily.

D. π-π Scattering

The amplitude $M^{(0)}(q_1,q_2q_3)^{abcd}$ is precisely the π-π scattering amplitude. If we use Eq. (50) for the contact term, then we have a reasonably good description of low-energy π-π scattering. We shall restrict ourselves here to the S-wave scattering lengths.[24]

We must make some kind of model for the σ terms in Eq. (43). If we ignore Γ_Σ and Σ, then we are effectively assuming the lack of a strong S-wave interaction. Crossing symmetry requires

$$L^{abcd}+L^{acbd}+L^{adbc}=\Lambda(P_0+\tfrac{2}{5}P_2)_{ab;\,cd},$$

where P_0 and P_2 are isospin 0 and 2 projection operators. Adopting Weinberg's classification[25] of the transformational properties of σ^{abc}, we have

$$\Lambda=-i[3N(N+2)-4],$$

where N is an integer. For $N=1$, we have the usual σ model where σ^{ab} is an isotopic scalar. With these as-

[22] It should be noted, however, that there is still a direct $(\pi)^4$ coupling in Eq. (41) even if $C_A=\frac{1}{2}C_V$. This term arises from the σ contribution via Eq. (17). This effect has also been noted by Wess and Zumino, Ref. 10.

[23] I. S. Gerstein, B. W. Lee, H. T. Nieh, and H. J. Schnitzer, Phys. Rev. Letters 19, 1064 (1967).

[24] S. Weinberg, Phys. Rev. Letters 19, 616 (1966).

[25] See Eq. (6.9) of S. Weinberg, Ref. 10.

sumptions, the π-π scattering amplitude is

$$T(s,t,u)^{abcd}=iF_\pi^{-4}\delta^{ab}\delta^{cd}\{F_\pi^2[s+\tfrac{1}{5}(N(N+2)-8)m_\pi^2]+\tfrac{1}{4}\xi C_A m_{A_1}^{-2}[2ut-s(u+t)]$$
$$+\tfrac{1}{4}C_A^2C_V^{-1}m_{A_1}^{-4}(1+\delta)^2[ut(u+t)-s(u^2+t^2)]-\tfrac{1}{2}C_A m_{A_1}^{-2}(1+\delta)[2ut-s(u+t)]+\tfrac{1}{4}[C_AC_V^{-1}m_{A_1}^{-2}(1+\delta)u-1]^2$$
$$\times C_V u(t-s)/(m_\rho^2-u)+\tfrac{1}{4}[C_AC_V^{-1}m_{A_1}^{-2}(1+\delta)t-1]^2C_V t(u-s)/(m_\rho^2-t)\}$$
$$+iF_\pi^{-4}\delta^{ad}\delta^{bc}(t\leftrightarrow s)+iF_\pi^{-4}\delta^{ac}\delta^{bd}(u\leftrightarrow s)\,, \quad (53)$$

which yields, for the s-wave scattering length,

$$a=0.11m_\pi^{-1}\{[1+\tfrac{1}{4}N(N+2)]P_0-\tfrac{1}{5}[4-\tfrac{1}{2}N(N+2)]P_2\}\,, \quad (54)$$

in agreement with Weinberg's result.

Our calculation shows that Eq. (54) should be interpreted as one contribution to the pion scattering lengths. If they turn out to be larger than Eq. (54) predicts, then the rest is attributable to the σ terms we have neglected. It is interesting, as has been remarked before, that the only large contributions can come from these terms.[26]

E. π-ρ Scattering

The scattering of pions from ρ mesons can be constructed from the amplitude $M^{(2)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd}$, whose structure is displayed in Eq. (38). For example, the amplitude for physical ρ mesons and off-shell pions is

$$\lim_{q_3^2\to m_\rho^2}\lim_{q_4^2\to m_\rho^2}(m_\pi^2-q_1^2)(m_\pi^2-q_2^2)(m_\rho^2-q_3^2)(m_\rho^2-q_4^2)\bar{T}^{(2)}(q_1,q_2,q_3)_{\lambda\sigma}{}^{abcd}\epsilon_\rho{}^\lambda(q_3)\epsilon_\rho{}^\sigma(q_4)\,, \quad (55)$$

where $\bar{T}^{(2)}$ is related to $M^{(2)}$ as in Eq. (12), and $\epsilon_\rho{}^\lambda(q)$ is the ρ-meson polarization vector satisfying $q_\lambda\epsilon_\rho{}^\lambda(q)=0$.

We use the primitive function (49) to construct a model for the low-energy scattering and again neglect Γ_Σ as in the π-π model. With these assumptions, the s-wave scattering lengths are

$$a=0.22m_\pi^{-1}(1+m_\pi/m_\rho)^{-1}\left\{2P_0+P_1-P_2+\frac{1}{4}\frac{m_\pi}{m_\rho}\left(\frac{m_\rho^2C_A}{m_{A_1}^2C_V}\right)\left(\frac{3m_{A_1}^2-m_\rho^2}{m_{A_1}^2-m_\rho^2}\right)(2+\delta)^2(2P_0-3P_1-P_2)\right.$$
$$\left.+\frac{m_\pi^2}{m_\rho^2}\left(\frac{m_\rho^2C_A}{m_{A_1}^2C_V}\right)\frac{m_\rho^2[m_{A_1}^2+(1+\delta)m_\rho^2]}{(m_{A_1}^2-m_\rho^2)^2}(2+\delta)(2P_0+P_1-P_2)\right\}\,, \quad (56)$$

where we have obtained $O(m_\pi^2/m_\rho^2)$ modifications to the usual soft-pion predictions.[24] Again, $P_{0,1,2}$ are the isospin projection operators. For a numerical estimate we set $m_A^2=2m_\rho^2$ and $C_A=\frac{1}{2}C_V$, which gives

$$a=0.22m_\pi^{-2}(1+m_\pi/m_\rho)^{-1}\{(2P_0+P_1-P_2)[1+\tfrac{1}{4}(m_\pi/m_\rho)^2(3+\delta)(2+\delta)]$$
$$+(2P_0-3P_1-P_2)\tfrac{5}{16}(m_\pi/m_\rho)(2+\delta)^2\}\,. \quad (57)$$

For the typical values of δ, obtained from the A_1 width[5] of $-1\leq\delta\leq-\frac{1}{2}$, we find a 6–12% change in the prediction of a_0 and a_2 but a more substantial shift in a_1 from the $m_\pi=0$ results. This illustrates the well-known fact that there are important corrections to the current-algebra predictions when there is a low-lying, direct-channel s-wave resonance. Here the corrections occur because of the $\pi+\rho\to A_1$ resonance; notice that these terms in Eq. (57) vanish when $\delta=-2$, which corresponds to longitudinal decay of $A_1\to\rho+\pi$.[5]

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[26] We have recently received a report by R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor [Phys. Rev. Letters **20**, 475 (1968)] in which they study π-π scattering up to 1 BeV using a formalism which should yield a scattering amplitude identical to Eq. (53), together with a model for the σ terms. They obtain a good fit to the experimental data with $\xi=0$. We would like to thank these authors for a conversation which clarified the relation between their approach and ours.