TABLE I. Comparison between calculated and observed partial decay rates of 1^+ mesons.

Decay modes	Γ(theory) (MeV)	Γ(experiment) (MeV)
1. $A_1 \to \rho\pi$	**130**(input)	30–130
2. $K_A \to K^*\pi$	50.7	dominant modes
3. $K_A \to \rho K$	15.3	
4. $K_A \to \omega K$	5.4	$<2.1\pm1.4$
5. $E \to \bar{K}^*K+K^*\bar{K}$	34.2	39.2 ± 7.9
6. $\pi(1640) \to \rho\pi$	30.05	$<40\pm8$
7. $K_A(1800) \to K^*\pi$	14.3	28 ± 10
8. $K_A(1800) \to \rho K$	3.89	5.6 ± 2
9. $K_A(1800) \to \omega K$	3.54	8 ± 2

In the evaluation of the above matrix elements, we have followed the procedure outlined in Ref. 4.

The calculated amplitudes (11) and (12) are relativistically invariant. To obtain the partial decay rates, we square the amplitude and sum over the initial spin state and then multiply with the invariant phase space and the square of the $SU(3)$ isoscalar factor. The calculated rates and the corresponding observed rates[10] have been displayed in Table I. The agreement seems to be good. The crucial test of our calculation depends, however, on clear experimental information of the spin and parity assignments of these particles.

ACKNOWLEDGMENTS

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Soft-Photon Theory of Nucleon-Nucleon Bremsstrahlung*

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As a consequence of a soft-photon theorem, part of the amplitude for nucleon-nucleon bremsstrahlung depends only on phase parameters measurable in elastic N-N scattering and on the static electromagnetic properties of the nucleon. The corresponding cross section is computed in a fully covariant and gauge-independent fashion, and is found to be in quantitative agreement with most of the existing experimental data.

I. INTRODUCTION

In recent years, several measurements of nucleon-nucleon bremsstrahlung have been performed at low and intermediate energies.[1–5] These experiments are of importance to the theory of nuclear forces, since the electromagnetic field can probe the interaction in areas not available in elastic scattering. The electromagnetic interaction is weak and can normally be treated in lowest order of perturbation theory, which facilitates the theoretical interpretation of the results. Along with these advantages comes the fact that the electromagnetic current is conserved, which leads to soft-photon theorems.[6,7] In this paper, we shall apply the soft-photon technique to proton-proton and neutron-proton bremsstrahlung in order to determine to what extent the interesting aspects are masked by the low-energy behavior of the radiation.

II. SOFT-PHOTON THEOREM

We consider an expansion of the bremsstrahlung amplitude in powers of the frequency ω of the radiated photon and shall demonstrate that the two leading terms depend only on the elastic (mass-shell) properties of the N-N interaction. This was first done by Low.[6] It is necessary to repeat the proof in order to obtain the model-independent expression for fermion-fermion scattering. The model-independent amplitude has previously been given for p-p bremsstrahlung without derivation.[8]

The manipulations necessary to obtain the low-frequency limit of the bremsstrahlung amplitude are

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[2] J. C. Thompson, S. I. H. Naqvi, and R. E. Warner, Phys. Rev. **156**, 1156 (1967); I. Slaus, J. Verba, J. Richardson, R. Carlson, W. van Oers, and L. August, Phys. Rev. Letters **17**, 536 (1966); R. E. Warner, Can. J. Phys. **44**, 1225 (1966); M. L. Halbert, D. L. Mason, and L. C. Northcliffe, Rev. Mod. Phys. **39**, 716 (1967); A. Bahnsen and R. L. Burman, Phys. Letters **26B**, 585 (1968).

[3] K. W. Rothe, P. F. M. Koehler, and E. H. Thorndike, Phys. Rev. **157**, 1247 (1966).

[4] P. F. M. Koehler, K. W. Rothe, and E. H. Thorndike, Phys. Rev. Letters **18**, 933 (1967).

[5] J. C. Young, F. P. Brady, and C. Badrinathan, Phys. Rev. Letters **20**, 750 (1968).

[6] F. E. Low, Phys. Rev. **110**, 974 (1958).

[7] F. E. Low, Phys. Rev. **96**, 1428 (1954).

[8] E. M. Nyman, Phys. Letters **25B**, 135 (1967).

reduced to a minimum by using the following theorem, which is due to Adler and Dothan.[9]

Theorem: Let $M_\mu(x,k_\nu)$ and $M_\mu{}^{\mathrm{I}}(x,k_\nu)$ be four-vector functions of a number of independent variables x and the four-vector k_ν, and let $M_\mu{}^{\mathrm{II}}(x)$ be a function of the same variables excluding k_ν, such that

$$M_\mu(x,k_\nu)=M_\mu{}^{\mathrm{I}}(x,k_\nu)+M_\mu{}^{\mathrm{II}}(x)+O(k). \tag{1}$$

Then, provided

$$k^\mu M_\mu(x,k_\nu)=k^\mu M_\mu{}^{\mathrm{I}}(x,k_\nu), \tag{2}$$

it follows that

$$M_\mu{}^{\mathrm{II}}(x)=0. \tag{3}$$

Proof: It follows from Eqs. (1) and (2) that $k^\mu M_\mu{}^{\mathrm{II}}(x)=0$ or $O(k^2)$. By inspection, we can exclude the latter alternative, concluding that

$$k^\mu M_\mu{}^{\mathrm{II}}(x)=0. \tag{4}$$

Since each component of k is an independent variable, Eq. (3) follows. [The proof can easily be generalized to the case where the assumptions (1) and (2) are only assumed to hold for lightlike vectors k_ν.]

To lowest order in the electromagnetic interaction, the (Feynman) amplitude B for bremsstrahlung is of the form

$$B=\epsilon^\mu M_\mu, \tag{5}$$

where M_μ contains the electromagnetic current operator. It follows from current conservation that

$$k^\mu M_\mu=0, \tag{6}$$

where $k_\mu=(\omega,\mathbf{k})$ is the four-momentum of the radiation with polarization ϵ_μ. We shall use the above theorem in the special case when each side of Eq. (2) vanishes and obtain the model-independent low-frequency amplitude, using the following prescription: We first split the matrix element M_μ into known and unknown parts;

$$M=M_{\mathrm{known}}(k)+M_{\mathrm{unknown}}(k),$$

both of which depend on k, then demonstrate that the unknown part is regular as $\omega\to 0$;

$$M_{\mathrm{unknown}}(k)=M_{\mathrm{unknown}}(0)+O(\omega),$$

and finally make sure that the current in the known part is conserved. As a consequence of the theorem, it now follows that the low-frequency limit of the unknown matrix element vanishes;

$$M_{\mathrm{unknown}}(0)=0.$$

Since the bremsstrahlung amplitude is known to behave like $1/\omega$ in the soft-photon limit, this procedure uniquely determines two leading terms in an expansion of the bremsstrahlung amplitude in powers of the radiated frequency.

