of the interaction between pions and nucleons; indeed, our results do not depend on any particular model. Nevertheless, a model serves to make our approximations explicit, and the one described above is chosen for simplicity.

Within this model, we may write the analog of Eq. (4.18) for the pion-nucleon vertex in the ladder approximation.

$$\Gamma_1(M^2,(\tfrac{1}{2}P+q)^2;\ (\tfrac{1}{2}P-q)^2)\gamma_5+\Gamma_2(M^2,(\tfrac{1}{2}P+q)^2;\ (\tfrac{1}{2}P-q)^2)(\tfrac{1}{2}\boldsymbol{P}+\boldsymbol{q}-M)\gamma_5=\frac{if_Nf_\pi}{16\pi^4}\int d^4k\frac{1}{(k-q)^2-m^2}\frac{1}{(\tfrac{1}{2}P+k)^2-M^2}$$
$$\times\frac{1}{(\tfrac{1}{2}P-k)^2-\mu^2}(\tfrac{1}{2}\boldsymbol{P}+\boldsymbol{k}+M)[\Gamma_1(M^2,(\tfrac{1}{2}P+k)^2;\ (\tfrac{1}{2}P-k)^2)\gamma_5+\Gamma_2(M^2,(\tfrac{1}{2}P+k)^2;\ (\tfrac{1}{2}P-k)^2)(\tfrac{1}{2}\boldsymbol{P}+\boldsymbol{k}-M)\gamma_5]. \quad (6.3)$$

The spin algebra here is easily worked out, and the limit as $s_1=(\frac{1}{2}P+q)^2$ becomes large can then be taken. We find

$$\Gamma_1(M^2,s_1;s_2)\to C_1/s_1 \quad \text{and} \quad \Gamma_2(M^2,s_1;s_2)\to C_2/s_1^2 \quad (6.4)$$

as $s_1\to\infty$. The same asymptotic behavior in s_2 holds if we let $s_2\to\infty$.

We can now go back to Eq. (6.1). After the spin algebra is calculated and separate equations for F_1 and F_2 are obtained, the asymptotic behavior resulting from Eqs. (6.4) comes out to be

$$F_1(t)\sim(\ln t)^2/t^2 \quad \text{and} \quad F_2(t)\sim 1/t^2.$$

Thus essentially the same conclusions obtain here as in the spinless case.

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Study of Bound-State Solutions to the Pion-Nucleon Bethe-Salpeter Equation*

KLAUS D. ROTHE†
Lawrence Radiation Laboratory, University of California, Berkeley, California 94720
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The Bethe-Salpeter equation describing the interaction of pseudoscalar mesons and nucleons via pseudoscalar coupling is solved numerically for energies below the elastic threshold by use of variational techniques. We consider only the "ladder" approximation with a local potential corresponding to the exchange of an elementary nucleon. Simple generalizations of this form of the interaction are considered as well. In the absence of a cutoff, this leads to a marginally singular integral equation. We examine in detail the boundary conditions to be imposed on the solutions in order to lead to a discrete eigenvalue spectrum. The study of this problem is considerably simplified at zero total c.m. energy, where the (Wick-rotated) equation is invariant under four-dimensional rotations. In order to take full advantage of this symmetry, we construct a new set of spinor spherical harmonics belonging to the representations $(\frac{1}{2}(n\pm1),\frac{1}{2}n)$ and $(\frac{1}{2}n,\frac{1}{2}(n\pm1))$ of the four-dimensional rotation group. The discussion is then extended to the general case, in which we examine briefly the formal structure of the $E\neq0$ solutions.

I. INTRODUCTION

IN recent years there has been renewed interest in the relativistic two-body equations of Salpeter and Bethe.[1] In the absence of a theory of the strong interactions, these off-shell equations provide at least a means for performing dynamical calculations within a manifestly covariant framework. However, even in the "ladder" approximation, in which we retain only the lowest-order term in the expansion of the interaction in powers of G^2 (the square of the coupling constant), the equation has for some time been considered intractable, the difficulties being largely due to the presence of a degree of freedom in the equation, the "relative time," which has no analog in nonrelativistic quantum mechanics. The numerical program initiated by Schwartz[2] demonstrated, however, that the (Wick-rotated) Bethe-Salpeter (BS) equation, in the ladder approximation, could be solved accurately by conventional numerical techniques. This led to renewed interest in the BS equation as a computational tool, a number of calculations having extended since then the bound-state calculation by Schwartz to the elastic scattering region[3] and as far as the second inelastic threshold.[4]

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† Present address: Institut für theoretische Physik der Universität Heidelberg, Heidelberg, Germany.
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In this paper we continue this numerical program by making a quantitative investigation of the "pion-nucleon"[5] bound states in conventional pseudoscalar meson theory, the dynamical framework being provided by the BS equation in the ladder approximation (see Fig. 1). This problem has received little attention within this particular framework, although it has been extensively studied by the dispersion-theoretic techniques of S-matrix theory, in which it has served as a prototype of so-called "bootstrap" calculations. In the present calculation we consider the 3-3 resonance (which we refer to as the N^*) and the nucleon as dynamical states of the πN system. Since we restrict ourselves to the bound-state problem, we are left with the nucleon as the only dynamical state of immediate physical interest. If the usual arguments given within a dispersion-theoretic framework[6] should serve as an indication, we expect the N^* to contribute here the dominant force. However, the inclusion of the N^* exchange would force us to introduce a cutoff right from the start. Since it was our original aim, if possible, not to introduce any cutoff into the calculation, we have considered here only the nucleon exchange force.[7] This choice of interaction still leads to a marginally singular integral equation. It is well known that the eigenvalue spectrum of such singular (non-Fredholm) integral equations may be continuous, rather than discrete. An example of this is provided by the nucleon-nucleon BS equation,[8] whose eigenvalue spectrum was found to be continuous unless the integral equation was supplemented with additional boundary conditions not already contained in it. We are faced here with a similar situation. It is for this reason that we devote a substantial part of this paper to a detailed study of the boundary conditions to be imposed on the solutions. We find that the behavior of the BS wave function near the light cone is critically dependent on the strength of the potential, and that a proper treatment of these boundary conditions is imperative for our numerical calculation to be successful. The problem of giving a proper treatment of the equation near the light cone has also been encountered in connection with the nucleon-nucleon problem, and has been studied at great length in a ϕ^4 field theory.[9] In our case the discussion of this problem is, however, considerably more complicated because of the presence of spin.

The subject material has been arranged as follows: In Sec. II we motivate field theoretically the precise form of the BS equation of interest, defining all the relevant quantities. Section III is devoted to a study of the asymptotic behavior of the solutions to the Wick-rotated integral equation. In Sec. IV we present some mathematical results which will be needed later on. In particular, we construct a new set of spinor spherical harmonics belonging to the irreducible representations $(\frac{1}{2}n\pm\frac{1}{2},\frac{1}{2}n)$ and $(\frac{1}{2}n,\frac{1}{2}n\pm\frac{1}{2})$ of the four-dimensional rotation group. The main properties of these functions are discussed and a number of useful formulas are presented. In Sec. V we discuss in detail the behavior of the solutions near the origin of the (Wick-rotated) Euclidean space, studying the BS equation in its differential form. No attempt is made in this section to select among the "regular" and "irregular" solutions. If a physical interpretation of the BS amplitude is to be avoided, the criterion for such a selection must come directly from the integral equation itself. This is discussed in Sec. VI. The numerical techniques and results of our calculation are presented in Secs. VII and VIII. A number of related topics are left for the Appendices, where we present, in in addition to some mathematical details relating to the conventional spinor spherical harmonics, a proof of elastic unitarity; and the group theoretical results which we use in Sec. IV.

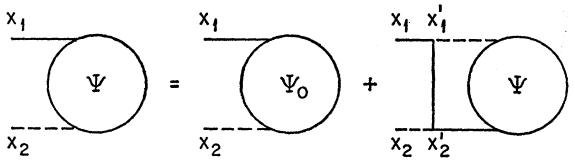

FIG. 1. Integral equation (in the ladder approximation) for the BS wave function for E above the elastic threshold.

We follow in general the notation of Ref. 3. Three-vectors are represented by boldface type, and natural units ($\hbar=c=1$) are used throughout this work.

II. BETHE-SALPETER EQUATION

Since we are concerned only with the bound-state problem, it will be particularly convenient to work with the relativistic two-body "wave function" for the pion and nucleon[5] in a state $|\beta\rangle$, as defined[10] by

$$\Psi^i(x_1,x_2)=\langle 0|\,T(\psi(x_1)\phi^i(x_2))\,|\beta\rangle, \qquad (2.1)$$

where $\phi^i(x)$ and $\psi(x)$ are the Heisenberg fields of the pion and nucleon, respectively, $|0\rangle$ is the physical vacuum, and T is the time-ordering operator; $\psi(x)$ is an eight-component object in the combined spin and isotopic spin space of the nucleon. We take the interaction Hamiltonian density to have the conventional form

$$\mathcal{H}(x)=\sum_i \tfrac{1}{2}iG[\bar{\psi}(x),\gamma_5\tau^i\psi(x)]\phi^i(x), \qquad (2.2)$$

where τ^i are the usual Pauli matrices, and G is referred to as the "pseudoscalar coupling constant." Experimentally, $G^2/4\pi\approx 14$. Restricting ourselves to the

[5] Throughout this paper we refer to the pseudoscalar meson as the "pion" and to the spin-$\frac{1}{2}$ particle as the "nucleon," although we do not restrict the masses to have the experimentally measured values.

[6] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963).

[7] In spite of this we were still forced in several instances to introduce a cutoff.

[8] J. S. Goldstein, Phys. Rev. **91**, 1516 (1953).

[9] A. Bastai, L. Bertocchi, S. Fubini, G. Furlan, and M. Tonin, Nuovo Cimento **30**, 1512 (1963); A. Bastai, L. Bertocchi, G. Furlan, and M. Tonin, *ibid.* **30**, 1532 (1963).

