Model of Elastic High-Energy Scattering*

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A model for elastic high-energy scattering is presented, using the eikonal picture, or optical model, generalized to the case of two hadrons of finite extension going through each other. Two-dimensional Fourier transforms and a two-dimensional impact parameter **b** are introduced in the discussion. The relationship with Glauber's theory is analyzed. A comparison with experiment is presented.

I. INTRODUCTION

HIGH-ENERGY elastic scattering experiments indicate that as the incoming energy $\to\infty$, the differential cross section approaches a limit

$$f(t)=\lim d\sigma/dt\,, \tag{1}$$

where $-t$ is the square of the 3-momentum transfer in the c.m. system. Assuming (1) to be true, it is important to interpret the meaning of the function $f(t)$.

In this paper we explore a model for such an interpretation in which the two incoming particles are considered as two objects of finite spatial extension which "go through" each other with attenuation. Elastic scattering then results from the propagation of the attenuated part of the incoming wave function. The spirit of this model is thus closely related to that of the optical model of Fernbach, Serber, and Taylor,[1] and also to that of the Glauber model.[2] But in the present case we need to consider, instead of the propagation of waves through a nucleus of finite extension, the propagation of two objects through each other. A preliminary report[3] of this model has already appeared in the literature. While that report is complete in itself, the present paper gives further developments of the mathematical formalism, the relationship with Glauber's theory, and additional detailed comparison with experiments.

The attenuation of two objects going through each other will be approximated through a product estimation of the opaqueness. This approximation is suggested by the heuristic relationship given by Wu and Yang[4] that the p-p scattering cross section is proportional to the fourth power of the proton form factor:

$$f(t)=(\mathrm{const})F_1{}^4(t)\,. \tag{2}$$

[Van Hove[5] has given this relationship a different but related meaning through the quark model. See also the recent paper by Abarbanel, Drell, and Gilman.[6]] We shall see, indeed, that for small opaqueness, our model yields (2) as a first approximation.

There are also corrections due to higher approximations. These corrections are experimentally in the right direction to bring (2) into good agreement with experiments, especially at high values of momentum transfer. Theoretically, these corrections are similar to the Glauber terms[2] in the theory of nucleon-nucleus scattering. (See Sec. V.) In this connection, it is useful to emphasize that (a) the good experimental fit[7] of p-α scattering or p-C scattering is precisely due to such correction terms, and (b) the corrections (see Sec. V) are due to the shielding of the back of the scatterer by the front part. That such shielding exists in a p-nucleus scattering in Glauber's model is easy to accept. If it is confirmed that such shielding also exists for p-p scattering, one would have an additional strong verification of the usefulness of a geometrical picture of p-p scattering.

II. EIKONAL PICTURE AND FOURIER TRANSFORM

We neglect spin throughout this paper. (See a discussion in Ref. 3.) The differential cross section is

$$d\sigma/dt=\pi|a|^2\,, \tag{3}$$

where

$$a=\lambda\!\!\!^{-2}\sum_l(2l+1)P_l(\cos\theta)\tfrac{1}{2}(1-S) \tag{4}$$

is given by the usual partial-wave expansion. At large energies and small angles we can replace

$$P_l(\cos\theta)\to J_0(b\sqrt{-t})\,, \tag{5}$$

where

$$b=\lambda\!\!\!^{-}(l+\tfrac{1}{2})\,. \tag{6}$$

The existence of the limit (1) is then merely the statement that S, the transmission coefficient (or the S matrix for given angular momentum), is a function only of the impact parameter:

$$S=S(b)\,. \tag{7}$$

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[1] S. Fernbach, R. Serber, and T. B. Taylor, Phys. Rev. **75**, 1352 (1949).

[2] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Interscience Publishers, Inc., New York, 1959), Vol. 1.

[3] T. T. Chou and C. N. Yang, in *High Energy Physics and Nuclear Structure*, edited by G. Alexander (North-Holland Publishing Co., Amsterdam, 1967), pp. 348–359. This paper also contains a number of remarks on the way the scattering amplitudes approach limits at high energies.