[9] S. L. Adler and Y. Dothan, Phys. Rev. **151**, 1267 (1966).

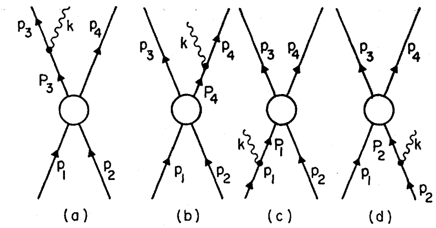


FIG. 1. These graphs give a singular contribution to the N-N bremsstrahlung amplitude in the soft-photon limit, since the virtual nucleon momenta P_i in this limit approach the mass shell. The "circles" represent N-N scattering with one particle in turn off its mass shell.

According to this prescription, we may arbitrarily add or subtract terms which are regular in the soft-photon limit (i.e., approach a unique value independent of how the components of $\mathbf{k}$ approach zero). At the end of these manipulations current conservation must be restored; the expression so obtained will give the desired soft-photon amplitude.

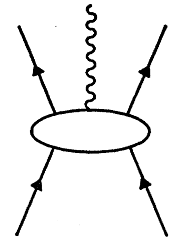
We separate the bremsstrahlung amplitude into two parts, depending on whether the photon is produced by one of the external nucleon lines or between strong interactions, as in Figs. 1 and 2. The graphs in Fig. 1 are singular in the soft-photon limit, since the virtual nucleon in that case approaches its mass shell. All graphs of the type illustrated in Fig. 2 are regular in the soft-photon limit[10]; we therefore need not consider them here. In order to discuss the graphs in Fig. 1, we first consider N-N scattering without energy loss.

The work of Goldberger *et al.*[11] provides a convenient representation for the elastic N-N amplitude. There are five independent phase parameters, and we define five kinematical invariants:

$$\begin{aligned}
T_1&=(\bar{u}_3u_1)(\bar{u}_4u_2),\\
T_2&=\tfrac{1}{2}(\bar{u}_3\sigma_{\mu\nu}u_1)(\bar{u}_4\sigma^{\mu\nu}u_2),\\
T_3&=(\bar{u}_3i\gamma_5\gamma_\mu u_1)(\bar{u}_4i\gamma_5\gamma^\mu u_2),\\
T_4&=(\bar{u}_3\gamma_\mu u_1)(\bar{u}_4\gamma^\mu u_2),\\
T_5&=(\bar{u}_3\gamma_5u_1)(\bar{u}_4\gamma_5u_2).
\end{aligned} \tag{7}$$

The u's are positive-energy spinors describing the nucleons 1, 2 and 3, 4 in the initial and final states,

FIG. 2. This graph summarizes radiation from internal lines in the N-N interaction. The contribution is not singular in the soft-photon limit.



[10] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961).

[11] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

respectively, in the notation of Bjorken and Drell.[12] We shall replace (7) by a shorter notation;

$$T_\alpha = (\bar{u}_3 t_\alpha u_1)(\bar{u}_4 t_\alpha u_2), \quad (8)$$

which suppresses the summation indices for some values of α. Sums over α will be indicated explicitly.

The invariant functions F_α for elastic scattering depend on two scalar variables; we shall use

$$\begin{aligned} \nu &= p_1p_2 + p_3p_4, \\ \Delta &= p_1p_3 + p_2p_4, \end{aligned} \quad (9)$$

where p_i stands for the four-momentum of the ith nucleon. The Feynman amplitude A for elastic scattering is of the form

$$A = \sum_{\alpha=1}^{5} F_\alpha(\nu,\Delta) T_\alpha. \quad (10)$$

In the Appendix, we give a string of equations which relate the invariant functions $F_\alpha(\nu,\Delta)$ to phase shifts. This is simply a matter of inverting some of the equations in Ref. 11.

In Fig. 1, the "circles" represent N-N scattering with one particle in turn off its mass shell. The form (10) is not general enough to cover this possibility. The process requires three scalar variables; in addition to ν and Δ, we use the variable P_i^2, where P_i is the four-momentum of the virtual nucleon line. (Capital letters are used for virtual momenta.) These invariant functions reduce to those describing elastic scattering for $P_i^2 = m^2$, where m is the nucleon mass (we ignore the n-p mass difference):

$$F_\alpha(\nu,\Delta,P_i^2) \to F_\alpha(\nu,\Delta) \quad \text{as} \quad P_i^2 \to m^2. \quad (11)$$

When one of the particles that take part in the scattering corresponds to an internal line in a graph, we must remove one of the spinors in the definition (7) or (8) of the invariants T_α. It is then possible to construct more than five independent invariants. By definition, the extra ones must vanish on the mass shell when multiplied into the missing spinor. Considering the amplitude needed in Fig. 1(a), we can readily construct such invariants (the other cases are treated analogously):

$$T_{\alpha+5} = (\boldsymbol{P}_3 - m) t_\alpha u_1 \bar{u}_4 t_\alpha u_2, \quad (12)$$

where t_α is any of the matrices that appear in Eq. (7). It is known from nonrelativistic considerations[13] that there is in fact only one extra invariant; we can therefore fix the value of α in Eq. (12). For other values of α, this expression must then be a linear combination of the invariants actually used. Our arguments will not rely on this property; they hold independently for each value of α in Eq. (12).

[12] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964). We use the notation $\gamma_\mu p^\mu \equiv \boldsymbol{p}$.

[13] A. H. Cromer and M. I. Sobel, Phys. Rev. **152**, 1351 (1966); **162**, 1112 (1967).

The electromagnetic vertex of the nucleon also takes a more general form when one particle is virtual. We use the following representation for the vertex function Γ_μ needed in Fig. 1(a):

$$\begin{aligned} \Gamma_\mu(P_3^2) = {} & \gamma_\mu f_1(P_3^2) - i\sigma_{\mu\nu}k^\nu f_2(P_3^2) \\ & + [\gamma_\mu f_3(P_3^2) - i\sigma_{\mu\nu}k^\nu f_4(P_3^2)](\boldsymbol{P}_3 - m), \end{aligned} \quad (13)$$

which reduces to the static vertex on the mass shell. In this limit, the form factors f_1 and f_2 give the charge and anomalous magnetic moment of the nucleon. The other two functions, f_3 and f_4, only contribute away from the mass shell.