[10] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

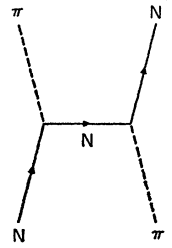

FIG. 2. Diagram representing the nucleon exchange force.

ladder approximation (see Fig. 1), we arrive from the definition (2.1) at the following integral equation for $\Psi^i(x_1,x_2)$ in the scattering region:

$$\Psi^i(x_1,x_2)=\Psi_0{}^i(x_1,x_2)+\frac{G^2}{(4\pi)^2}\sum_i\int d^4x_1'd^4x_2'G_1(x_1-x_1')\times G_2(x_2-x_2')I(x_1'-x_2')[2P^{ij}(\tfrac{3}{2})-P^{ij}(\tfrac{1}{2})]\times\Psi^j(x_2',x_1'),\quad(2.3)$$

$$x_1\equiv(\mathbf{r}_1,t_1),\quad x_2\equiv(\mathbf{r}_2,t_2).$$

Here $G_1(x)$ and $G_2(x)$ are the one-particle Green's functions for the nucleon and pion, respectively,[11]

$$G_1(x)=[m_1+i\gamma\cdot\partial]\Delta(x;m_1),\qquad G_2(x)=\Delta(x;m_2),\quad(2.4)$$

where $\Delta(x;m)$ is the usual invariant function,

$$\Delta(x;m)=(m/4\pi^2|x|)K_1(m|x|).$$

Here $K_1(z)$ is the first-order modified Bessel function,[12] and $|x|=(\mathbf{r}^2-t^2)^{1/2}$. $I(x)$ is the "exchange potential" corresponding to the Born amplitude of Fig. 2, and is given by

$$I(x)=4M[M-i\gamma\cdot\partial]K_1(M|x|)/|x|;\quad(2.5)$$

$\Psi_0(x_1,x_2)$ is the wave function for the free pion and nucleon,

$$\Psi_0(x_1,x_2)=U(\mathbf{k}_1)e^{ik_1\cdot x_1}e^{ik_2\cdot x_2},\quad(2.6)$$

where $k_i=(\mathbf{k}_i,\omega_i)$, $\omega_i=(\mathbf{k}_i{}^2+m_i{}^2)^{1/2}$, and $U(\mathbf{k})_{\alpha\sigma}\equiv U_\alpha(\mathbf{k},\sigma)$ is the nucleon Dirac spinor with the covariant normalization $\bar{U}(\mathbf{k})U(\mathbf{k})=1$ $[\bar{U}(\mathbf{k},\sigma)=U^\dagger(\mathbf{k},\sigma)\gamma^0]$. The matrices $P(I)_{\alpha\beta}{}^{ij}$ are the usual isotopic spin projection operators,

$$P(\tfrac{1}{2})=\tfrac{1}{3}(1-\mathbf{t}\cdot\boldsymbol{\tau}),\qquad P(\tfrac{3}{2})=\tfrac{1}{3}(2+\mathbf{t}\cdot\boldsymbol{\tau}),$$

where $\boldsymbol{\tau}$ and $\mathbf{t}$ are the isotopic spin operators of the nucleon and meson, respectively (in the Cartesian basis, $t_{jk}{}^i=-i\epsilon_{ijk}$). Using the fact that $P(I)P(I')=\delta_{II'}P(I)$, we arrive immediately at the integral equation for the two-body wave function in a state of definite total isotopic spin I:

$$\Psi^I(x_1,x_2)=\Psi_0{}^I(x_1,x_2)+\lambda^I\int d^4x_1'd^4x_2'G_1(x_1-x_1')\times G_2(x_2-x_2')I(x_1'-x_2')\Psi^I(x_2',x_1').\quad(2.7)$$

Here

$$\lambda^I=\eta(I)G^2/(4\pi)^2,\qquad\eta(\tfrac{1}{2})=-1,\quad\eta(\tfrac{3}{2})=2.\quad(2.8)$$

We have omitted all other isotopic spin labels, since $\Psi^I(x_1,x_2)$ is only a function of the total isotopic spin I. In the future we omit this label as well.

We proceed now to eliminate one of the integration variables in Eq. (2.7) by making use of the translational invariance of the theory. To this end we consider the canonical transformations

$$P=p_1+p_2,\qquad X=\mu_1x_1+\mu_2x_2,\qquad p=\mu_2p_1-\mu_1p_2\equiv(\mathbf{p},p^0),\qquad x=x_1-x_2\equiv(\mathbf{r},t),\quad(2.9)$$

where μ_1 and μ_2 are constants subject to the condition

$$\mu_1+\mu_2=1.\quad(2.10)$$

Since we are dealing with an exchange potential, it turns out to be convenient to make the choice $\mu_1=\mu_2=\frac{1}{2}$.[13] This is the choice made throughout this work, unless explicitly stated otherwise.

Translational invariance allows us to write

$$\Psi(x_1,x_2)=\psi(x)e^{iK\cdot X}.$$

Substitution into (2.7) leads, then, to the following integral equation for $\psi(x)$ (we now label the wave function by the momentum of the incident wave):

$$\psi_{\mathbf{k}}(x)=U(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{12}t/2}-\lambda\int d^4x'G(x-x')I(x')\psi_{\mathbf{k}}(-x'),\quad(2.11)$$

where $\omega_{12}\equiv\omega_1-\omega_2$ and $G(x)$ is the two-particle Green's function for the pion and nucleon. In the c.m. system of the pion and nucleon

$$G(x)=[m_1+i\gamma\cdot\partial+\tfrac{1}{2}\gamma^0E]G_0(x),\quad(2.12)$$

where $E=\omega_1+\omega_2$, and

$$G_0(x)=\int\frac{d^4q}{(2\pi)^4}\times\frac{e^{iq\cdot x}}{[\mathbf{q}^2-(q^0+\frac{1}{2}E)^2+m_1{}^2-i\epsilon][\mathbf{q}^2-(q^0-\frac{1}{2}E)^2+m_2{}^2-i\epsilon]}\quad(2.13)$$

is, of course, just the two-particle Green's function in the absence of spin. Because of the poles of the integrand

[11] Throughout this paper we label the variables referring to the nucleon and pion by the indices 1 and 2, respectively.

[12] See, for instance, G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1962), 2nd ed.

[13] This choice turns out to be undesirable in the case in which the masses of the "pion" and "nucleon" are very different. This point is discussed in Sec. VII B.

it is desirable for the purpose of our later discussion to cast the integral (2.13) into a different form. Thus, making use of the familiar Feynman parametrization procedure we arrive (as in Ref. 3) at

$$G_0(x-x')=\frac{i}{8\pi^2}\int_{-\omega_1}^{\omega_2}\frac{d\beta}{E}e^{-i(\beta+\frac{1}{2}\omega_{12})(t-t')}\times K_0[(\beta^2-\mathbf{k}^2)^{1/2}|x-x'|], \quad (2.14)$$

where $K_0(z)$ is the zeroth-order modified Bessel function, and (see Ref. 3) $-\frac{1}{2}\pi\leqslant\arg[(\beta^2-\mathbf{k}^2)^{1/2}|x-x'|]\leqslant\frac{1}{2}\pi$; $\mathbf{k}^2$, ω_1, and ω_2 are related to the c.m. energy E by

$$\mathbf{k}^2=(1/4E^2)[E^2-(m_1+m_2)^2][E^2-(m_1-m_2)^2],$$
$$\omega_1=[E^2+(m_1^2-m_2^2)]/2E, \quad \omega_2=[E^2-(m_1^2-m_2^2)]/2E, \quad (2.15)$$

and

$$\omega_{12}=\omega_1-\omega_2=(m_1^2-m_2^2)/E. \quad (2.16)$$

We may rewrite Eq. (2.11) as the differential equation

$$[m_1-i\gamma\cdot\partial-\tfrac{1}{2}\gamma^0E][\Box^2-iE\partial_0+\tfrac{1}{4}E^2-m_2^2]\psi(x)=\lambda I(x)\psi(-x), \quad (2.17)$$
$$\gamma\cdot\partial=\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+\gamma^0\partial_0, \quad \Box^2=\boldsymbol{\nabla}^2-\partial_0^2,$$

supplemented with suitable boundary conditions. It is this equation which we solve in the calculational part of this paper.

Finally we would like to remark that unless additional information is required which is not already contained in the integral equation (2.11), we do not need to give a physical interpretation to the two-body wave function (2.1), but may regard it merely as one possible way of formulating the given mathematical problem. Thus, with the definition of the scattering amplitude,

$$\langle\mathbf{k}_1'\sigma',\mathbf{k}_2'|T|\mathbf{k}_1\sigma,\mathbf{k}_2\rangle=\delta^4(K'-K)(E/2\pi\omega_1\omega_2)f_{\sigma'\sigma}(\mathbf{k}',\mathbf{k}),$$

where T is related to the S matrix in the usual way,

$$S=1-iT,$$

we easily establish the connection between $f(\mathbf{k}',\mathbf{k})$ and and the solution $\psi_{\mathbf{k}}(x)$ to the integral equation (2.11):

$$f(\mathbf{k}',\mathbf{k})=\frac{i\lambda m_1}{4\pi E}\bar{U}(\mathbf{k}')\int d^4x\, e^{-ik'\cdot x}I(x)\psi_{\mathbf{k}}(-x), \quad (2.18)$$

the numerical factors being fixed by simply looking at the Born term.

III. ASYMPTOTIC BOUNDARY CONDITIONS

Since it is very difficult to solve numerically the eigenvalue problem in the form of Eq. (2.17), we should like to consider instead the corresponding eigenvalue problem for $\varphi(\mathbf{r},\tau)$, the analytic continuation of $\psi(\mathbf{r},t)$ to the imaginary time axis: $\varphi(\mathbf{r},\tau)=\psi(\mathbf{r},-i\tau)$. Assuming the solution to be an analytic function of t, we arrive at the differential equation

$$[m_1-i\gamma\cdot\partial-\tfrac{1}{2}\gamma^0E]\times[\Box^2+E\partial_4+\tfrac{1}{4}E^2-m_2^2]\varphi(x)=\lambda V(x)\varphi(-x) \quad (3.1)$$

for $\varphi(\mathbf{r},\tau)$, where

$$V(x)=I(\mathbf{r},-i\tau)=(4M^2/R)[K_1(MR)+i\gamma\cdot\hat{x}K_2(MR)]. \quad (3.2)$$

Here $R=(\mathbf{r}^2+\tau^2)^{1/2}$, and $K_n(MR)$ is the nth-order modified Bessel function with the asymptotic property

$$K_n(z)\sim(\pi/2z)^{1/2}e^{-z} \quad \text{as} \quad |z|\to\infty. \quad (3.3)$$

We have introduced the notation $\hat{x}=(x_\alpha/R)$,

$$(x_\alpha)=(\mathbf{r},\tau), \quad (\partial_\alpha)=(\boldsymbol{\nabla},\partial_4), \quad \partial_4=\partial/\partial\tau, \quad \gamma_4=i\gamma^0, \quad (3.4)$$

scalar products being now defined in terms of an Euclidean metric:

$$\gamma\cdot\partial=\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+\gamma_4\partial_4, \quad \Box^2=\boldsymbol{\nabla}^2+\partial_4^2, \quad x^2=\mathbf{r}^2+\tau^2. \quad (3.5)$$

The corresponding analytic continuation of the integral equation (2.11) has already been discussed in the literature.[3,14] We do not repeat the arguments here except to state that for suitable restrictions (to be given at the end of this section) on the c.m. energy E, we may carry out this analytic continuation by setting $t=\tau\exp(-i\theta)$, $t'=\tau'\exp(-i\theta)$ in Eq. (2.11) and letting θ range from zero to $\frac{1}{2}\pi$. Denoting the analytically continued solution by $\phi_{\mathbf{k}}(x)$, we arrive then at the integral equation

$$\phi_{\mathbf{k}}(x)=U(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-\frac{1}{2}\omega_{12}\tau}-\lambda\int d^4x'H(x-x')V(x')\phi_{\mathbf{k}}(-x'), \quad (3.6)$$

where $d^4x=d^3\mathbf{r}d\tau$, and

$$H(x)=[m_1+i\gamma\cdot\partial+\tfrac{1}{2}\gamma^0E]H_0(x), \quad H_0(\mathbf{r},\tau)=-iG_0(\mathbf{r},-i\tau). \quad (3.7)$$

An explicit expression for $H_0(\mathbf{r},\tau)$ may be obtained directly from (2.14):

$$H_0(x)=\frac{1}{8\pi^2}\int_{-\omega_1}^{\omega_2}\frac{d\beta}{E}e^{-(\beta+\frac{1}{2}\omega_{12})\tau}K_0[(\beta^2-\mathbf{k}^2)^{1/2}R]. \quad (3.8)$$

In the bound-state region $\phi_{\mathbf{k}}(x)$ will have poles at (one hopes) a discrete set of values of E (for a given value of λ). At these values of E, the residue of $\phi_{\mathbf{k}}(x)$ at the pole, $\varphi(x)$, satisfies the homogeneous integral equation

$$\varphi(x)=-\lambda\int d^4x'\,H(x-x')V(x')\varphi(-x'). \quad (3.9)$$

[14] G. C. Wick, Phys. Rev. 96, 1124 (1954).