[4] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).

[5] L. Van Hove, in Stony Brook Report, 1966 (unpublished).

[6] H. D. I. Abarbanel, S. D. Drell, and F. J. Gilman, Phys. Rev. Letters **20**, 280 (1968).

[7] W. Czyż and L. Leśniak, Phys. Letters **24B**, 227 (1967); R. H. Bassel and C. Wilkin, Phys. Rev. Letters **18**, 871 (1967); T. T. Chou, Phys. Rev. **168**, 1594 (1968).

Thus,

$$a\to\int_0^\infty(1-S)J_0(b\sqrt{-t})bdb=(2\pi)^{-1}\int_0^\infty(1-S)\int_0^{2\pi}\times\exp[ib(\sqrt{-t})\cos\phi]d\phi b\,db. \quad (8)$$

This equation[1,2] can be expressed in terms of a two-dimensional Fourier transform: We introduce the two-dimensional momentum transfer vector $\boldsymbol{\kappa}$ and impact parameter $\mathbf{b}$ in the plane perpendicular to the incoming beam:

$$\boldsymbol{\kappa}=(\kappa_x,\kappa_y),\quad \boldsymbol{\kappa}^2=-t, \quad (9)$$

$$\mathbf{b}=(b_x,b_y),\quad \mathbf{b}^2=b^2. \quad (10)$$

Denote by ϕ the angle between $\boldsymbol{\kappa}$ and $\mathbf{b}$. Then the right side of (8) is

$$(2\pi)^{-1}\int\int[1-S(\mathbf{b})]\exp(i\boldsymbol{\kappa}\cdot\mathbf{b})d^2b, \quad (11)$$

which is the Fourier transform of $1-S(\mathbf{b})$. We shall denote the Fourier transform of a function $X(\mathbf{b})$ of $\mathbf{b}$ by

$$\langle X\rangle\equiv(2\pi)^{-1}\int\int X(\mathbf{b})\exp(i\boldsymbol{\kappa}\cdot\mathbf{b})d^2b. \quad (12)$$

$\langle X\rangle$ is, of course, a function of $\boldsymbol{\kappa}$. Thus (8) becomes

$$a=\langle 1-S\rangle \quad (13)$$

or

$$\langle a\rangle=1-S, \quad (14)$$

where we denote by the same symbol $\langle\ \rangle$ the inverse Fourier transform.

Note that the above discussion is closely related to that of the scattering of waves by a disk-shaped scatterer.[1] We remark that in many discussions of the optical model a potential V is introduced. Actually the introduction of a potential is often unnecessary. For the present discussion, to arrive at (13) we need only the concept of a local transmission coefficient S. The rest is Huygen's principle. The two-dimensional Fourier transform that occurs in (13) is typical of considerations based on Huygen's principle. For example, the two-dimensional Fourier transform is a most useful concept in laser optics.

III. SIMPLE PROPERTIES OF FOURIER TRANSFORMS

We define folding integrals $\otimes$ in either $\mathbf{b}$ space or $\boldsymbol{\kappa}$ space:

$$X\otimes Y|_{\mathbf{b}}=(2\pi)^{-1}\int\int X(\mathbf{b}-\mathbf{b}')Y(\mathbf{b}')d^2b', \quad (15)$$

$$a\otimes c|_{\boldsymbol{\kappa}}=(2\pi)^{-1}\int\int a(\boldsymbol{\kappa}-\boldsymbol{\kappa}')c(\boldsymbol{\kappa}')d^2\kappa'. \quad (16)$$