Before constructing the contribution from the graphs in Fig. 1, we notice that the propagator of the virtual nucleon line will give rise to the well-known $1/\omega$ singularity. We have, for instance (dropping a factor i),

$$\frac{1}{\boldsymbol{P}_3 - m} = \frac{\boldsymbol{P}_3 + m}{P_3^2 - m^2} = \frac{\boldsymbol{P}_3 + m}{2p_3k}, \quad (14)$$

since $P_3 = p_3 + k$, $p_3^2 = m^2$, and $k^2 = 0$. However, in the case of the "extra" invariants in the N-N amplitude and the electromagnetic vertex, the propagator will cancel against the factor $(\boldsymbol{P}_3 - m)$, which was necessary to prevent them from contributing on the mass shell. These parts of the bremsstrahlung amplitude are therefore nonsingular in the soft-photon limit and need not be kept when constructing the soft-photon amplitude.

The form factors f_1 and f_2 (which are not the same as those measured in e-p scattering) have been shown to be analytic by Bincer.[14] They therefore enjoy the following expansion:

$$f(P^2) = f(m^2) + (P^2 - m^2)[f'(m^2) + O(\omega)], \quad (15)$$

where the prime denotes differentiation. Note that the mass-shell pole will cancel from the contribution of the second term in (15); only the first term gives a singular contribution in the soft-photon limit. Following the prescription of leaving out nonsingular contributions, we may therefore replace the form factors by their static values (i.e., the charge and anomalous moment). Assuming that the dependence of the scattering amplitude on the virtual masses P_i^2 is differentiable, we can also replace the invariant functions by their known values on the mass shell.

We have now seen that all unknown contributions to the bremsstrahlung amplitude are regular to the soft-photon limit; following the prescription, they have been subtracted from the amplitude. Denoting the static vertex of the nucleon by Γ_μ, we have (dropping factor $-i$)

$$\begin{aligned} \Gamma_\mu = \Gamma_\mu{}^p &= e[\gamma_\mu - (\lambda_p/2m)i\sigma_{\mu\nu}k^\nu] \quad \text{(protons)} \\ &= \Gamma_\mu{}^n = -e(\lambda_n/2m)i\sigma_{\mu\nu}k^\nu, \quad \text{(neutrons)} \end{aligned} \quad (16)$$

[14] A. M. Bincer, Phys. Rev. **118**, 855 (1960).

with

$$e^2/4\pi\equiv\alpha\cong1/137\,,\quad \lambda_p\cong1.7928\,,\quad \lambda_n\cong-1.9128\,. \tag{17}$$

(Note that the photon momentum is defined as outgoing; this differs from the usual convention in e-p scattering.) Conservation of momentum at the electromagnetic vertices in Fig. 1 gives us

$$P_1=p_1-k\,,\quad P_2=p_2-k\,,\quad P_3=p_3+k\,,\quad P_4=p_4+k\,. \tag{18}$$

After the modifications made so far, the expression to be used in the bremsstrahlung amplitude is

$$\begin{aligned}M_\mu=\sum_{\alpha=1}^{5}\Bigg\{&\left(\bar u_3t_\alpha\frac{1}{P_1-m}\Gamma_\mu u_1\right)(\bar u_4t_\alpha u_2)F_\alpha(\nu_1,\Delta_1)\\&+(\bar u_3t_\alpha u_1)\left(\bar u_4t_\alpha\frac{1}{P_2-m}\Gamma_\mu u_2\right)F_\alpha(\nu_2,\Delta_2)\\&+\left(\bar u_3\Gamma_\mu\frac{1}{P_3-m}t_\alpha u_1\right)(\bar u_4t_\alpha u_2)F_\alpha(\nu_3,\Delta_3)\\&+(\bar u_3t_\alpha u_1)\left(\bar u_4\Gamma_\mu\frac{1}{P_4-m}t_\alpha u_2\right)F_\alpha(\nu_4,\Delta_4)\Bigg\}\,,\end{aligned} \tag{19}$$

where ν_i and Δ_i are obtained from Eq. (9) by replacing p_i by P_i, and $u_i=u(p_i)$ are solutions to the Dirac equation for the external momenta. If the functions $F_\alpha(\nu,\Delta)$ were independent of their arguments, Eq. (19) would correspond to a conserved current and give the desired low-frequency amplitude. This is in general not the case, and we must restore current conservation by adding a nonsingular term. To do this, we expand around the point ν,Δ defined in terms of the external momenta in Eq. (9). The leading part in this expansion is proportional to $F_\alpha(\nu,\Delta)$, with the same argument in all terms, and gives a divergenceless contribution. The next term need only be computed to lowest order in ω, and in this case, for example, the first line in Eq. (19) can be treated as follows:

$$\bar u_3\Gamma_\mu{}^p\frac{1}{P_3-m}\longrightarrow\bar u_3\gamma_\mu\frac{p_3+m}{P_3{}^2-m^2}=\bar u_3\frac{p_{3\mu}}{p_3k}\,. \tag{20}$$

Since we have

$$\begin{aligned}\nu_3&=\nu+p_4k\,,\\\Delta_3&=\Delta+p_1k\,,\end{aligned} \tag{21}$$

the derivative terms from this part of (19) are

$$\sum_\alpha T_\alpha\left[\frac{\partial F_\alpha}{\partial\nu}\frac{p_4k}{p_3k}p_{3\mu}+\frac{\partial F_\alpha}{\partial\Delta}\frac{p_1k}{p_3k}p_{3\mu}\right]. \tag{22}$$

It is easily seen that the following expression differs from (22) by a nonsingular amount in the soft-photon limit and is divergenceless:

$$\sum_\alpha T_\alpha\left[\frac{\partial F_\alpha}{\partial\nu}\left(\frac{p_4k}{p_3k}p_{2\mu}-p_{4\mu}\right)+\frac{\partial F_\alpha}{\partial\Delta}\left(\frac{p_1k}{p_3k}p_{3\mu}-p_{1\mu}\right)\right]. \tag{23}$$

The terms added to accomplish this result are uniquely determined to lowest order in ω. Similar manipulations on the other parts of Eq. (19) lead to the following expression for p-p bremsstrahlung:

$$\begin{aligned}M_\mu{}^{\mathrm I}=\sum_{\alpha=1}^{5}\Bigg\{&\left[\bar u_3\left(\Gamma_\mu{}^p\frac{1}{p_3+k-m}t_\alpha+t_\alpha\frac{1}{p_1-k-m}\Gamma_\mu{}^p\right)u_1\bar u_4t_\alpha u_2+\bar u_3t_\alpha u_1\bar u_4\left(\Gamma_\mu{}^p\frac{1}{p_4+k-m}t_\alpha+t_\alpha\frac{1}{p_2-k-m}\Gamma_\mu{}^p\right)u_2\right]F_\alpha(\nu,\Delta)\\&+e[(kp_2-kp_1)(p_1/kp_1-p_2/kp_2)_\mu+(kp_4-kp_3)(p_3/kp_3-p_4/kp_4)_\mu]T_\alpha\frac{\partial}{\partial\nu}F_\alpha(\nu,\Delta)\\&+e[(kp_3-kp_1)(p_1/kp_1-p_3/kp_3)_\mu+(kp_4-kp_2)(p_2/kp_2-p_4/kp_4)_\mu]T_\alpha\frac{\partial}{\partial\Delta}F_\alpha(\nu,\Delta)\Bigg\}\,.\end{aligned} \tag{24}$$

It follows from the theorem in this section that the above function differs from the true bremsstrahlung matrix element only by terms which vanish in the soft-photon limit. In this limit, the sum of the unknown terms which were excluded from the amplitude is equal to the terms added when replacing (19) by (24).