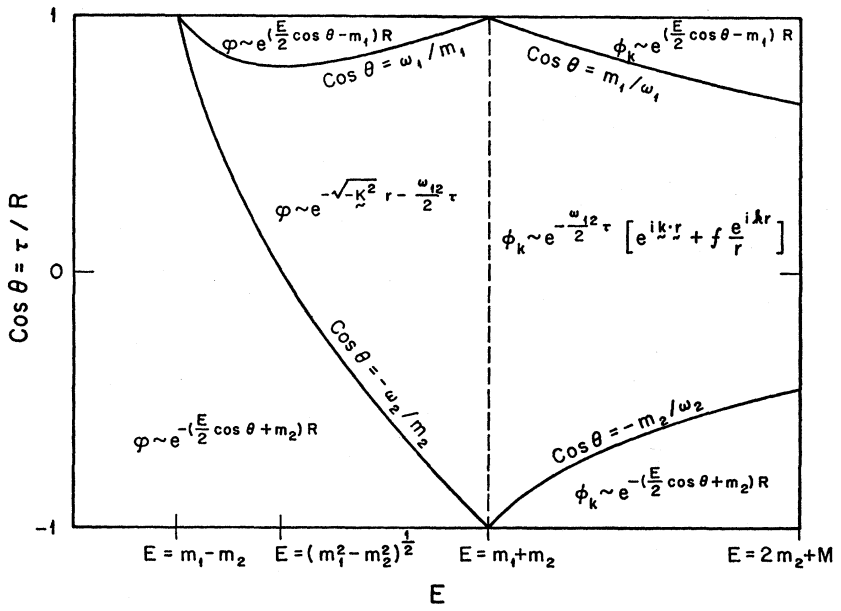

FIG. 3. Leading asymptotic behavior of the solution to the (Wick rotated) BS equation for the c.m. energy E below the first inelastic threshold and for the choice $\mu_i=\frac{1}{2}$ in (2.9). (The scale corresponds to $m_1=1.0$, $m_2=0.6$, $M=1.6$.)

The asymptotic boundary conditions to be imposed on the solutions to (3.1) are those of $\phi_k(x)$ (in the scattering region) and $\varphi(x)$ (in the bound-state region) as deduced from the integral equations (3.6) and (3.9). Although the asymptotic properties of Bethe-Salpeter amplitudes have already been discussed in the literature, we review here briefly the results, with particular emphasis on the unequal mass kinematics of the problem.

Making use of (3.3), we have, in the limit $R\to\infty$,

$$H_0(x)\sim\frac{1}{8\pi^2}\int_{-\omega_1}^{\omega_2}\frac{d\beta}{E}\left(\frac{\pi}{2(\beta^2-\mathbf{k}^2)^{1/2}R}\right)^{1/2}e^{-g(\beta)R}, \quad (3.10)$$

where $g(\beta)=(\beta+\frac{1}{2}\omega_{12})\cos\theta+(\beta^2-\mathbf{k}^2)^{1/2}$, $\cos\theta=\tau/R$. We note that $(\beta^2-\mathbf{k}^2)^{1/2}>0$ for $E<m_1+m_2$ and β within the integration interval (even if $m_1\neq m_2$!) Making use of the "method of steepest descent,"[15] we readily deduce the asymptotic form of the integral (3.10). Introducing the definition

$$A_i(R,\cos\theta)=-\frac{1}{8\pi E}\left(\frac{m_i}{2\pi R}\right)^{1/2}\frac{e^{(\frac{1}{2}E\cos\theta-m_i)R}}{(\omega_i-m_i\cos\theta)R}, \quad (3.11)$$

we summarize the results as follows (we assume $(m_1\geq m_2)$:

(1) $0\leqslant E\leqslant m_1-m_2$. Here [see (2.15)], $\omega_1>0$, $\omega_2<0$, $\mathbf{k}^2>0$; $g(\beta)$ takes on its smallest value for $\beta=\omega_2$, in, dependent of the value of $\cos\theta$, so that $H_0(x)$ exhibits the asymptotic behavior $A_2(R,-\cos\theta)$ for $-1\leqslant\cos\theta\leqslant 1$.

(2) $m_1-m_2\leqslant E\leqslant(m_1^2-m_2^2)^{1/2}$. Here $\omega_1>0$, $\omega_2<0$, $\mathbf{k}^2<0$; $g(\beta)$ has a relative minimum within the integration interval at $\beta=-(-\mathbf{k}^2)^{1/2}\cot\theta$ for $-\omega_2/m_2\leqslant\cos\theta\leqslant\omega_1/m_1$. We distinguish therefore three domains in the $(E,\cos\theta)$ plane, the boundaries being determined by the conditions $g'(-\omega_1)=0$ and $g'(\omega_2)=0$:

(2a) $\omega_1/m_1\leqslant\cos\theta\leqslant 1$; $\quad H_0(x)\sim A_1(R,\cos\theta)$,

(2b) $-1\leqslant\cos\theta\leqslant-\omega_2/m_2$; $\quad H_0(x)\sim A_2(R,-\cos\theta)$,

(2c) $-\omega_2/m_2\leqslant\cos\theta\leqslant\omega_1/m_1$;

$$H_0(x)\sim(1/8\pi E)(1/r)\exp[-(-\mathbf{k}^2)^{1/2}r-\tfrac{1}{2}\omega_{12}\tau].$$

(3) $(m_1^2-m_2^2)^{1/2}\leqslant E\leqslant m_1+m_2$. Here $\omega_1>0$, $\omega_2>0$, $\mathbf{k}^2<0$; the same conclusions hold here as in (2).

(4) $E\geqslant m_1+m_2$. Here $\omega_1>0$, $\omega_2>0$, $\mathbf{k}^2>0$; setting $k=+(\mathbf{k}^2)^{1/2}$ we have

$$H_0(x)\sim(1/8\pi E)(1/r)\exp(ikr-\tfrac{1}{2}\omega_{12}\tau)+A_1(R,\cos\theta)+A_2(R,-\cos\theta), \quad (3.12)$$

as in Ref. 3. The leading asymptotic behavior of $H_0(x)$ for (a) $m_1/\omega_1\leqslant\cos\theta\leqslant 1$, (b) $-1\leqslant\cos\theta\leqslant-m_2/\omega_2$, and (c) $-m_2/\omega_2\leqslant\cos\theta\leqslant m_1/\omega_1$ may be read off directly from (3.12), and is the same as that given in (2a), (2b), and (2c), respectively [with $(-\mathbf{k}^2)^{1/2}=-ik$].

The above results are summarized in Fig. 3, the asymptotic form of the solutions being the same as that of $H_0(x)$. (We have exhibited only the exponential dependence of these asymptotic forms. To be specific we chose $m_1\geqslant m_2$, corresponding to the physical case of interest.) Following the same type of reasoning as given in Ref. 3, we find, with the aid of Fig. 3, that the integration in Eqs. (3.6) and (3.9) does indeed converge at infinity, provided that $E<2m_2+M$ and $E<2m_1+M$; that is, for the c.m. energy E below the first inelastic threshold. For $E\geqslant m_1+m_2$, these restrictions imply of course the stability conditions $m_1<m_2+M$ for the "nucleon," and $m_2<m_1+M$ for the "pion."

[15] See, for example, P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Co., New York, 1953), Part I, p. 437.

Before concluding this discussion we should like to comment briefly on the negative energy spectrum of Eq. (3.1). Ignoring for the moment the question of boundary conditions, we readily verify that if $\varphi(x)$ is a solution to Eq. (3.1) for $E=E_0$ ($E_0>0$ to be specific), then $\gamma_5\varphi(-x)$ is also a solution for $E=-E_0$, at the same value of the coupling constant. In fact, as we will see later on, the same statement can be made about the eigenvalue problem. This shows that the equation admits a positive and a negative energy spectrum whenever the equation has solutions at all. As will become evident later on, when we restate these observations for the partial-wave projections of $\varphi(x)$, this is precisely the statement of the familiar "MacDowell symmetry."[16] It arises here as a consequence of the transformation properties of the equation under space-time inversion, but can be shown to follow also from more general principles.[17] Note that as a function of x, the solutions for $E=-E_0$ have the same asymptotic behavior as the solutions $\varphi(-x)$ for $E=E_0$.

IV. MATHEMATICAL PRELIMINARIES

A. Some Useful Formulas

We derive here some useful formulas relating to the four-dimensional spinor spherical harmonics defined in (A3). Specifically, we are concerned with products of the form $QY_n{}^{l\pm M}(\hat{R})$, where Q is any of the "operators" τ, ∂_4, $(\boldsymbol{\sigma}\cdot\mathbf{r})$, and $(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})$. The evaluation of $\tau Y_n{}^{l\pm M}$ and $\partial_4 Y_n{}^{l\pm M}$ has already been given in Ref. 2. We state here the result[18]:

$$\tau Y_n{}^{l\pm}(\hat{R})=R[A_{n+1}{}^l Y_{n+1}{}^{l\pm}(\hat{R})+A_n{}^l Y_{n-1}{}^{l\pm}(\hat{R})], \quad (4.1)$$

$$\partial_4 Y_n{}^{l\pm}(\hat{R})=A_{n+1}{}^l Y_{n+1}{}^{l\pm}(\hat{R})(d/dR-n/R) +A_n{}^l Y_{n-1}{}^{l\pm}(\hat{R})[d/dR+(n+2)/R], \quad (4.2)$$

where

$$A_n{}^l=\left[\frac{(n-l)(n+l+1)}{4n(n+1)}\right]^{1/2}. \quad (4.3)$$

The evaluation of $(\boldsymbol{\sigma}\cdot\mathbf{r})Y_n{}^{l\pm}$ is also straightforward. With the aid of Eq. (A6) and the recursion relations for the Gegenbauer polynomials $C_n{}^l(t)$,[19]

$$(n+l)C_n{}^l(t)=l[C_n{}^{l+1}(t)-C_{n-2}{}^{l+1}(t)],$$
$$4l(n+l+1)(1-t^2)C_n{}^{l+1}(t) =[(n+2l)(n+2l+1)C_n{}^l(t)-(n+1)(n+2)C_{n+2}{}^l(t)],$$

we obtain, after some algebra,

$$-i(\boldsymbol{\sigma}\cdot\mathbf{r})Y_n{}^{l\pm}(\hat{R})=R[B_n{}^{l\pm}Y_{n+1}{}^{(l\pm1)\mp}(\hat{R}) -C_n{}^{l\pm}Y_{n-1}{}^{(l\pm1)\mp}(\hat{R})], \quad (4.4)$$

where

$$B_n{}^{l+}=\left[\frac{(n+l+2)(n+l+3)}{4(n+1)(n+2)}\right]^{1/2},$$
$$B_n{}^{l-}=\left[\frac{(n-l+1)(n-l+2)}{4(n+1)(n+2)}\right]^{1/2}, \quad (4.5a)$$

and

$$C_{n+1}{}^{(l\pm1)\mp}=B_n{}^{l\pm}. \quad (4.5b)$$

To evaluate $(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})Y_n{}^{l\pm}$, we consider first the Fourier transform of this product as obtained with the aid of the expansion ($p\cdot x=\mathbf{p}\cdot\mathbf{x}+p_4x_4$),

$$\delta_{\alpha\beta}e^{ip\cdot x}=(2\pi)^2\sum_{l\pm,M,n} i^n\frac{J_{n+1}(PR)}{PR}Y_n{}^{l\pm M}(\hat{P})_\alpha{}^*Y_n{}^{l\pm M}(\hat{R})_\beta,$$

and recover the desired result by taking the inverse Fourier transform. Proceeding in this manner, and making use of (4.4) and the recursion relations

$$J_{n+2}(PR)/R=-(d/dR-n/R)J_{n+1}(PR)/PR,$$
$$J_n(PR)/R=[d/dR+(n+2)/R]J_{n+1}(PR)/PR$$

for the Bessel functions, we obtain

$$-i(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})Y_n{}^{l\pm}(\hat{R})=B_n{}^{l\pm}Y_{n+1}{}^{(l\pm1)\mp}(\hat{R})(d/dR-n/R) -C_n{}^{l\pm}Y_{n-1}{}^{(l\pm1)\mp}(\hat{R})[d/dR+(n+2)/R], \quad (4.6)$$

where the coefficients $B_n{}^{l\pm}$ and $C_n{}^{l\pm}$ have already been defined in (4.5).