It is easy to prove

$$\langle X\rangle\langle Y\rangle=\langle X\otimes Y\rangle, \quad (17)$$

$$\langle a\rangle\langle c\rangle=\langle a\otimes c\rangle. \quad (18)$$

Using these, we can express a of (13) in terms of

$$s=-\langle\ln S\rangle \quad (19)$$

and vice versa by taking the Fourier transform of

$$\langle a\rangle=1-S=1-\exp[\ln S]=-\ln S-\frac{1}{2!}[\ln S]^2-\frac{1}{3!}[\ln S]^3-\cdots,$$

and

$$-\ln S=-\ln(1-\langle a\rangle)=\langle a\rangle+\tfrac{1}{2}[\langle a\rangle]^2+\tfrac{1}{3}[\langle a\rangle]^3+\cdots.$$

The results are

$$a=s-\frac{1}{2!}s\otimes s+\frac{1}{3!}s\otimes s\otimes s-\cdots, \quad (20)$$

and

$$s=a+\tfrac{1}{2}a\otimes a+\tfrac{1}{3}a\otimes a\otimes a+\cdots. \quad (21)$$

IV. PRODUCT APPROXIMATION OF OPAQUENESS

Equations (20) and (21) are results of the eikonal picture, *independent* of any other approximations. They related the scattering amplitude a with the quantity s which in turn is related through (19) to the transmission coefficient S. Now the transmission coefficient $S(\mathbf{b})$ can be given a physical interpretation as follows. Consider a slab of thickness g. If the slab absorbs and disperses an incoming wave, the transmission coefficient for the wave through the slab would be

$$S=\exp(-\alpha g).$$

Thus the quantity $-\ln S$ is proportional to the thickness of the slab and could be considered as the opaqueness of the slab to the wave. For the scattering of waves by a spherically symmetrical object, the quantity $\langle s\rangle=-\ln S(\mathbf{b})$ is similarly the opaqueness at the impact parameter $\mathbf{b}$.

For the collision of two hadrons we are thus faced with the problem of evaluating the opaqueness at the impact parameter $\mathbf{b}$. The picture that we are pursuing leads[3] naturally to the concept that this opaqueness should be some mean opaqueness of the target as it appears to the different parts of the incoming particle.

To be concrete, each hadron is assumed to have a certain internal structure represented by a density of opaqueness $\rho(x,y,z)$. ρ is assumed to be spherically symmetrical. [*Note added in proof.* A related but different ρ function was discussed in N. Byers and C. N. Yang, Phys. Rev. **142**, 976 (1966).] To a point inside of the incoming particle, the target appears as a disk with a two-dimensional density of opaqueness

$$D(x,y)=\int_{-\infty}^{\infty}\rho(x,y,z)dz. \qquad (22)$$

For the collision between a proton and a pion, for example, we argue[3] that the resultant opaqueness at an impact parameter **b** is

$$\langle s\rangle=-\ln S(\mathbf{b})=K_{\pi p}\int\int D_\pi(\mathbf{b}-\mathbf{b}')D_p(\mathbf{b}')d^2b' =2\pi K_{\pi p}D_\pi\otimes D_p, \qquad (23)$$

where $K_{\pi p}=\text{const}$. Notice that this expression is symmetrical with respect to switching the pion and proton. It is linear in both D_π and D_p, which are obvious requirements to be satisfied.

Equation (23) is our assumption. It expresses the following approximation: The attenuation of the probability amplitude accompanying the process of two hadrons going through each other is governed by the local opaqueness within each hadron.

Taking the Fourier transform of (23), one obtains

$$s_{\pi p}=2\pi K_{\pi p}\langle D_p\rangle\langle D_\pi\rangle.$$

Similarly,

$$s_{pp}=2\pi K_{pp}\langle D_p\rangle^2,\quad s_{\pi\pi}=2\pi K_{\pi\pi}\langle D_\pi\rangle^2. \qquad (24)$$