The matrix element for n-p bremsstrahlung is obtained in an analogous manner. Assigning indices 2 and 4 to the neutron, we find

$$\begin{aligned}M_\mu{}^{\mathrm I}=\sum_{\alpha=1}^{5}\Bigg\{&\left[\bar u_3\left(\Gamma_\mu{}^p\frac{1}{p_3+k-m}t_\alpha+t_\alpha\frac{1}{p_1-k-m}\Gamma_\mu{}^p\right)u_1\bar u_4t_\alpha u_2+\bar u_3t_\alpha u_1\bar u_4\left(\Gamma_\mu{}^n\frac{1}{p_4+k-m}t_\alpha+t_\alpha\frac{1}{p_2-k-m}\Gamma_\mu{}^n\right)u_2\right]F_\alpha(\nu,\Delta)\\&+e[(p_2k\;p_1-p_1k\;p_2)_\mu/p_1k+(p_4k\;p_3-p_3k\;p_4)_\mu/p_3k]\frac{\partial F}{\partial\nu}+e(p_3k-p_1k)(p_1/p_1k-p_3/p_3k)_\mu\frac{\partial F}{\partial\Delta}\Bigg\}\,.\end{aligned} \tag{25}$$

When the elastic amplitude shows resonances narrow compared with the energy loss, the expansion in the variables ν and Δ is not justified. An example of this is bremsstrahlung in nuclear reactions. It has been shown

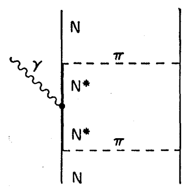


FIG. 3. The electromagnetic properties of the nucleon resonances will enter in the N-N bremsstrahlung amplitude through mechanisms of this kind.

by Feshbach and Yennie[15] that one can still obtain the $1/\omega$ part of the amplitude in a reasonably model-independent way. Except at very low energies, the N-N interaction is free of resonances. It is therefore justified to make the calculation exact to one more power of ω, although the expansion in ν and Δ could otherwise be avoided. The Feshbach-Yennie approach has been used in p-p bremsstrahlung by Felsner.[16]

We have seen that two powers in the expansion in ω depend only on elastic-scattering parameters and static electromagnetic properties of the nucleon. It is easy to see that the next term contains processes which are quite different in nature. In the $O(\omega)$ term, for instance, the magnetic moments of all the nucleon resonances will contribute through graphs like Fig. 3. We do not mean to imply that mechanisms of this kind are the main contribution to the $O(\omega)$ part of the bremsstrahlung amplitude—the role of the nucleon resonances in generating the nuclear force is still subject to discussion.

III. RESULTS

A. General

When the bremsstrahlung *cross section* is expanded in ω, the leading term can easily be shown to be proportional to the corresponding elastic cross section.[12] This (infrared-divergent) part of the bremsstrahlung cross section therefore has the same dependence on the nucleonic polarizations as elastic N-N scattering. After summing over the γ-ray polarizations, the infrared part of the cross section takes the following form[12]:

$$d\sigma_{NN\gamma}=\frac{1}{2\omega}\frac{d^3\mathbf{k}}{(2\pi)^3}(-J_\mu J^\mu)d\sigma_{NN}\,,$$

where J_μ is given by

$$J_\mu=e(p_1/kp_1-p_3/kp_3)_\mu\,, \qquad (p\text{-}n)$$

$$J_\mu=e(p_1/kp_1+p_2/kp_2-p_3/kp_3-p_4/kp_4)_\mu\,, \qquad (p\text{-}p)$$

for pn and pp bremsstrahlung, respectively, and $d\sigma_{NN}$ is the elastic differential cross section. The nonrelativistic limit of $\mathbf{J}$ is proportional to the velocity v of (say) the incident nucleon for n-p bremsstrahlung. The corresponding term in the p-p bremsstrahlung amplitude vanishes, since here $\mathbf{J}$ is proportional to v^2. This is an anomaly in any system where the center of charge coincides with the center of mass; the normally dominating dipole radiation is then absent. Higher electric multipoles are accompanied by additional powers of v, and magnetic transitions do not contribute in lowest order of ω.

At low energies, the infrared part of the bremsstrahlung cross section is adequate as an order-of-magnitude estimate of the full cross section. The ratio between the p-p and n-p bremsstrahlung cross sections is therefore small at nonrelativistic energies and behaves like v^2 in the low-energy limit. However, even outside the region where the Coulomb force is important, there are large differences between the elastic pp and np differential cross sections, which can mask this behavior.

B. Proton-Proton Bremsstrahlung

A number of p-p bremsstrahlung experiments have been performed in the so-called Harvard geometry.[1,2] In these experiments, events are identified as bremsstrahlung without necessarily observing the radiation. The two final-state protons are detected by small counters Ω_1 and Ω_2 at equal angles θ in a plane on opposite sides of the beam. (See Fig. 4.) The energies of the outgoing protons are observed, which allows a determination of the angle ψ between the γ ray and the beam. We use ψ as a variable rather than one proton energy, since this elminates a kinematical singularity in phase space at each end of the allowed range of the energy. The cross section in this geometry is customarily reported in the laboratory frame, and is given by

$$\frac{d\sigma}{d\Omega_1 d\Omega_2 d\psi}=\tfrac{1}{4}\sum_{\text{spins}}\sum_{\epsilon}|M_\mu\epsilon^\mu|^2\frac{1}{2}\frac{1}{(2\pi)^5}\frac{m^3}{|\mathbf{p}_1|E_3E_4}\times\frac{\mathbf{p}_3{}^2\mathbf{p}_4{}^2}{\sin 2\theta-v_3\sin(\theta+\psi)-v_4\sin(\theta-\psi)}\,, \qquad (26)$$

where the energies and velocities of the final protons are E_3, E_4, and v_3, v_4, respectively. When evaluating this expression, we approximate M_μ by the model-independent part $M_\mu{}^{\mathrm{I}}$, given in Eq. (24). The quantity $M_\mu{}^{\mathrm{I}}\epsilon^\mu$ is invariant under Lorentz transformations and can therefore be evaluated in any frame of reference. All noninvariant quantities in (26) are to be evaluated in the laboratory system. We use units where $\hbar=c=1$.