B. Basis Functions for Representations $(\frac{1}{2}n\pm\frac{1}{2},\frac{1}{2}n)$ and $(\frac{1}{2}n,\frac{1}{2}n\pm\frac{1}{2})$ of O_4

Construction of Basis Functions

In this section we make use of the group-theoretical results of Appendix B in order to construct explicitly the basis functions for the irreducible representations $(\frac{1}{2}n\pm\frac{1}{2},\frac{1}{2}n)$ and $(\frac{1}{2}n,\frac{1}{2}n\pm\frac{1}{2})$ of O_4, the four-dimensional rotation group. These functions are simple linear combinations of the four-dimensional spherical harmonics defined in (A3) and are particularly suited for our discussion of Eq. (3.1) at $E=0$. As we indicate in Appendix B, we may obtain these basis functions by simply projecting out, with the aid of the projection operators (B4), the desired components from the spherical harmonics (A3). Specifically [we choose to label here the spherical harmonics (A3) by J, l, M, and n],

$$Z_{n\pm}{}^{JM}\equiv NP_{n\pm}Y_n{}^{Jl M},$$
$$\tilde{Z}_{n\pm}{}^{JM}\equiv\tilde{N}\tilde{P}_{n\pm}Y_n{}^{Jl M}, \quad (4.7)$$

where $P_{n\pm}$ and $\tilde{P}_{n\pm}$ are the projection operators (B4) for the irreducible spaces of the representations $(\frac{1}{2}n\pm\frac{1}{2},\frac{1}{2}n)$ and $(\frac{1}{2}n,\frac{1}{2}n\pm\frac{1}{2})$, respectively, and N, $\tilde{N}$ are

[16] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).

[17] J. D. Stack (Ph.D. thesis), University of California Radiation Laboratory Report No. UCRL-16115, 1965 (unpublished).

[18] We omit in general the label M for the magnetic quantum number unless its presence is strictly required, in order to simplify the notation. We also frequently omit the argument of the spherical harmonics.

[19] We take the definitions from W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea Publishing Co., New York, 1949).

normalization constants. One may readily verify that it is immaterial whether we apply the projection operators to $Y_n{}^{l+}$ or $Y_n{}^{(l+1)-}$ (same J). After proper normalization we end up with identical results, as it has to be. To be specific, we apply the projection operators to $Y_n{}^{l+}$. Making use of

$$(\boldsymbol{\sigma}\cdot\mathbf{L})Y_n{}^{l+}=lY_n{}^{l+},$$
$$(\boldsymbol{\sigma}\cdot\mathbf{L})Y_n{}^{l-}=-(l+1)Y_n{}^{l-},$$

where the operators L_i are the usual generators of infinitesimal rotations in three-dimensional space, we obtain, with the aid of the results of Sec. IV A,

$$(\boldsymbol{\sigma}\cdot\mathbf{A})Y_n{}^{l+}=\tfrac{1}{2}\{lY_n{}^{l+}+[(n-l)(n+l+2)]^{1/2}Y_n{}^{(l+1)-}\},$$
$$(\boldsymbol{\sigma}\cdot\mathbf{B})Y_n{}^{l+}=\tfrac{1}{2}\{lY_n{}^{l+}-[(n-l)(n+l+2)]^{1/2}Y_n{}^{(l+1)-}\},$$

where the operators A_i and B_i have been defined in (B1). Defining

$$D_n\equiv(2n+2)^{-1/2},$$

we obtain from (4.7)

$$\begin{aligned}Z_{n+}{}^{JM}&=D_n[(n+l+2)^{1/2}Y_n{}^{l+}+(n-l)^{1/2}Y_n{}^{(l+1)-}],\\ Z_{n-}{}^{JM}&=D_n[(n-l)^{1/2}Y_n{}^{l+}-(n+l+2)^{1/2}Y_n{}^{(l+1)-}]\end{aligned}\tag{4.8a}$$

as basis functions for $(\frac{1}{2}n\pm\frac{1}{2},\frac{1}{2}n)$ representation, and

$$\begin{aligned}\tilde{Z}_{n+}{}^{JM}&=D_n[(n+l+2)^{1/2}Y_n{}^{l+}-(n-l)^{1/2}Y_n{}^{(l+1)-}],\\ \tilde{Z}_{n-}{}^{JM}&=D_n[(n-l)^{1/2}Y_n{}^{l+}+(n+l+2)^{1/2}Y_n{}^{(l+1)-}]\end{aligned}\tag{4.8b}$$

as basis functions for the $(\frac{1}{2}n,\frac{1}{2}n\pm\frac{1}{2})$ representation. The quantum numbers J and M labeling $Z_{n\pm}{}^{JM}$ and $\tilde{Z}_{n\pm}{}^{JM}$ take on the values $J=\frac{1}{2},\frac{3}{2},\cdots,(n\pm\frac{1}{2})$ and $M=-J,-J+1,\cdots,J$; we are thus left with $(n+1)\times[(n\pm1)+1]$ basis functions for each of the irreducible spaces $(\frac{1}{2}n\pm\frac{1}{2},\frac{1}{2}n)$ and $(\frac{1}{2}n,\frac{1}{2}n\pm\frac{1}{2})$, in agreement with the dimensionality of these representations as given in Appendix B.

Properties

We list here a number of properties of the functions (4.8), which we need in our later work. To begin with, we note the orthonormality property,

$$\sum_\alpha\int d\Omega_R\, Z_{n\pm}{}^{JM}(\hat{R})_\alpha{}^*Z_{n\pm'}{}^{J'M'}(\hat{R})_\alpha=\delta_{JJ'}\delta_{MM'}\delta_{n\pm,n\pm'},\tag{4.9}$$

and the completeness relation,

$$\sum_{J,M,n\pm}Z_{n\pm}{}^{JM}(\hat{R})_\alpha{}^*Z_{n\pm}{}^{JM}(\hat{R}')_\beta=\delta_{\alpha\beta}\delta(\Omega_{R'}-\Omega_R),\tag{4.10}$$

where $d\Omega_R$ is the differential element of solid angle defined in (A2). The functions $\tilde{Z}_{n\pm}{}^{JM}$ satisfy exactly the same relations. Under space-time reflection

$$\begin{aligned}Z_{n\pm}{}^{JM}(-\hat{R})&=(-)^nZ_{n\pm}{}^{JM}(\hat{R}),\\ \tilde{Z}_{n\pm}{}^{JM}(-\hat{R})&=(-)^n\tilde{Z}_{n\pm}{}^{JM}(\hat{R}).\end{aligned}\tag{4.11}$$

From (4.8) we see that these functions are not all independent. In fact, $Z_{n\pm}{}^{JM}$ and $\tilde{Z}_{n\pm}{}^{JM}$ are related by space inversion

$$\begin{aligned}\tilde{Z}_{n\pm}{}^{JM}(\hat{R})&=(-)^{J-(1/2)}Z_{n\pm}{}^{JM}(\hat{R}_p),\\ \hat{R}_p&=(-\mathbf{r},\tau)/R.\end{aligned}\tag{4.12}$$

By construction we have, of course,

$$\begin{aligned}\mathfrak{A}^2Z_{n\pm}{}^{JM}&=\tfrac{1}{2}(n\pm1)[\tfrac{1}{2}(n\pm1)+1]Z_{n\pm}{}^{JM},\\ \mathfrak{B}^2Z_{n\pm}{}^{JM}&=\tfrac{1}{2}n(\tfrac{1}{2}n+1)Z_{n\pm}{}^{JM},\end{aligned}\tag{4.13}$$

the notation being the same as in Appendix B. The corresponding formulas for the functions $\tilde{Z}_{n\pm}{}^{JM}$ are obtained from (4.13) by the substitutions $\mathfrak{A}_i\rightarrow\mathfrak{B}_i$ and $\mathfrak{B}_i\rightarrow\mathfrak{A}_i$. When the quantum numbers J and M are of no relevance (as in the case of O_4 symmetry, for example), it is convenient to define the functions

$$X^{n\pm}(\hat{R}',\hat{R})_{\alpha\beta}=4\pi^2\sum_{JM}Z_{n\pm}{}^{JM}(\hat{R}')_\alpha Z_{n\pm}{}^{JM}(\hat{R})_\beta{}^*,\tag{4.14a}$$

$$\tilde{X}^{n\pm}(\hat{R}',\hat{R})_{\alpha\beta}=4\pi^2\sum_{JM}\tilde{Z}_{n\pm}{}^{JM}(\hat{R}')_\alpha\tilde{Z}_{n\pm}{}^{JM}(\hat{R})_\beta{}^*.\tag{4.14b}$$

Since $X^{n\pm}(\hat{R}',\hat{R})$ and $\tilde{X}^{n\pm}(\hat{R}',R)$ share a number of properties, it is useful to introduce $W^{n\pm}(\hat{R}',\hat{R})$ to represent either one of them. We have then evidently,

$$W^{n\pm}(\hat{R}',\hat{R})^\dagger=W^{n\pm}(\hat{R},\hat{R}').$$

From (4.9) we obtain the orthonormality relations

$$\int\frac{d\Omega_{R''}}{4\pi^2}W^{n\pm'}(\hat{R}'',\hat{R}')^\dagger W^{n\pm}(\hat{R}'',\hat{R})=\delta_{n\pm,n\pm'}W^{n\pm}(\hat{R}',\hat{R}),\tag{4.15}$$

and the addition theorem takes the form

$$\sum_\pm W^{n\pm}(\hat{R}',\hat{R})=2(n+1)C_n{}^1(\cos\gamma),\tag{4.16}$$

where $\cos\gamma$ is defined in (A10).