Equations (20), (21), and (24) form the complete mathematical statement of our model. To deduce experimental consequences, we need to relate the functions $\langle D_p\rangle$, $\langle D_\pi\rangle$ to observable quantities. Now

$$\langle D\rangle|_{\kappa_x,\kappa_y}=(2\pi)^{1/2}\langle\rho\rangle|_{\kappa_x,\kappa_y,0},$$

where $\langle\rho\rangle$ is the three-dimensional Fourier transform of the density ρ. In a very rough way we shall identify ρ with the charge distribution inside the hadron. (If the very strong interaction inside of a hadron is thought of as causing complete "mixing," this rough guess may not be entirely wrong.) We thus write

$$F_1(\kappa^2)=\text{const}\langle D\rangle, \qquad (25)$$

where F_1 is the charge form factor of the hadron. [However, because of the uncertainty of the reasoning behind this formula, we do not know whether

$$G_M(\kappa^2)=\text{const}\langle D\rangle \qquad (26)$$

would be a more correct approximation.]

V. RELATIONSHIP WITH GLAUBER THEORY

Equations (20), (21), (24), and (25) lead to

$$[F_1(\kappa^2)]_p{}^2=(\text{const})[a_{pp}(\boldsymbol{\kappa})+\tfrac{1}{2}a_{pp}(\boldsymbol{\kappa})\otimes a_{pp}(\boldsymbol{\kappa}) +\tfrac{1}{3}a_{pp}(\boldsymbol{\kappa})\otimes a_{pp}(\boldsymbol{\kappa})\otimes a_{pp}(\boldsymbol{\kappa})+\cdots] \qquad (27a)$$

and

$$a_{pp}(\boldsymbol{\kappa})=\Delta_p(\boldsymbol{\kappa})-\frac{1}{2!}\Delta_p(\boldsymbol{\kappa})\otimes\Delta_p(\boldsymbol{\kappa}) +\frac{1}{3!}\Delta_p(\boldsymbol{\kappa})\otimes\Delta_p(\boldsymbol{\kappa})\otimes\Delta_p(\boldsymbol{\kappa})-\cdots, \qquad (27b)$$

where

$$\Delta_p(\boldsymbol{\kappa})=(\text{const})[F_1(\kappa^2)]_p{}^2.$$

These equations explicitly give the relation between the infinite-energy scattering amplitude a_{pp} and the form factor F_1.

If one drops all terms but the first on the right side of Eqs. (27), one obtains (2) which was first proposed in Ref. 4. We shall refer to the dropped terms as "corrections." These correction terms correspond to the additional terms in Glauber's model[2] of, say, pion-nucleus scattering. [In fact, the terms on the right side of (27b) correspond, term by term, to those in Glauber's theory.] The relationship between our model and Glauber's theory can be understood as follows: Consider the scattering of a point particle "π" by a "nucleus" in Glauber's theory, and assume (a) that the nucleons in the nucleus each scatter infinitesimally, (b) that there are infinitely many of these infinitesimal nucleons, and (c) the infinitesimal nucleons have, each of them, a very small dimension compared with that of the nucleus. (In other words, the nucleus is a droplet of some finely granulated scattering medium.) The transmission coefficient through the nucleus in Glauber's theory is

$$S=\text{average of }(S_1S_2\cdots), \qquad (28a)$$

where S_1, S_2, etc., are the transmission coefficients through the individual nucleons. In the limit of infinitely many nucleons, one obtains

$$S(\mathbf{b})=\exp\left[-(\text{const})\int\rho(b_x,b_y,z)dz\right], \qquad (28b)$$

where ρ is the density of nucleons. This reduces exactly to (23) if the density ρ_π there is taken to be a δ function.

The importance of the correction terms in Glauber's theory was emphasized recently.[7,8] (See Sec. I.) Their importance in the present model is illustrated in Sec. VI.

VI. COMPARISON WITH EXPERIMENT

To compare with experimental data, we need $a_{pp}(\boldsymbol{\kappa})$, i.e., the limit of the scattering amplitude at infinite energies. Two sample possibilities are tried. (In both of these we take a_{pp} to be real, i.e., the usually defined scattering amplitude pure imaginary.)