Note that we have approximated the bremsstrahlung *amplitude* by two terms of its expansion in ω, which are independent of the off-shell behavior of the p-p inter-

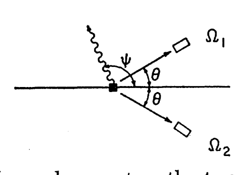


FIG. 4. In the Harvard geometry, the two final-state nucleons are detected by small counters Ω_1 and Ω_2 at equal angles θ in a plane on opposite sides of the beam. By measuring the energies of the nucleons, the direction and energy of the γ ray can be inferred. All quantities are given in the laboratory reference frame.

[15] H. Feshbach and D. R. Yennie, Nucl. Phys. **37**, 150 (1962).
[16] G. Felsner, Phys. Letters **25B**, 290 (1967).

action. When the true amplitude is squared to get the cross section, two leading powers in ω are also model-independent. The third term consists of two parts, the square of the second term in the amplitude, which is model-independent and included here, and the interference between the model-dependent term and the $1/\omega$ part of the amplitude. Because of the absence of the dipole moment, the $1/\omega$ amplitude is anomalously small. This suppresses the model-independent terms as well as the model-dependent interference term. An important part of the cross section may therefore arise from the square of magnetic parts of the amplitude. Although such a term is of the same order in ω as some model-dependent terms, its presence does not indicate a breakdown of the soft-photon method.

Writing the bremsstrahlung amplitude as

$$B=a/\omega+b+c\omega+\cdots,$$

the a/ω term contains only electric multipoles, whereas the magnetic moments of the nucleons contribute to b. The third term, $c\omega$, is model-dependent and considered unknown in this paper. We have

$$d\sigma\sim B^2=a^2/\omega^2+2ab/\omega+(b^2+2ac)+\cdots.$$

We see that in the limit when the infrared electric term is absent, i.e., $a=0$, the third term in the cross section becomes model-independent, while the two leading terms vanish. This is the case in n-n bremsstrahlung. Although a is proportional to v^2 at low energies in p-p bremsstrahlung, the magnetic term (b^2) does not dominate the cross section, since ω also vanishes in this limit.

We have included the contribution from the Coulomb force to the invariant functions F_α. This does not make the calculation exact to any higher order in e^2, but is likely to be an improvement. We have used the phase parameters given by Arndt and MacGregor.[17] In some cases, phase shifts are needed at energies below 24 MeV, which is outside the energy range considered in Ref. 17.

TABLE I. Proton-proton bremsstrahlung cross sections in the Harvard geometry. The experimental numbers are obtained from Refs. 1–3.

Lab energy (MeV)	θ (deg)	$d\sigma/d\Omega_1d\Omega_2$ (μb/sr^2) Present theory	Experiment
20	35	1.11	1.3 ±0.4
30	35	1.71	1.85±0.25
35.5	30	1.11	3.0 ±0.8
46	30	1.66	3.46±0.6
48	30	1.76	2.12±0.36
48	35	2.91	3.04±0.44
61.8	30	2.52	2.2 ±1.2
158	30	7.9	10.6 ±2.1
158	35	11.8	14.0 ±2.8
204	30	10.0	13.0 ±2.4
204	35	15.3	14.0 ±2.7

[17] R. A. Arndt and M. H. MacGregor, Phys. Rev. 141, 873 (1966).

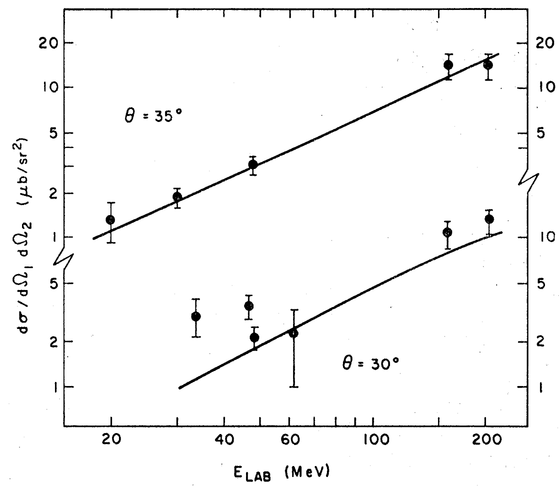


FIG. 5. Proton-proton bremsstrahlung cross sections in the Harvard geometry. The experimental numbers are obtained from Refs. 1–3.

In these cases, we have used phase shifts generated from the Hamada-Johnston potential.[18] In the region where the two phase-shift tables are joined, the difference between the corresponding bremsstrahlung cross sections is only a few percent.

It would be a laborious task to square Eq. (24) and work out the traces resulting from the spin summations by hand. An attempt was made to use a computer code developed by Hearn[19] for such problems. This approach was also abandoned because the final answer contained an exceedingly large number of terms. The difficulty was avoided by using a numerical representation for all γ matrices and spinors. This makes it possible to perform the sum over α indicated in Eq. (24) before squaring the expression. In this approach, calculations for arbitrarily polarized particles involve no further complication.

At low energies the experimental determination of the γ-ray angle ψ from the proton energies is often quite uncertain. We have therefore integrated over this variable ($0\leq\psi<2\pi$). In Table I, we summarize the experimental data in the Harvard geometry. The numbers predicted by the soft-photon theory as outlined here are also given. The same numbers are shown in Fig. 5. For $\theta=35°$, the agreement is remarkably good. The case $\theta=30°$ corresponds to harder γ rays and is therefore farther from the mass shell. Here there is a slight tendency for the soft-photon theory to predict smaller cross sections than what is found experimentally. It would be desirable to have data for smaller values of θ to see if this trend is real. The cross section decreases rapidly with θ, however, and this makes such experiments difficult at low energies.

At 158 MeV, angular distributions of the γ ray are available.[1] These have previously[8] been compared with

[18] T. Hamada and I. D. Johnston, Nucl. Phys. 34, 382 (1962).
[19] A. C. Hearn, Comm. A.C.M. 9, 573 (1966).

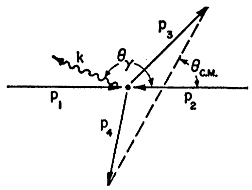


FIG. 6. The polar angles θ_γ and $\theta_{\text{c.m.}}$ are used in the Rochester geometry. The corresponding azimuthal angles φ_γ and $\varphi_{\text{c.m.}}$ are not shown here. All quantities refer to the center-of-momentum frame.

the soft-photon theory and found to be in good agreement.