In the remaining part of this section we give a list of formulas which will be particularly useful in our later work. Thus, introducing the notation

$$\begin{aligned}&(\sigma_\alpha)=(\boldsymbol{\sigma},\sigma_4),\quad(\tilde{\sigma}_\alpha)=(\boldsymbol{\sigma},-\sigma_4);\quad\sigma_4=i,\\ &\sigma\cdot x=\boldsymbol{\sigma}\cdot\mathbf{r}+i\tau,\quad\sigma\cdot\partial=\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}+i\partial_4,\quad\text{etc.},\end{aligned}\tag{4.17}$$

and making use of the results obtained in Sec. IV A, we readily derive the following formulas [we suppress the quantum numbers J and M, which here play no particular role; our notation is $\hat{x}=(x_\alpha/R)$]:

$$\begin{aligned}-i(\sigma\cdot\hat{x})Z_{n\pm}&=\tilde{Z}_{(n\pm1)\mp},\\ i(\tilde{\sigma}\cdot\hat{x})\tilde{Z}_{n\pm}&=Z_{(n\pm1)\mp},\\ -i(\sigma\cdot\partial)Z_{n+}&=\tilde{Z}_{(n+1)-}(d/dR-n/R),\\ -i(\sigma\cdot\partial)Z_{n-}&=\tilde{Z}_{(n-1)+}[d/dR+(n+2)/R],\\ i(\tilde{\sigma}\cdot\partial)\tilde{Z}_{n+}&=Z_{(n+1)-}(d/dR-n/R),\\ i(\tilde{\sigma}\cdot\partial)\tilde{Z}_{n-}&=Z_{(n-1)+}[d/dR+(n+2)/R].\end{aligned}\tag{4.18}$$

Combining the above results we obtain the additional formulas:

$$\begin{aligned}(\tilde{\sigma}\cdot\partial)(\sigma\cdot\hat{x})Z_{n+}&=Z_{n+}[d/dR+(n+3)/R],\\(\tilde{\sigma}\cdot\partial)(\sigma\cdot\hat{x})Z_{n-}&=Z_{n-}[d/dR-(n-1)/R],\\(\tilde{\sigma}\cdot\hat{x})(\sigma\cdot\partial)Z_{n+}&=Z_{n+}(d/dR-n/R),\\(\tilde{\sigma}\cdot\hat{x})(\sigma\cdot\partial)Z_{n-}&=Z_{n-}[d/dR+(n+2)/R],\end{aligned}\tag{4.19}$$

the corresponding formulas for the functions $\tilde{Z}_{n\pm}{}^{JM}$ being obtained by the simple replacement $\sigma_\alpha\rightarrow\tilde{\sigma}_\alpha$ and $Z_{n\pm}{}^{JM}\rightarrow\tilde{Z}_{n\pm}{}^{JM}$ in (4.19). We may write down immediately the corresponding results for the spinor functions (4.14). We will not do so except for noting, in particular, the formula

$$(\sigma\cdot\hat{x}')X^{n\pm}(\hat{R}',\hat{R})(\tilde{\sigma}\cdot\hat{x})=\tilde{X}^{(n\pm1)\mp}(\hat{R}',\hat{R}),\tag{4.20}$$

which we need later on.

V. BOUNDARY CONDITIONS AT THE ORIGIN

In this section we deduce the behavior of the solutions to (3.1) near the origin of the four-dimensional Euclidean space. The results will include the so-called "regular" and "irregular" solutions. No attempt is made in this section to justify the selection of one or the other. Aside from physical considerations which we do not go into in this paper, the criterion for such a selection must come from a study of the integral equation (3.9) itself. This is left for Sec. VI, where we find, in fact, that even the integral equation (3.9) does not necessarily exclude the irregular solutions.

The study of the behavior of the solutions to (3.1) at $R=0$ is rather involved, if we consider the general case in which $E\neq0$. A great deal of insight into this question may be obtained, however, by considering first the problem at zero total c.m. energy E, where the equation simplifies considerably. If the behavior of the solutions at $R=0$ should turn out to be independent of E, as one might expect on the basis of some familiar examples, it would be sufficient to consider this special case, although the generalization of the results to the case in which $E\neq0$ is not immediate, as we shall see.

A. $E=0$

For $E=0$, Eq. (3.1) is invariant with respect to the transformations of the group O_4. In order to take full advantage of this symmetry it is convenient to work in the representation in which

$$\gamma_\alpha=\begin{pmatrix}0 & -\tilde{\sigma}_\alpha\\ \sigma_\alpha & 0\end{pmatrix},\quad \gamma_5=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

the matrices σ_α and $\tilde{\sigma}_\alpha$ having been defined already in (4.17). We call this the "Weyl representation"; γ_α and γ_5 are matrices in the direct sum space of the representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ of O_4. The reason for working in this direct sum space is of course the required invariance of the theory under space inversion. Indeed, as (4.12) demonstrates explicitly, the operation of space inversion takes the representation $(\frac{1}{2}n\pm\frac{1}{2},\frac{1}{2}n)$ into $(\frac{1}{2}n,\frac{1}{2}n\pm\frac{1}{2})$, and vice versa. Denoting by $D^{1/2}(R)$ the representation matrices in the direct sum space $[(\frac{1}{2},0)\oplus(0,\frac{1}{2})]$, we have the following transformation law for the γ matrices:

$$D^{1/2}(R)\gamma_\alpha D^{1/2}(R^{-1})=R^{-1}{}_{\alpha\beta}\gamma_\beta.$$

It follows that for $E=0$, Eq. (3.1) is invariant under four-dimensional rotations, so that its solutions may be labeled by the eigenvalues of the complete set of commuting generators of O_4. We write the $E=0$ equation in the form

$$\mathfrak{D}_0(\partial;\lambda)\varphi(x)=0,\tag{5.1}$$

where

$$\mathfrak{D}_E(\partial;\lambda)=[m_1-i\gamma\cdot\partial-\tfrac{1}{2}\gamma^0E][\Box]^2+E\partial_4+\tfrac{1}{4}E^2-m_2{}^2]-\lambda V(x)\mathcal{P}(x).\tag{5.2}$$

Here $V(x)$ is the "potential" (3.2) and $\mathcal{P}(x)$ is the operator defined by

$$\mathcal{P}(x):(\mathbf{r},\tau)\rightarrow(-\mathbf{r},-\tau).$$

Introducing

$$\mathcal{P}(\mathbf{r}):(\mathbf{r},\tau)\rightarrow(-\mathbf{r},\tau),\tag{5.3}$$

we define the "parity" operators in three- and four-space:

$$\Pi_3=-i\gamma_4\mathcal{P}(\mathbf{r}),\quad \Pi_4=\gamma_5\mathcal{P}(x).\tag{5.4}$$

We note the commutation relations

$$[\Pi_{3,4},\mathfrak{D}_0(\partial;\lambda)]=0,\quad [\Pi_3,\Pi_4]\neq0.\tag{5.5}$$

Because of these commutation relations we may choose the solutions of Eq. (5.1) to be also eigenfunctions of Π_3 or Π_4, but not of both. In particular, the eigenfunctions of Π_4 will have to be of the form[20]

$$\varphi_{n\pm}{}^{JM}(x)=\begin{pmatrix}F_{n\pm}(R)Z_{n\pm}{}^{JM}\\ G_{n\pm}(R)\tilde{Z}_{(n\pm1)\mp}{}^{JM}\end{pmatrix},\tag{5.6}$$

with space-time parity $(-)^n$. Substituting (5.6) into (5.1) and using the results of Sec. IV, we arrive at the following set of coupled differential equations for the radial functions $F_{n+}(R)$ and $G_{n+}(R)$:

$$\begin{aligned}m_1K^nF_{n+}(R)+[d/dR+(n+3)/R]K^{n+1}G_{n+}(R)&=(-)^n\lambda(4M^2/R)[K_1(MR)F_{n+}(R)+K_2(MR)G_{n+}(R)],\\m_1K^{n+1}G_{n+}(R)+(d/dR-n/R)K^nF_{n+}(R)&=(-)^{n+1}\lambda(4M^2/R)[K_1(MR)G_{n+}(R)+K_2(MR)F_{n+}(R)],\end{aligned}\tag{5.7}$$

where

$$K^n=d^2/dR^2+(3/R)(d/dR)-n(n+2)/R^2-m_2{}^2.\tag{5.8}$$

Equations (5.7) exhibit explicitly the degeneracy of

[20] We do not label the radial functions by the quantum numbers J and M, since they are independent of them.

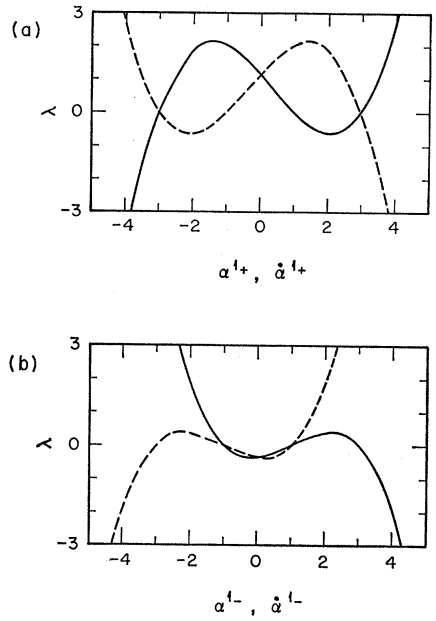

FIG. 4 (a,b) A plot of the solutions $\alpha^{1\pm}$ (solid curve) and $\dot{\alpha}^{1\pm}$ (dashed curve) to the indicial equations (5.11) as a function of the coupling constant λ.

the solutions with respect to the quantum numbers J and M. In exactly the same manner we arrive at a set of coupled differential equations for $F_{n-}(R)$ and $G_{n-}(R)$. It is, however, simpler to note that if $\varphi_{n+}{}^{JM}(x)$ is a solution of (5.1) with eigenvalue λ, then so is $\varphi_{(n+1)-}{}^{JM} \propto \Pi_3\varphi_{n+}{}^{JM}$, on account of the commutation relations (5.5). That is, we may obtain the differential equations for $F_{n-}(R)$ and $G_{n-}(R)$ by simply noting that (we make a convenient choice of the proportionality constant)

$$F_{(n\pm1)\mp}=G_{n\pm}. \tag{5.9}$$

We conclude therefore that for a given value of n, the combined set of solutions $\varphi_{n+}{}^{JM}$ and $\varphi_{(n+1)-}{}^{JM}$ [with J and M taking on the values $J=\frac{1}{2}, \frac{3}{2}, \cdots, (n+\frac{1}{2})$, $M=-J, -J+1, \cdots, J$] is $2(n+1)(n+2)$-fold degenerate. For eigenfunctions of Π_3 the additional twofold degeneracy expressed by (5.9) would correspond to a degeneracy with respect to the two values of the orbital angular momentum coupling to a given total angular momentum J.