Possibility A: $\lim(d\sigma/dt)_{pp}=79.04e^{10.3t}$ mb $(\text{BeV}/c)^{-2}$, (29)

i.e.,

$$a_{pp}=8.04e^{-5.15\kappa^2}\ (\text{BeV}/c)^{-2}, \qquad (30)$$

$$\langle a_{pp}\rangle=0.78e^{-0.0485b^2}. \qquad (31)$$

Possibility B:

$$\lim(d\sigma/dt)_{pp}=79.04(e^{5.15t}+0.015e^{2t})^2\text{ mb }(\text{BeV}/c)^{-2}, \qquad (32)$$

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[8] R. J. Glauber, in *High Energy Physics and Nuclear Structure*, edited by G. Alexander (North-Holland Publishing Co., Amsterdam, 1967), pp. 311–338.

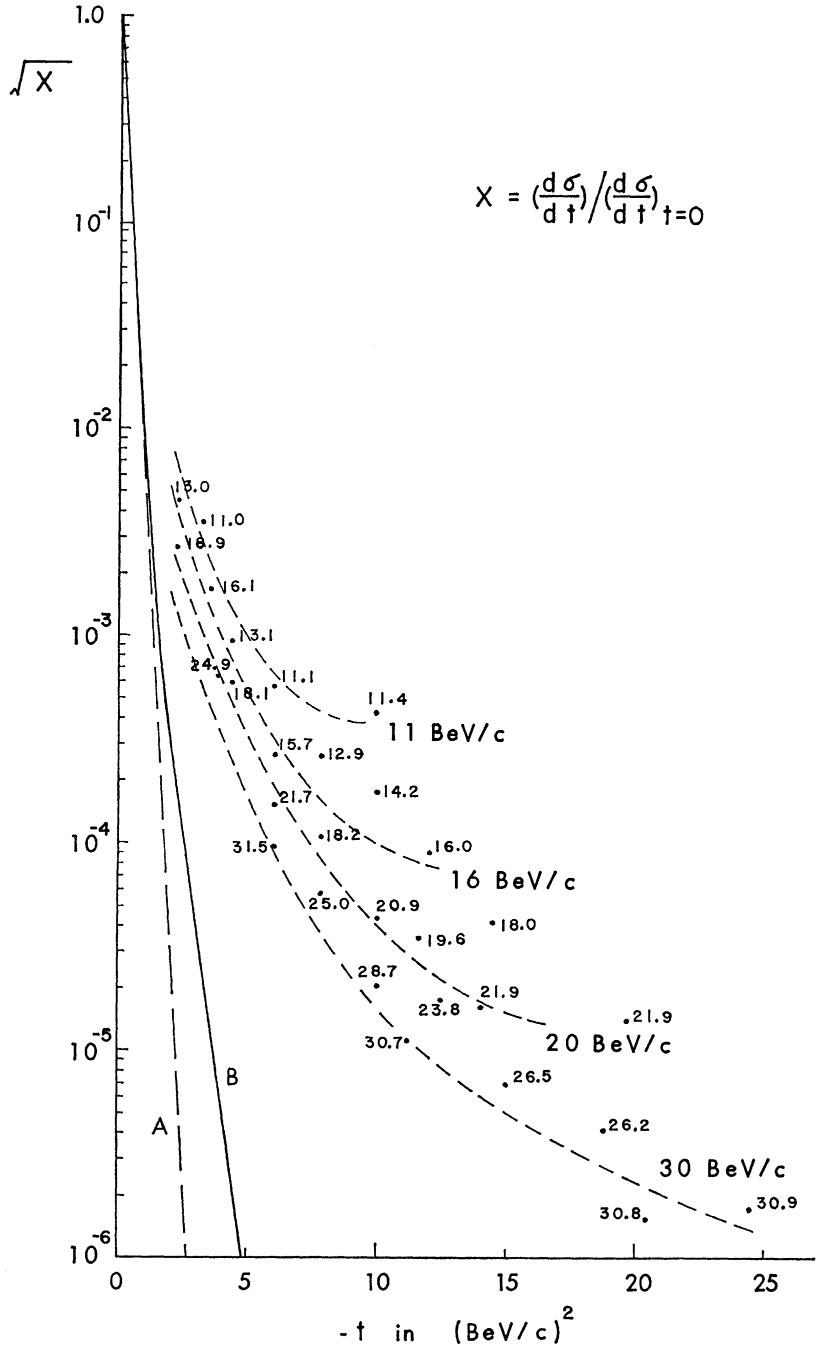

FIG. 1. Two sample possibilities A and B for the limit of the scattering amplitude at infinite energies. Existing data for large $-t$ values [G. Cocconi *et al.*, Phys. Rev. 138, B165 (1965)] are also shown.

i.e.,

$$a_{pp}=8.04e^{-5.15\kappa^2}+0.121e^{-2\kappa^2}\ (\mathrm{BeV}/c)^{-2}, \quad (33)$$

$$\langle a_{pp}\rangle=0.78e^{-0.0485b^2}+0.03e^{-0.125b^2}. \quad (34)$$

Both fit existing data for $0\leqslant -t<1$ (BeV/c)^2. Comparison with existing data for large $-t$ values are shown in Fig. 1. [In the interesting paper of Abarbanel, Drell, and Gilman,[6] the following possibility was discussed:

$$\lim(d\sigma/dt)_{pp}\cong(d\sigma/dt) \text{ at lab energy}=30\ \mathrm{BeV}/c. \quad (35)$$

While this is not demonstrably inconsistent with existing data on pp scattering, it seems to us that the concomitant "precipitous" flattening out (to use the expression of Ref. 6) of $d\sigma/dt$ versus E_L (=laboratory incoming energy) at $E_L\approx 30$ BeV/c is not likely to be borne out by experiments at higher energies. It would be most interesting to put this point to experimental test.]

Comparison with the experimental proton form factor F_1 through (27) is given in Fig. 2 for possibilities A and