Measurements in a more general geometry have been performed at Rochester,[3] detecting all particles in the final state. We define $\theta_{\text{c.m.}}$ and $\varphi_{\text{c.m.}}$ as the polar angles of the vector $\mathbf{p}_3-\mathbf{p}_4$ in the center-of-momentum frame (such that $\varphi_{\text{c.m.}}=0$ denotes the horizontal plane) and $E_\gamma\equiv\omega$, θ_γ, φ_γ as the polar components of the γ-ray momentum $\mathbf{k}$, in the same frame. (See Fig. 6.) In the soft-photon limit, $\theta_{\text{c.m.}}$ approaches the usual angle in elastic scattering. The differential cross section is given by

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}d\Omega_\gamma dE_\gamma}=\frac{1}{4}\sum_{\text{spins}}\sum_{\epsilon}|M_\mu\epsilon^\mu|^2$$

$$\times\frac{1}{4}\frac{1}{(2\pi)^5}\frac{E_1}{|\mathbf{p}_1-\mathbf{p}_2|}\frac{E_\gamma m^2q^3}{q^2(E_3+E_4)-2\mathbf{k}\cdot\mathbf{q}(E_3-E_4)}, \quad (27)$$

where $\mathbf{q}=\mathbf{p}_3-\mathbf{p}_4$, and all quantities are to be evaluated in the center-of-momentum frame. Most of the experiment[3] was done with the γ counter in the horizontal plane, with $\theta_\gamma=108°$ (90° in the laboratory). It was found that the cross section was largest around $\theta_{\text{c.m.}}=90°$, $\varphi_{\text{c.m.}}=0$. In Fig. 7, we give the theoretical angular distribution for $E_\gamma=60$ MeV, and it does indeed show this general feature. The decrease as one goes away from $\varphi_{\text{c.m.}}=0$ is not so sharp as the experiment indicated; the distribution is also not symmetric around $\theta_{\text{c.m.}}=90°$. (The experimental numbers were presented in a way which masks any such asymmetry.) Figure 8 shows the energy spectrum of the radiation after integration over the proton angles. The over-all normalization does not agree with the experiment.[3] Note, however, that the experimental numbers are based on data for $|\cos\theta_{\text{c.m.}}|\lesssim 0.6$ and $|\varphi_{\text{c.m.}}|\lesssim 50°$, which corresponds to only $\frac{1}{3}$ of the unit sphere.

Rothe *et al.*[3] performed their experiment with a polarized proton beam, finding a spin dependence fairly similar to that of elastic p-p scattering. For sufficiently soft photons, the dependence on any of the proton spins will indeed be the same as in elastic scattering. Radiation from the magnetic moments start altering these ratios above $E_\gamma\approx 20$ MeV, and as seen in Fig. 9, at $E_\gamma=80$ MeV the asymmetry has changed its sign.

C. Neutron-Proton Bremsstrahlung

A measurement of the n-p bremsstrahlung cross section, using a neutron beam, has been performed.[5] Experimental results for protons incident on a deuterium target are also available. Koehler *et al.*[4] have extracted an n-p bremsstrahlung cross section from their deuteron data at 197 MeV. They consider γ rays above 40 MeV and integrate over the final-state nucleon angles. The result is given as a function of the γ-ray angle θ_γ in Fig. 10, together with a curve based on Eqs. (25) and (27). The theoretical curve is seen to be below the lower experimental limit. A significantly smaller p-d bremsstrahlung cross section was, however, observed at 140 MeV by Edington and Rose.[20]

The neutron-beam experiment[5] was performed in the Harvard geometry. The obtained cross section is larger than the soft-photon prediction, as shown in Fig. 11.

IV. DISCUSSION

The bremsstrahlung experiments are intended to be measurements of N-N interactions off the mass shells in a case where complications due to the presence of more than two strongly interacting particles can be ignored.

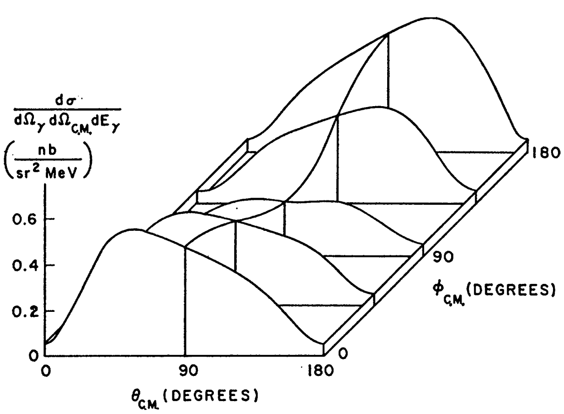


FIG. 7. Theoretical p-p bremsstrahlung cross section at 204 MeV in the Rochester geometry for an unpolarized beam. Shown is the cross section as function of the angles $\theta_{\text{c.m.}}$ and $\varphi_{\text{c.m.}}$ for $E_\gamma=60$ MeV, $\theta_\gamma=108°$, and $\varphi_\gamma=0$.

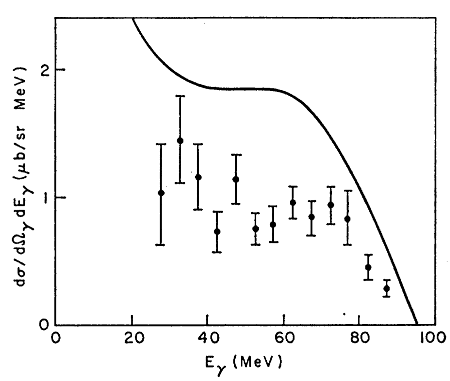


FIG. 8. Proton-proton bremsstrahlung in the Rochester geometry at 204 MeV. Shown is the energy spectrum of radiation at the angle $\theta_\gamma=108°$ from the beam direction. The experimental numbers are obtained from Ref. 3; only statistical errors are shown.

[20] J. A. Edgington and B. Rose, Nucl. Phys. 89, 523 (1966).

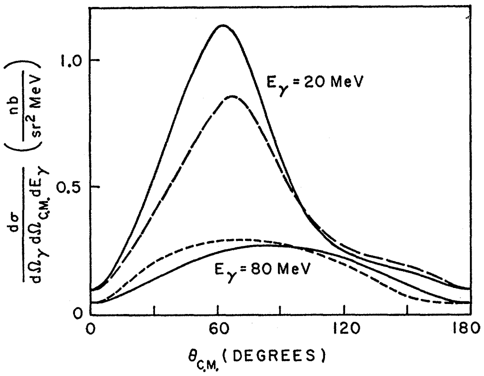


FIG. 9. Coplanar p-p bremsstrahlung cross section at 204-MeV laboratory energy in the Rochester geometry for a polarized beam. The two spin states ("up" and "down") are denoted by solid and dashed lines for γ-ray energies of 20 and 80 MeV. The cross section is plotted against the angle $\theta_{c.m.}$ for $\varphi_{c.m.} = \varphi_\gamma = 0$ and $\theta_\gamma = 108°$. All quantities are given in the center-of-momentum frame.