We now turn our attention to a study of the behavior of $F_{n\pm}(R)$ and $G_{n\pm}(R)$ as $R\to 0$. Although the potential (3.2) has a logarithmic singularity at $R=0$, we have for the leading contribution, $V(x)\to 8(\gamma\cdot\hat{x})R^{-3}$ as $R\to 0$. Hence taking this limit in Eqs. (5.7) we arrive at two coupled differential equations of the Fuchsian type, with $R=0$ a regular, singular point. We conclude therefore that the six independent solutions (for a given value of λ, J, M, and $n\pm$) to Eq. (5.7) exhibit the behavior

$$F_{n\pm}{}^{i}(R)\sim R^{\alpha_i{}^{n\pm}},\quad G_{n\pm}{}^{i}(R)\sim R^{\beta_i{}^{n\pm}} \quad (\text{with } i=1,\,6), \tag{5.10}$$

as $R\to 0$, provided that $(\alpha_i-\alpha_j)$ and $(\beta_i-\beta_j)$ are not integers for $i\neq j$. We must have, however, $\alpha_i-\beta_i=m$ (integer). In fact, a more detailed examination of the equations shows that m can have only the values $m=\pm1$. The α_i's and β_i's are then found to satisfy the indicial equations

$$\begin{aligned}&[\alpha^{n+}-n][\alpha^{n+}-(n+2)][\alpha^{n+}+(n+2)]=(-)^{n+1}8\lambda,\\ &\qquad\qquad \beta^{n+}=\alpha^{n+}+1,\\ &[\dot{\alpha}^{n+}+n][\dot{\alpha}^{n+}-(n+2)][\dot{\alpha}^{n+}+(n+2)]=(-)^{n}8\lambda,\\ &\qquad\qquad \dot{\beta}^{n+}=\dot{\alpha}^{n+}-1,\end{aligned} \tag{5.11a}$$

and

$$\begin{aligned}&[\alpha^{n-}-n][\alpha^{n-}+n][\alpha^{n-}-(n+2)]=(-)^{n}8\lambda,\\ &\qquad\qquad \beta^{n-}=\alpha^{n-}-1,\\ &[\dot{\alpha}^{n-}+n][\dot{\alpha}^{n-}-n][\dot{\alpha}^{n-}+(n+2)]=(-)^{n+1}8\lambda,\\ &\qquad\qquad \dot{\beta}^{n-}=\dot{\alpha}^{n-}+1.\end{aligned} \tag{5.11b}$$

We note the following useful relations:

$$\begin{aligned}&\alpha^{n\pm}=\alpha^{(n\pm1)\mp}\mp1,\\ &\dot{\alpha}^{n\pm}=\dot{\alpha}^{(n\pm1)\mp}\pm1.\end{aligned} \tag{5.12}$$

For the purpose of later discussions it is also convenient to distinguish among the six independent solutions to Eqs. (5.11) by a subscript m, where m denotes the integral value which the solution in question approaches as $\lambda\to 0$. Thus

$$\alpha_m{}^{n\pm}\to m \quad \text{as} \quad \lambda\to 0. \tag{5.13}$$

Note that $\dot{\alpha}_{-m}{}^{n\pm}=-\alpha_m{}^{n\pm}$; finally we note that except for some isolated values of λ, including $\lambda=0$, the solutions to Eqs. (5.11) do indeed satisfy the above stated conditions for the validity of the ansatz (5.10). This can also be seen from Fig. 4 for the special case $n=1$, where we present a plot of $\alpha^{1\pm}$ and $\dot{\alpha}^{1\pm}$ as a function of the coupling strength λ.

B. $E\neq 0$

We now turn our attention to the actual case of interest, that is, when $E\neq 0$. In this case it is still desirable to expand the solution in terms of the four-dimensional spinor spherical harmonics, although this leads now to an *infinite* set of coupled third-order ordinary differential equations. The situation is therefore much more complicated and the discussion of the boundary conditions at the origin is correspondingly more involved.

We begin by restating the results of Sec. V, A in a somewhat different form, the reason for this being twofold:

(a) In the $E\neq 0$ case it turns out to be more convenient to work in the "Dirac representation" related to the "Weyl representation" by a unitary transforma-

tion. In this representation

$$\gamma^0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \boldsymbol{\gamma}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix},\quad \gamma_5=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

(b) If $E\neq 0$ the space-time parity is no longer a good quantum number, whereas Π_3 still commutes with the operator $\mathfrak{D}_E(\partial;\lambda)$. It is therefore desirable to restate our earlier conclusions in terms of the simultaneous eigenfunctions of $\mathfrak{D}_E(\partial;\lambda)$ and Π_3.

We may obtain the $E=0$ eigenfunctions of Π_3 by simply applying the projection operator $P_\pm=\frac{1}{2}(1\pm\Pi_3)$ (three-parity ± 1) to the $E=0$ solutions (5.6). Doing so, and going to the Dirac representation, we arrive, making use of the results (4.8), at the following expressions for the $E=0$ solutions[18]:

$$\chi_n{}^{l+}(x)=\begin{pmatrix}\left[\frac{2(n+l+2)}{n+1}\right]^{1/2}F(R)Y_n{}^{l+}+\left[\frac{2(n-l+1)}{n+2}\right]^{1/2}G(R)Y_{n+1}{}^{l+}\\ \left[\frac{2(n-l)}{n+1}\right]^{1/2}F(R)Y_n{}^{(l+1)-}-\left[\frac{2(n+l+3)}{n+2}\right]^{1/2}G(R)Y_{n+1}{}^{(l+1)-}\end{pmatrix} \tag{5.14a}$$

with three-parity $(-)^l$, and

$$\chi_n{}^{(l+1)-}(x)=(-)^n\Pi_4\chi_n{}^{l+}(x) \tag{5.14b}$$

with three-parity $(-)^{l+1}$. Here $F(R)\equiv F_{n+}(R)$ and $G(R)\equiv G_{n+}(R)$, so that

$$\begin{aligned}&F(R)\sim R^{\alpha^{n+}},\quad G(R)\sim R^{\alpha^{n+}+1},\\ &F(R)\sim R^{\dot\alpha^{n+}},\quad G(R)\sim R^{\dot\alpha^{n+}-1}\end{aligned} \tag{5.15}$$

as $R\to 0$, where α^{n+} and $\dot\alpha^{n+}$ are the solutions to the indicial equations (5.11a). The result (5.14b) together with the commutation relations (5.5) shows that $\chi_n{}^{l+}$ and $\chi_n{}^{(l+1)-}$ are solutions to Eq. (5.1) for the same eigenvalue λ. This corresponds to the twofold degeneracy already noted in connection with (5.9). Hence, with the additional degeneracy in l and the magnetic quantum number M, we have again a set of $2(n+1)(n+2)$ degenerate $E=0$ solutions.

If $E\neq 0$, the partial differential equation (3.1) is no longer separable and we obtain now an infinite set of coupled ordinary differential equations for the radial functions in the expansion of the $E\neq 0$ solution in terms of the spherical harmonics (A3). This infinite set of equations no longer decouples (for $E=0$ this set did decouple, as is evident from the form of the $E=0$ solutions[21]) and the study of the boundary conditions at the origin is correspondingly more involved. We will show, however, that if we develop the solution as a power series in E, then,[22] to first order in E, we are left with only a finite set of coupled equations for the radial functions. Indeed, consider the solutions $\phi_k(x)$ to Eq. (3.6). For E below the elastic threshold and at a given value of λ, $\phi_k(x)$ is expected to have a discrete set of poles in E, corresponding to the eigenvalues E_m of Eq. (3.9). Considering in particular one such pole at $E=E_0$, we write

$$\phi_k(x)=\tilde\phi_E(x)/(E-E_0). \tag{5.16}$$

The bound-state wave function $\varphi(x)$ corresponding to the eigenvalue $E=E_0$ was defined as $\tilde\phi_E(x)$ evaluated at the pole; assuming that $\tilde\phi_E(x)$ can be developed into a power series in E with radius of convergence $r_0\geq|E_0|$, and noting that in the limit $E\to 0$ we must recover the results (5.14), we conclude that to first order in E_0 (we omit the magnetic quantum number M)

$$\varphi_n{}^{l\pm}(x)\approx\chi_n{}^{l\pm}(x)+E_0\psi_n{}^{l\pm}(x), \tag{5.17}$$

where $\chi_n{}^{l\pm}(x)$ are the $E=0$ solutions (5.14) and $\psi_n{}^{l\pm}(x)$ is independent of E_0. The general form of $\psi_n{}^{l\pm}$ will be determined below. The $E\neq 0$ solutions are of course no longer degenerate with respect to J and l. Note, however, that both $\varphi_n{}^{l\pm}$ and $\chi_n{}^{l\pm}$ are labeled by the same quantum numbers, implying a one-to-one correspondence between the $E=0$ and $E\neq 0$ solutions.

In order to determine the form of $\psi_n{}^{l\pm}(x)$, we substitute (5.17) into (3.1), keeping only the first-order terms in E_0. We arrive then at the following set of coupled equations for $\chi_n{}^{l\pm}$ and $\psi_n{}^{l\pm}$:

$$\mathfrak{D}_0(\partial;\lambda)\chi_n{}^{l\pm}(x)=0, \tag{5.18a}$$

$$\begin{aligned}\mathfrak{D}_0(\partial;\lambda)\psi_n{}^{l\pm}(x)=&(\tfrac{1}{2}\gamma^0[\Box^2-m_2{}^2]\\ &-[m_1-i\boldsymbol{\gamma}\cdot\boldsymbol{\partial}]\partial_4)\chi_n{}^{l\pm}(x).\end{aligned} \tag{5.18b}$$

Here (5.18a) is of course just the $E=0$ equation; (5.18b) is of the form of an "inhomogeneous" $E=0$ equation, the inhomogeneous term being already known from the solution of Eq. (5.18a). This suggests that we seek a solution of such form as to lead to only a finite set of coupled differential equations. An examination of the inhomogeneous term in (5.18b) shows that $\psi_n{}^{l\pm}(x)$ must

[21] For more details see Klaus D. Rothe (Ph.D. thesis), University of California Radiation Laboratory Report No. UCRL-17671, 1967 (unpublished).

[22] I should like to thank Dr. L. Schlessinger for suggesting this approach to the problem.

in fact be of the specific form[18]

$$\psi_n{}^{l+}(x)=\begin{pmatrix}\left[\frac{2(n+l+1)}{n}\right]^{1/2}F_1(R)Y_{n-1}{}^{l+}+G_1(R)Y_n{}^{l+}\\ \left[\frac{2(n-l-1)}{n}\right]^{1/2}F_1(R)Y_{n-1}{}^{(l+1)-}+\tilde{G}_1(R)Y_n{}^{(l+1)-}\end{pmatrix}$$
$$+\begin{pmatrix}F_2(R)Y_{n+1}{}^{l+}+\left[\frac{2(n-l+2)}{n+3}\right]^{1/2}G_2(R)Y_{n+2}{}^{l+}\\ \tilde{F}_2(R)Y_{n+1}{}^{(l+1)-}+\left[\frac{2(n+l+4)}{n+3}\right]^{1/2}G_2(R)Y_{n+2}{}^{(l+1)-}\end{pmatrix}; \quad (5.19)$$

the *formal* structure of $\psi_n{}^{(l+1)-}(x)$ being the same as that of $\Pi_4\psi_n{}^{l+}(x)$. The coefficients of Y_{n-1} and Y_{n+2} have been chosen so that when the operator $\mathfrak{D}_0(\partial;\lambda)$ is applied to $\psi_n{}^{l\pm}(x)$ in (5.18b), the coefficients of Y_{n-2} and Y_{n+3} in the resulting expressions vanish identically. Thus we find that the substitution of (5.19) into (5.18b) leads to a set of *eight* coupled equations for the *six* radial functions in (5.19), only six of the eight equations being, in fact, independent.

Although we have demonstrated it only to first order in E_0, our results suggest that, subject to the assumed analyticity properties of $\tilde{\phi}_E(x)$, we may obtain the $E\neq 0$ solution $\varphi_n{}^{l\pm}(x)$ perturbatively by starting from the $E=0$ solution $\chi_n{}^{l\pm}(x)$, with only a *finite* number of terms contributing to the partial-wave expansion of $\varphi_n{}^{l\pm}(x)$ to a given order in E_0. As for the radius of convergence r_0 of the expansion of $\tilde{\phi}_E(x)$ in powers of E, we note that certainly $r_0<E_1$, where E_1 is the first excited state in the bound-state spectrum of Eq. (3.9).