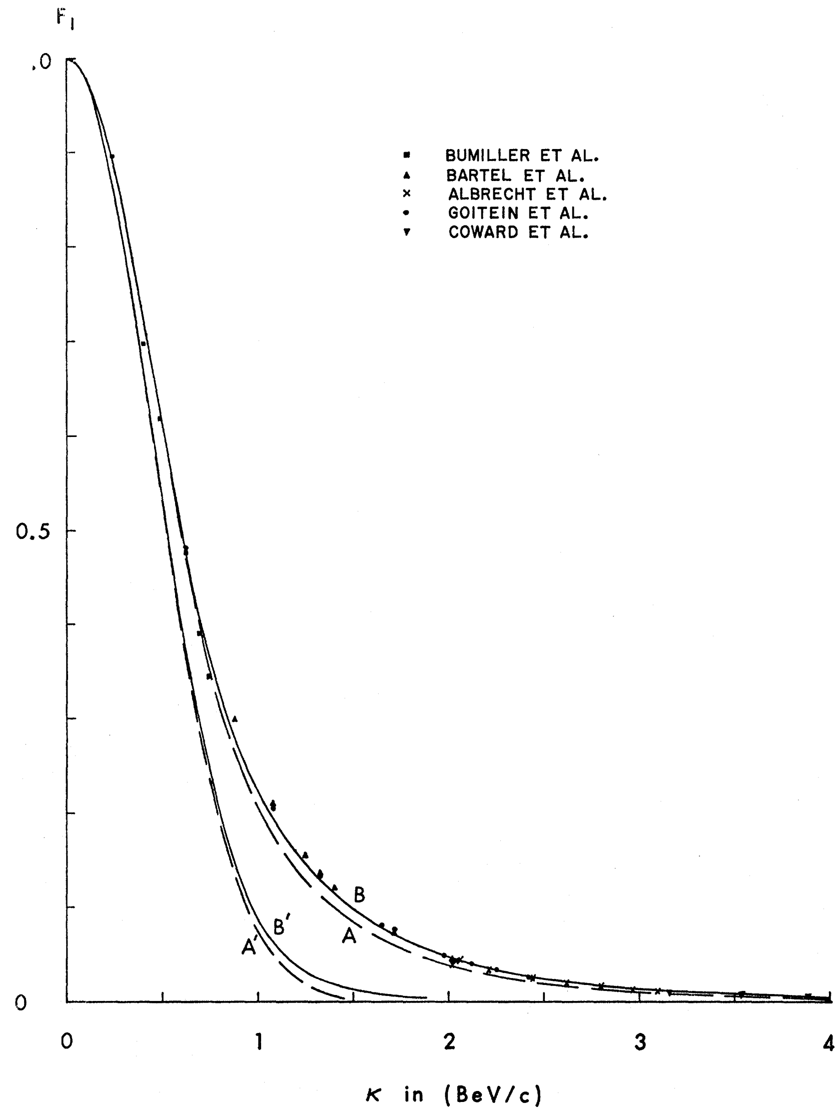

FIG. 2. The charge form factor F_1 of proton versus κ. Curves A and B are obtained by using Eq. (27a) and the two possibilities, Eqs. (30) and (33), mentioned in the text. Curves A′ and B′ are the two corresponding cases with the correction terms in Eq. (27a) deleted. The experimental points are those of the form factor F_1 of the proton taken from the following references: L. N. Hand *et al.*, Rev. Mod. Phys. **35**, 335 (1963) for $\kappa^2<0.6$ (BeV/c^2); W. Bartel *et al.*, Phys. Rev. Letters **17**, 608 (1966) for $0.6<\kappa^2<3.5$ (BeV/c)^2; W. Albrecht *et al.*, *ibid.* **17**, 1192 (1966) for $\kappa^2>3.5$ (BeV/c)^2; M. Goitein *et al.*, *ibid.* **18**, 1016 (1967); D. H. Coward *et al.*, *ibid.* **20**, 292 (1968).

B. It is seen that F_1 is not very sensitive to the difference between A and B. Both agree with F_1 quite well. To appreciate the importance of the correction terms in (27) (i.e., the terms after the first on the right side) we delete them and plot in Fig. 2 F_1 for the two possibilities A and B, obtaining curves A′ and B′. They give rather poor fits for $\kappa^2\sim 1$ (BeV/c)^2. For the significance of the comparison between A′, B′, and A, B, see Secs. I and V. It should be emphasized that no adjustable parameters have been used in writing (27). The fit of curves A and B to the experimental points in Fig. 2 is thus a *no-parameter fit.*

To push this fit to the extreme, and examine very small values of F_1 at large κ^2, is, in principle, a procedure not wholly justifiable in view of the rather simple assumptions underlying the present model. It is nevertheless tempting to make such an examination, and the result appears in Fig. 3. It is clear that with a limiting value of $(d\sigma/dt)_{pp}$ not very different from that given in possibility B [Eq. (32)], (27a) would be in very good agreement with present experimental information on F_1.

VII. ASYMPTOTIC LIMIT

In general, it is not useful to discuss asymptotic limits of approximate formulas. However, it is of mathematical interest to study the asymptotic limits, assuming (20), (21), (24), and (25) to be exact. In that case since $F_1(\kappa^2)$ is analytic in $\boldsymbol{\kappa}^2$ at $\boldsymbol{\kappa}^2=0$, $-\langle \ln S_{pp}\rangle$ has the same property and $\ln S_{pp}$ must be less than any inverse power of b for large b.