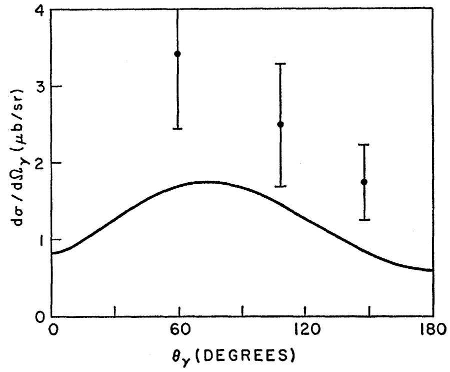


FIG. 10. Cross section for production of a γ ray with an energy above 40 MeV in proton-neutron scattering at 197-MeV laboratory energy, as function of the γ-ray angle θ_γ, in the center-of-momentum frame. The experimental data are obtained from Ref. 4.

The fact that the electromagnetic form factors of the nucleon depend on the distance from the mass shell is, however, a manifestation of just this. It is therefore important to have an idea of the importance of these variations. Effects of this kind can be seen in scattering of γ rays by protons. Note that there is a soft-photon theorem for Compton scattering also,[7] to the effect that the internal structure of the proton enters only in second and higher orders of the photon frequency. For soft photons the cross section approaches the Powell[21] value, which depends only on the static electromagnetic properties of the proton.

Experimental and theoretical results for proton Compton scattering are given by Hyman *et al.*[22] We note that the calculated cross sections start depending on the assumed structure of the proton near threshold for pion production. This is also where the measured cross section starts deviating from the Powell value. The situation in Compton scattering is similar to bremsstrahlung; in both cases, the coefficients of two leading powers of ω depend only on static properties of the nucleon. It is therefore probably justifiable to ignore the nucleonic structure in bremsstrahlung calculations provided that the energy loss is below the pion mass. All bremsstrahlung experiments which have been done so far satisfy this criterion.

The relativistic nature of the electromagnetic vertex also deserves a comment. A comparison between the covariant vertex of a point (Dirac) proton with an anomalous magnetic moment and its nonrelativistic limit has been made for Compton scattering. Pugh *et al.*[23] give the result of an unpublished calculation by Walecka, where the kinematics was treated relativistically, but proton wave functions and vertices were represented by nonrelativistic (two-component) approximations. Walecka's calculation was compared with the Powell cross section for γ-ray laboratory energies of 94 and 130 MeV. It was concluded that the nonrelativistic vertex is inadequate, giving errors of 10–20%. This approximation therefore needs justification already in the case of the Harvard experiments,[1] where the maximum γ-ray energy was above 70 MeV. The error increases with the energy loss and could easily be confused with the off-shell behavior of strong interactions. In the present paper, all calculations have been done in a fully relativistic manner.

A number of authors have considered N-N bremsstrahlung from the standpoint of potential theory.[24,25] This approach is unavoidably nonrelativistic, although the kinematics can easily be treated exactly. The form factors have, for lack of knowledge, been replaced by the static values. With one exception,[26] other mesonic effects have not been considered at all. At the lower

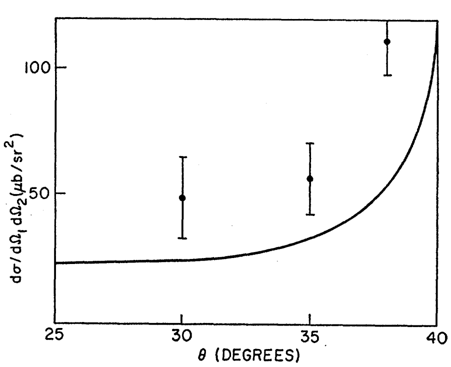


FIG. 11. Neutron-proton bremsstrahlung cross section at 200 MeV in the Harvard geometry. The experimental points are from Ref. 5; only statistical errors are shown.

[21] J. L. Powell, Phys. Rev. **75**, 32 (1949).
[22] L. G. Hyman, R. Ely, D. H. Frisch, and M. A. Wahlig, Phys. Rev. Letters **3**, 93 (1959).
[23] G. E. Pugh, R. Gomez, D. H. Frisch, and S. G. Janes, Phys. Rev. **105**, 982 (1957).
[24] V. Brown, Phys. Letters **25B**, 506 (1967).
[25] W. A. Pearce, W. A. Gale, and I. M. Duck, Nucl. Phys. **B3**, 241 (1967); see also references quoted here.
[26] Y. Ueda, Phys. Rev. **145**, 1214 (1966). This paper finds that radiation via a π-ρ-γ vertex is negligible.

energies, these are quite acceptable sacrifices, and by generating the off-shell matrix elements from a potential, one can perform a calculation to *all orders* in the radiated energy. This approach makes it possible to compare the off-shell behaviors of different potentials which give (more or less) the same elastic scattering amplitude. To the extent that these calculations are gauge invariant, they satisfy the low-energy theorem. Differences in the off-shell behavior therefore appear only in the $O(\omega)$ part of the amplitude. Differences in the on-shell behavior will of course, alter the leading terms as well.

The comparison between different potentials should be made when a power series in ω converges as poorly as possible. We therefore estimate the radius of convergence of such an expansion. It was shown by Bincer[14] that the electromagnetic form factors $f(P^2)$ have discontinuities at pion production threshold, i.e., for $P^2=(m+m_\pi)^2$. This reduces to the condition

$$E_\gamma \approx m_\pi . \tag{28}$$

Expanding the form factors in ω will therefore yield a divergent series above this energy. We expect the invariant functions $F_\alpha(\nu,\Delta,P^2)$ to have an analogous analytic structure in P^2. When the functions are computed from a potential the situation is slightly different, the pion mass in (28) might then be replaced by the inverse range of the force. In either case, the condition (28) is reproduced.

The invariant functions have also been expanded in ν and Δ, and they are known to have square-root branch points at the elastic threshold. It therefore follows that the power-series expansion in ω will also cease to converge when this point is reached. This is the condition

$$E_\gamma = E_{\max} , \tag{29}$$

where $E_{\max}$ is the largest possible energy loss. The condition (29) corresponds to $\theta=0$ in the Harvard geometry.

Because of the form factors, the theoretical translation of bremsstrahlung data near or above the pion-production threshold into a potential with a reliable off-shell behavior presents difficulties. Experiments for small values of θ in the Harvard geometry below pion-production threshold might, however, prove valuable.