Having determined the structure of the $E\neq 0$ solutions to first order in E, we examine now the behavior of the radial functions in (5.19) at $R=0$. This is a straightforward, although tedious, task, since we are dealing now with a finite set of coupled equations. Introducing the expansion[18]

$$\varphi_n{}^{l\pm}(x)=\sum_{k=l}^{\infty}\begin{pmatrix}F_{nk}{}^{l\pm}(R)Y_k{}^{l\pm}\\ G_{nk}{}^{l\pm}(R)\,Y_{k\pm1}{}^{(l\pm1)\mp}\end{pmatrix}, \quad (5.20)$$

where we have followed the notation (5.17) (the radial functions are of course independent of the magnetic quantum number M), we may summarize our findings by the statements

$$\begin{aligned}&F_{nk}{}^{l\pm}(R)\sim R^{\alpha_m{}^{n+}+|k-n|},\\ &G_{nk}{}^{l\pm}(R)\sim R^{\alpha_m{}^{n+}+|(k\pm1)-n|},\end{aligned} \quad (5.21a)$$

$$\begin{aligned}&F_{nk}{}^{l\pm}(R)\sim R^{\dot{\alpha}_m{}^{n+}-1+|k-(n+1)|},\\ &G_{nk}{}^{l\pm}(R)\sim R^{\dot{\alpha}_m{}^{n+}-1+|(k\pm1)-(n+1)|},\end{aligned} \quad (5.21b)$$

as $R\to 0$, where $\alpha_m{}^{n+}$ and $\dot{\alpha}_m{}^{n+}$ are the six solutions to the indicial equations (5.11a). Although we have shown only the first-order solutions (5.17) to exhibit the behavior (5.21) at $R=0$, we note that these boundary conditions are independent of J and M, and are in fact determined by the $E=0$ solution itself. We therefore assume the results (5.21) to be correct independent of the value of E.

VI. INTEGRAL EQUATION AND BOUNDARY CONDITIONS AT ORIGIN

In order to complete our discussion of boundary conditions, we need to examine which of the boundary conditions (5.21) are actually contained in the integral equation (3.9).

Our objective in this section is actually twofold: In addition to establishing which of the six independent solutions (for given values of J, l, M, and n) to Eq. (3.1) are also prospective solutions of the integral equation (3.9), it is also of some interest to learn how the boundary conditions (5.21) may be obtained directly from the integral equation itself. We restrict our discussion to the case in which $E=0$, since, as we have seen in Sec. V, this is entirely sufficient for our purpose. In this case the expression (3.7) for the "Wick-rotated" Green's function simplifies considerably, the result of taking the limit $E\to 0$ being

$$H_0(x)=\frac{1}{8\pi^2}\int_0^1 d\beta\, K_0[(m_1{}^2+\beta(m_1{}^2-m_2{}^2))^{1/2}R]. \quad (6.1)$$

For simplicity we take $m_1=m_2=m$. In that case we obtain, upon substituting (6.1) into (3.7),

$$H(x)=(m/8\pi^2)[K_0(m|x|)-i\gamma\cdot\hat{x}K_1(m|x|)]. \quad (6.2)$$

As in Sec. V A it is convenient to work in the "Weyl representation." The $E=0$ solutions are then of the form (5.6). In order to carry out the partial-wave analysis of Eq. (3.9) we make use of the

expansions

$$\delta_{\alpha\beta}K_0(m|x-x'|)=\sum_{n\pm}\frac{1}{4(n+1)}[\epsilon_n I_n(mR_<)K_n(mR_>) -2I_{n+2}(mR_<)K_{n+2}(mR_>)]W^{n\pm}(\hat{R},\hat{R}')_{\alpha\beta},$$
$$\delta_{\alpha\beta}\frac{K_1(m|x-x'|)}{|x-x'|}=\sum_{n\pm}\frac{I_{n+1}(mR_<)K_{n+1}(mR_>)}{mRR'}\times W^{n\pm}(\hat{R},\hat{R}')_{\alpha\beta}, \tag{6.3}$$

as obtained with the aid of the addition theorem (4.16). Here $R=(\mathbf{r}^2+\tau^2)^{1/2}$, $R_>=\max\{R,R'\}$, etc., and $e_n=2$ for all positive integers n except for $n=0$, where $e_0=1$; $I_n(x)$ and $K_n(x)$ are modified Bessel functions with the properties

$$I_n(x)\sim\Gamma(n+1)(\tfrac{1}{2}x)^n,$$
$$K_n(x)\sim\tfrac{1}{2}\Gamma(n)(\tfrac{1}{2}x)^{-n},$$
$$K_0(x)\sim-\ln x, \quad (n\neq 0) \quad \text{as} \quad x\to 0. \tag{6.4}$$

Making use of the expansions (6.3), and of (4.20), we obtain the following set of coupled integral equations for the radial functions $F_{n\pm}(R)$ and $G_{n\pm}(R)$ in (5.6):

$$F_{n\pm}(R)=2M^2\lambda(-)^{n+1}\int R'^2dR'\{[m/4(n+1)][\epsilon_n I_n(mR_<)K_n(mR_>)-2I_{n+2}(mR_<)K_{n+2}(mR_>)] \times[K_1(MR')F_{n\pm}(R')+K_2(MR')G_{n\pm}(R')]-[(1/R')I_{\bar{n}+1}(mR_<)K_{\bar{n}+1}(mR_>) -(1/R)I_{n+1}(mR_<)K_{n+1}(mR_>)][K_1(MR')G_{n\pm}(R')+K_2(MR')F_{n\pm}(R')]\};$$
$$G_{n\pm}(R)=2M^2\lambda(-)^{n+1}\int R'^2dR'\{[(1/R')I_{n+1}(mR_<)K_{n+1}(mR_>)-(1/R)I_{\bar{n}+1}(mR_<)K_{\bar{n}+1}(mR_>)] \times[K_1(MR')F_{n\pm}(R')+K_2(MR')G_{n\pm}(R')]-[m/4(\bar{n}+1)][\epsilon_n I_{\bar{n}}(mR_<)K_{\bar{n}}(mR_>) -2I_{\bar{n}+2}(mR_<)K_{\bar{n}+2}(mR_>)][K_1(MR')G_{n\pm}(R')+K_2(MR')F_{n\pm}(R')]\}, \tag{6.5}$$

where $\bar{n}=(n\pm1)$. In order to deduce the behavior of the radical functions at $R\approx0$ we again make the ansatz that in this limit $F_{n\pm}(R)\sim R^\alpha$ and $G_{n\pm}(R)\sim R^\beta$. In the limit $R\to0$ the dominant contribution to the integration in (6.5) comes from the neighborhood of $R'=R$, so that the leading term may be obtained by simply substituting for $F_{n\pm}(R)$ and $G_{n\pm}(R)$ their values at $R\approx0$. We arrive then at the following indicial equations for α and β:

$$4\lambda(-)^{n+1}\{[(\alpha+\bar{n}+1)(\alpha-\bar{n}-1)]^{-1} -[(\alpha+n+2)(\alpha-n)]^{-1}\}=1 \tag{6.6a}$$

with $\beta=\alpha+1$, and

$$4\lambda(-)^{n+1}\{[(\beta+\bar{n}+2)(\beta-\bar{n})]^{-1} -[(\beta+n+1)(\beta-n-1)]^{-1}\}=1 \tag{6.6b}$$

with $\alpha=\beta+1$. Comparing Eqs. (6.6) with the indicial equations (5.11) obtained earlier, we see that they are in fact identical. This, then, completes our first objective; that is, to learn how the boundary conditions (5.21) may be deduced directly from the integral equation itself. It remains now to be seen for which of the six solutions (5.21) the integral equation (6.5) remains well defined. An examination of the equations leading to (6.6) shows that we need to observe the inequalities

$$\alpha^{n+}>-(n+2), \quad \dot{\alpha}^{n+}>-n,$$
$$\alpha^{n-}>-n, \quad \dot{\alpha}^{n-}>-n \tag{6.7}$$

if the integration in (6.5) is not to diverge at $R'=0$. [Compare (6.7) with the $\lambda=0$ solutions to the indicial equations.]

At this point it is convenient to define what we will refer to as the regular and irregular solutions. We separate the solutions to the indicial equations (5.11) into two groups, depending on whether they take on positive (including zero) or negative values for $\lambda=0$. We will refer to these as the regular and irregular solutions, respectively. Thus, to take an example, the solutions $\alpha_n{}^{n+}$, $\alpha_{n+2}{}^{n+}$, and $\dot{\alpha}_{n+2}{}^{n+}$ to Eqs. (5.11a) belong to the first class, whereas $\alpha_{-(n+2)}{}^{n+}$, $\dot{\alpha}_{-(n+2)}{}^{n+}$, and $\dot{\alpha}_{-n}{}^{n+}$ belong to the second class. We make the corresponding classification of the six independent solutions (for given values of J, M, l, and n) to Eq. (3.1). Consider then a typical case such as represented in Fig. 4, where we have plotted $\alpha^{1\pm}$ and $\dot{\alpha}^{1\pm}$ as a function of the coupling constant λ. The intersection of a given line $\lambda=\lambda_1$ with the curves gives the six (possibly pairwise complex) solutions to the indicial equations. We see that the restrictions (6.7) do not necessarily exclude the irregular solutions. Thus if we take the $1+$ solutions in Fig. 4(a) as an example, the conditions (6.7) are seen to exclude only one of the irregular solutions for $\lambda>0$, whereas they exclude all three for $\lambda\leqslant0$. By merely counting the number of free parameters, we would therefore expect Eq. (3.9) to have a continuous eigenvalue spectrum for $\lambda>0$, unless we supplement the integral equation with additional boundary conditions. To this end we require the solutions to be regular at $R=0$. This choice of boundary conditions has the virtue that we treat the solutions for positive and negative values of the coupling constant in a like fashion. Another reason for favoring this choice is that for a nonsingular force the integral

equation would include without question only the regular solutions.

Having thus completed our study of the boundary conditions to be imposed on the solutions to Eq. (3.1), we should like to comment again briefly on the MacDowell symmetry (see Sec. III) in relation to the eigenvalue problem. Labeling the solutions to Eq. (3.1) by the corresponding values of the energy E, we may now restate our earlier conclusions as follows: If $\varphi_n{}^{l\pm}(x;E_0)$ is a solution of Eq. (3.1) for $E=E_0$ and coupling constant λ, then $\varphi_n{}^{(l\pm1)\mp}(x;-E_0)\propto\Pi_4\varphi_n{}^{l\pm}(x;E_0)$ is also a solution for $E=-E_0$, at the same value of λ. In fact both of these solutions are regular, if either one of them is. Combining this with our earlier remarks in Sec. III concerning their asymptotic behavior, we conclude that if $\varphi_n{}^{l\pm}(x;E_B)$ solves the *eigenvalue* problem for $E=E_B$, then $\varphi_n{}^{(l\pm1)\mp}(x;-E_B)$ as defined above, solves the eigenvalue problem for $E=-E_B$, at the same value of the coupling constant.[23] That is,

$$E_B(\lambda;l_\pm)=-E_B(\lambda;(l\pm1)\mp)\,, \tag{6.8}$$

which is a more familiar way of stating the MacDowell symmetry.