Another mathematical result that can be proved is that if for large $\boldsymbol{\kappa}^2$,

$$a \longrightarrow (\text{const})(\kappa^2)^m\,, \quad m=\text{negative integer,}$$

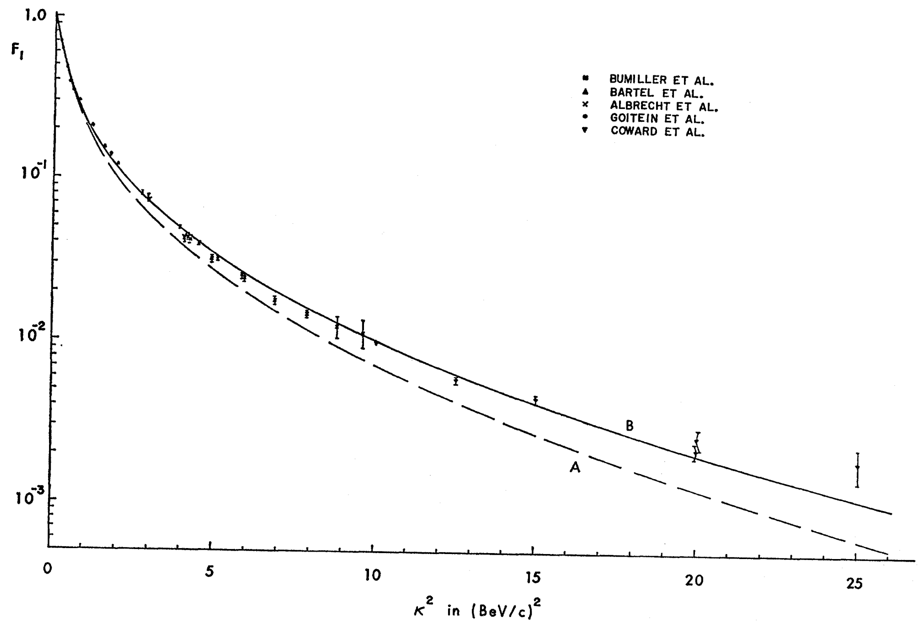

FIG. 3. The charge form factor F_1 of proton versus κ^2. Curves A and B are obtained by using Eq. (27a) and the two possibilities, Eqs. (30) and (33), mentioned in the text. The experimental points are taken from the references given in the caption of Fig. 2.

then (20), (21), (24), and (25) lead to the result that for large κ^2,

$$F_1(\kappa^2) \longrightarrow (\text{const})(\kappa^2)^{m/2},$$

which is consistent with the heuristic relation (2).

What about $F_1(\kappa^2)$ for negative values of κ^2? Analytic continuation is involved in this question and it is extremely dangerous to attempt to draw any conclusions starting from approximate relationships. We shall thus not explore this subject further.

VIII. PION FORM FACTOR

In Ref. 3 the model was also applied to $\pi\pi$ scattering and the pion form factor. (There is some recent experimental information[9] on the latter problem, but data for large momentum transfer are not available.) The important conclusions are that the rms radius of the pion is similar to that of the proton, both being about $\sim 0.73\times 10^{-13}$ cm, and that for large momentum transfers the pion form factor falls much faster than that of the proton. This "softness" of the pion may be related to the recently observed fact[10] that for 90° scattering of πp and pp at the same c.m. momentum, the former (πp) has a much lower cross section than the latter.

IX. CONCLUSION

(a) The value of $(d\sigma/dt)_{pp}$ from small t to large t changes by an enormous factor. At 30 BeV/c the factor is $\sim 10^{12}$. On the other hand, the experimentally observed proton form factor changes by a factor of $\sim$1 000 from $\kappa^2=0$ to $\kappa^2=25$ (BeV/c)^2. That these two factors are approximately related though (2) which, was heuristically conjectured in Ref. 4, seems to be very well borne out by experiments, at least to the zeroth approximation.

(b) However, corrections to (2) should be included because of shielding effects of the back of a target by the front. These correction terms lead to (27). These corrections are in the direction to bring the model into quite good agreement with experiment.

[9] C. W. Akerlof, W. W. Ash, K. Berkelman, C. A. Lichtenstein, A. Ramanauskas, and R. H. Siemann, Phys. Rev. **163**, 1482 (1967).

[10] Private communication from J. Orear of results of a recent Brookhaven National Laboratory—Cornell experiment.