A detailed comparison between the soft-photon theory and a potential-theory calculation of p-p bremsstrahlung by Brown[24] has been performed. Brown considered the Bryan-Scott[27] and Hamada-Johnston[18] potentials. It was found that the difference between the present approach and the potential-theory predictions is (a) of the same order of magnitude as the difference between the two potentials and (b) in no case larger than the corresponding experimental uncertainties. It will therefore be necessary to have more accurate experiments before the contribution from the model-dependent terms can be determined.

Note added in proof. Burnett and Kroll[28] have shown that the coefficients of two powers of ω in the unpolarized bremsstrahlung cross section can be expressed in terms of the corresponding elastic cross section. This differs from the present approach, as explained in Sec. III.

ACKNOWLEDGMENT

The author has benefited from the interest and encouragement of Professor J. D. Walecka. He would also like to thank a large number of theoretical and experimental groups for supplying information prior to its publication.

APPENDIX

We record here the equations used to compute the invariant functions F_α in the elastic N-N scattering amplitude (10). Apart from minor changes in notation and correction of an error, the equations are simply the inversion of corresponding formulas in Sec. IV of Ref. 11. The reader is referred to Stapp *et al.*[29] for expressions for the partial-wave amplitudes in terms of phase shifts. The Coulomb interaction is also discussed in that paper. We normalize the partial-wave amplitude so that the singlet amplitude with angular momentum J is of the form

$$f_0{}^J = \exp(i\delta_0{}^J)\sin\delta_0{}^J .$$

The triplet amplitude for $l=J$ will be called $f_1{}^J$, and the coupled partial waves are $f_{l=J-1}$, $f_{l=J+1}$, and $f_{J-1,J+1}$. We define

$$\begin{pmatrix} f_{11}{}^J \\ f_{22}{}^J \\ f_{12}{}^J \end{pmatrix} = \frac{1}{2J+1}\begin{pmatrix} J & J+1 & 2[J(J+1)]^{1/2} \\ J+1 & J & -2[J(J+1)]^{1/2} \\ [J(J+1)]^{1/2} & -[J(J+1)]^{1/2} & 1 \end{pmatrix} \times \begin{pmatrix} f_{l=J-1} \\ f_{l=J+1} \\ f_{J-1,J+1} \end{pmatrix} .$$

We let the energy and momentum of one nucleon in the center-of-mass system be E and p, and use the variable $z=\cos\theta$, where θ is the scattering angle. We can now add the partial waves

$$f_1(z) = (E/p)\sum_J (2J+1)P_J(z) f_0{}^J ,$$

$$f_2(z) = (E/p)\sum_J (2J+1) f_{11}{}^J P_J(z) ,$$

[27] R. A. Bryan and B. L. Scott, Phys. Rev. **164**, 1215 (1967).

[28] T. H. Burnett and N. M. Kroll, Phys. Rev. Letters **20**, 87 (1968).

[29] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

$$f_3(z)=(E/p)\sum_J [A_J(z)f_{22}^J+B_J(z)f_1^J],$$

$$f_4(z)=(E/p)\sum_J [B_J(z)f_{22}^J+A_J(z)f_1^J],$$

$$f_5(z)=(m/p)\sum_J C_J(z)f_{12}^J.$$

Here, $P_J(z)$ denotes a Legendre polynomial, and A, B, and C have the following expansions:

$$A_J(z)=\sum_{l=0}^{J}(2l+1)a_{lJ}P_l(z),$$

$$B_J(z)=\sum_{l=0}^{J}(2l+1)b_{lJ}P_l(z),$$

$$C_J(z)=\sum_{l=0}^{J}(2l+1)c_{lJ}P_l(z).$$

The coefficients in these expansions can conveniently be generated recursively:

$$a_{l,J+1}=-\frac{2J+3}{J+2}\left(b_{lJ}+\frac{J-1}{2J-1}a_{l,J-1}\right),$$

$$b_{l,J+1}=-\frac{2J+3}{J+2}\left(a_{lJ}+\frac{J-1}{2J-1}b_{l,J-1}-\delta_{lJ}\right),$$

$$c_{l-2,J}=c_{lJ},$$

with the initial conditions

$$a_{l1}=0,\qquad b_{l1}=\tfrac{3}{2}\delta_{l0},\qquad c_{JJ}=0,$$
$$a_{l2}=-\tfrac{5}{2}\delta_{l0},\; b_{l2}=0,\qquad c_{J-1,J}=-(2J+1)/[J(J+1)]^{1/2}.$$

For p-p (or n-n) scattering, we must add an exchange term, which will make the final amplitude antisymmetric. This corresponds to the substitution

$$f_\alpha(z)\to f_\alpha(z)+f_\alpha(-z),\quad (\alpha=1,2,3)$$
$$f_\alpha(z)\to f_\alpha(z)-f_\alpha(-z),\quad (\alpha=4,5).$$

This is equivalent to counting the allowed states twice when summing over J. After the transformation

$$\begin{pmatrix}\bar F_1\\ \bar F_2\\ \bar F_3\\ \bar F_4\\ \bar F_5\end{pmatrix}=\frac{\pi}{2E^2p^2}\begin{pmatrix} p^2 & -3E^2 & -(p^2+3E^2) & -z(3E^2+p^2) & -2zE^2(2E^2+m^2)/m^2\\ 0 & 0 & 0 & -4E^2 & -4(E^4/m^2)\\ -p^2 & -E^2 & -m^2 & -zm^2 & -2zE^2\\ 0 & 0 & 0 & 4E^2 & 4E^2\\ 3p^2 & -E^2 & 4E^2+3Em^2 & -z(E^2+3p^2) & -2zE^2(E^2+p^2)/m^2\end{pmatrix}\times\begin{pmatrix}f_1\\ f_2\\ f_3\\ f_4\\ f_5\end{pmatrix},$$

we obtain the invariant functions used in Eq. (2.3) in Ref. 11. The functions F_α in Eq. (10) of the present paper are obtained after one further linear transformation:

$$\begin{pmatrix}F_1\\ F_2\\ F_3\\ F_4\\ F_5\end{pmatrix}=\tfrac{1}{4}\begin{pmatrix} 3 & 6 & -4 & 4 & -1\\ -1 & 2 & 0 & 0 & -1\\ -1 & 0 & 6 & 2 & 1\\ -1 & 0 & -2 & 2 & 1\\ -1 & 6 & 4 & -4 & 3\end{pmatrix}\times\begin{pmatrix}\bar F_1\\ \bar F_2\\ \bar F_3\\ \bar F_4\\ \bar F_5\end{pmatrix}.$$