VII. NUMERICAL TECHNIQUES

A. General Treatment

In the preceding sections we have prepared the ground for doing the calculation we have in mind; that is, to solve Eq. (3.1) subject to the conditions that $\varphi(x)$ exhibit the asymptotic behavior of Fig. 3 and that it be regular at $R=0$. Writing Eq. (3.1) in the form

$$D_E(\partial)\varphi(x)=\lambda V(x)\varphi(-x)\,, \tag{7.1}$$

we easily verify that the quotient

$$[\lambda]=\frac{\int d^4x\,\phi^\dagger(-x)D_E(\partial)\phi(x)}{\int d^4x\,\phi^\dagger(x)V(-x)\phi(x)} \tag{7.2}$$

is stationary with respect to infinitesimal variations in $\phi(x)$ about the solution to (7.1). The integrals in (7.2) converge at infinity for E below the two-body elastic threshold, i.e., $E<m_1+m_2$. This can be easily verified with the aid of the results of Sec. III as summarized in Fig. 3. In the scattering region the integral in the numerator ceases to converge, however (the integral in the denominator continues to converge for E below the first inelastic threshold), so that the stationary expression (7.2) can be used only for calculations in the bound-state region. The choice of scalar product was of course dictated by the desired stationary properties. As an extra bonus we notice that the operators $D_E(\partial)$ and $V(x)$ are self-adjoint with respect to the scalar products chosen, so that we will have to deal only with Hermitian (actually real, symmetric) matrices in the calculation. We also note that $\lambda=\lambda^*$ as long as the denominator in (7.2) does not vanish. However, since $V(x)$ is not a positive definite operator, this is not guaranteed, and Eq. (7.1) does in fact have complex as well as real eigenvalues λ.

In Sec. VI we argued on the basis of the integral Eq. (3.9) that the solutions to Eq. (3.1) should be regular at $R=0$. The solutions to the eigenvalue problem are then in general a linear combination of the three independent regular solutions to Eq. (3.1) (for given values of the quantum numbers J, M, l, and n). Since a proper treatment of all three solutions at $R=0$ would have required the inversion of very large matrices, we have incorporated into our calculation only the boundary conditions

$$\begin{aligned}&F_{nk}{}^{l\pm}(R)\sim R^{\alpha_n n^+ + |k-n|}\,,\\ &G_{nk}{}^{l\pm}(R)\sim R^{\alpha_n n^+ + |(k\pm1)-n|}\quad\text{as}\quad R\to0\,,\end{aligned} \tag{7.3}$$

which represent the dominant contribution at $R=0$, at least for a restricted range of coupling constant values.

Now, it follows from our discussion in Sec. V that the specification of the angular momentum and parity of a particular channel of interest is insufficient to single out a unique solution, since we are left with the quantum number n labeling the solutions (5.20) as an additional degree of freedom. Since this quantum number has no direct physical interpretation, it follows that, to take an example, the "nucleon" could in principle be considered as a bound state in any one of the infinite number of $J^P=\frac{1}{2}^+$, $I=\frac{1}{2}$ channels distinguished by the quantum number n. In view of the role of this quantum number in labeling the solutions, we would expect on a merely intuitive basis that higher values of n would correspond to higher excited states. The stability of the nucleon under the strong interactions would then suggest that it should be interpreted as a bound state in the channel labeled by the lowest value of n. We have thus restricted our attention to a study of the solutions $\varphi_l{}^{l+}(x)$ and $\varphi_l{}^{(l+1)-}(x)$ as a logical starting point for our calculation. We will refer to these solutions simply as $\varphi^{l\pm}(x)$, and write

$$\varphi^{l\pm}(x)=\sum_{k=l}^{\infty}\begin{pmatrix}F_k{}^{l\pm}(R)Y_k{}^{l\pm}\\ G_k{}^{l\pm}(R)Y_{k\pm1}{}^{(l\pm1)\mp}\end{pmatrix}. \tag{7.4}$$

Then, according to our choice of boundary conditions (7.3) we will, in the actual numerical calculation, require the radial functions to exhibit the behavior

$$F_k{}^{l\pm}(R)\sim R^{\alpha_l l\pm+|k-l|}\,,\quad G_k{}^{l\pm}(R)\sim R^{\alpha_l l\pm\pm1+|k-l|} \tag{7.5}$$

as $R\to0$, where we have made use of (5.12) in order to put the results into a more symmetric form.

[23] This statement is not quite so empty as it may appear. To give an example, the Schrödinger equation is left unchanged by the substitution $l\to-(l+1)$. Hence $u^{-l-1}(r)$ solves the equation if $u^l(r)$ does. However, if $u^l(r)$ is regular at $r=0$, then $u^{-l-1}(r)$ is not, so that only $u^l(r)$ solves the eigenvalue problem.

Now, with the usual definition of the charge conjugation matrix,

$$C_{\alpha\beta}=(-)^{\alpha+(1/2)}\delta_{\beta,-\alpha}=\exp(-i\pi\sigma_2)_{\alpha\beta}, \quad (7.6)$$

we have

$$C\mathfrak{D}_E(\partial;\lambda)C^{-1}=\mathfrak{D}_E^*(\partial;\lambda^*),$$

where $\mathfrak{D}_E(\partial;\lambda)$ is the operator defined in (5.2). Noting also that

$$CY_n^{l\pm,M}(\hat{R})=(-)^{l\pm(1/2)-M}Y_n^{l\pm,-M}(\hat{R})^*,$$

we conclude that, for the bound-state problem, $F_k^{l\pm}(R)$ and $G_k^{l\pm}(R)$ in (7.4) may be chosen to be real, for λ real. Specifically, we took the radial trial functions to be of the form

$$F_k^{l\pm}(R)=R^{\alpha l^{l\pm}+(k-l)}\sum_{m=0}^{M}a_{km}R^m e^{-\gamma R},$$
$$G_k^{l\pm}(R)=R^{\alpha l^{l\pm}\pm 1+(k-l)}\sum_{m=0}^{M}b_{km}R^m e^{-\gamma R}, \quad (7.7)$$

where $k=l,\ K$. The actual solutions have logarithmic singularities at $R=0$ so that we could very likely have improved on the convergence of our calculation by including explicitly these logarithmic singularities in the expansion (7.7). We have not done so for sake of simplicity. In practice only fairly low values of K and M in (7.7) were considered; thus typically $2\leqslant K\leqslant 5$ and $0\leqslant M\leqslant 8$. The stationary expression (7.2) was then converted to a matrix equation for the expansion coefficients (a_{km},b_{km}) by performing all the integrations. This left us with a set of (real symmetric) matrices $D_E(k'm';km)_{ab}$ and $V(k'm';km)_{ab}$ corresponding to the differential operator and potential integrals, respectively, where m and k have the same meaning as in (7.7) and the indices a, b label the upper and lower components in the expansion (7.4).

In the calculations involving the singular potential (3.2) we were not free to consider arbitrary values of the coupling constant strength λ. Evidently the six solutions to the indicial equations (5.11a) are real for only a restricted range (λ_1,λ_2) of coupling-constant values. (We will refer to $\lambda_{1,2}$ as the critical values of the coupling constant λ.) Outside this range we have two pairs of complex conjugate solutions. In fact, as Fig. 4 illustrates, the regular and irregular solutions do or do not mix at $\lambda=\lambda_{1,2}$, depending on the sign of λ and the particular quantum numbers involved. Still, from a purely mathematical standpoint one would expect that, at least in principle, a well-defined eigenvalue problem could be formulated even for coupling constant strengths exceeding the critical value. However, for $|\lambda|>|\lambda_{1,2}|$ the solutions develop an essential singularity at $R=0$, oscillating infinitely fast in the limit $R=0$. Aside from presenting obvious numerical problems (the existence of an eigenvalue problem evidently depends critically on the choice of phase of the solution at $R=0$), it is very doubtful whether a meaningful physical interpretation can be given to the solution.[24] We have therefore restricted our attention to the coupling-constant range (λ_1,λ_2). In Table I we have listed the values of $\lambda_{1,2}$ corresponding to the solutions $\varphi^{l\pm}(x)$ for $l=0$ through 5.[25]

TABLE I. Values of the critical coupling constants λ_1 and λ_2 corresponding to the solution $\varphi^{l\pm}(x)$ for several values of the angular momentum and parity.

$l_\pm$	λ_1	λ_2
$0_+,1_-$	− 0.384	0.384
$1_+,2_-$	− 0.631	2.112
$2_+,3_-$	− 6.064	0.879
$3_+,4_-$	− 1.128	13.128
$4_+,5_-$	−24.192	1.377

Although the present calculation was originally intended to be free of arbitrary parameters, the range (λ_1,λ_2) of coupling-constant values considered in the singular problem was too restrictive in the $J^P=\frac{1}{2}^+$ channel (see Table I) to give a bound state, so that we needed to modify the nucleon exchange potential (3.2) appropriately if bound-state solutions were to be obtained at all. We considered therefore the two modifications

$$\lambda V(x)=(4M^2/R)[aK_1(MR)+bi\gamma\cdot\hat{x}K_2(MR)], \quad (7.8a)$$
$$V(x)=(4M^2/R)\times[K_1(MR)+i\gamma\cdot\hat{x}f(R,R_c)K_2(MR)]. \quad (7.8b)$$

Here R_c is a cutoff parameter and $f(R;R_c)$ is a cutoff function which was chosen to be of the form

$$f(R;R_c)=R/(R+R_c). \quad (7.9)$$

The bound-state solutions were thus studied as a function of the three masses m_1, m_2, and M, the parameters a and b appearing in (7.8a), and the cutoff parameter R_c. We were of course free to give a any desired value, whereas b was still restricted to the range (λ_1,λ_2). For $a=b$ the potential (7.8a) reduces of course to the form (3.2) corresponding to the exchange of an elementary nucleon. We also note that for the choice of potential (7.8b) the boundary conditions (7.5) need to be replaced by

$$F_k^{l\pm}(R)\sim R^k, \quad G_k^{l\pm}(R)\sim R^{k\pm 1} \quad \text{as} \quad R\to 0. \quad (7.10)$$

Before we proceed to a discussion of the numerical results, there are several "technical" points which seem worth mentioning. In practice it was found that the "optimum" value of the exponential parameter γ in the expansion (7.7) as determined in the course of the calculation was in general far off from the one suggested by the known asymptotic properties of the solution because of the short-range character of the force. The starting

[24] K. M. Case, Phys. Rev. **80**, 797 (1950).

[25] It follows from (5.12) and the property $\alpha_m=-\dot{\alpha}_{-m}$ that it is sufficient to calculate $\lambda_{1,2}$ for the first equation in (5.11a) only.

TABLE II. Computed values of E in the $J_P=\frac{3}{2}^+$, $I=\frac{3}{2}$ channel at several values of the exponential parameter γ and at several matrix sizes for $m_1=m_2=M=1.0$ and the choice of potential (7.8a) with $a=b=1.6$. The form of the trial function is specified following the notation (7.11). The computed eigenvalues are seen to converge to the same extrapolated value $E=1.649$ independent of the value of γ.

Form of trial function	Matrix size	($\gamma=0.5$) E	($\gamma=1.0$) E	($\gamma=1.5$) E
2; 10	6	1.31352	1.65943	1.74158
3; 210	12	1.40361	1.64251	1.67156
4; 3200	18	1.51648	1.65056	1.66522
5; 43100	26	1.58347	1.64919	1.65514
5; 54200	32	1.61660	1.64928	1.65211
5; 65300	38	1.63243	1.64931	1.65060
5; 76410	46	1.64045	1.64938	1.65019
5; 87520	54	1.64440	1.64942	1.64995
Extrapolated value		1.649	1.649	1.649

point of our numerical calculation therefore consisted in searching for a "good" value of the parameter γ by studying the convergence properties of the calculation at small but increasing matrix sizes. We then improved on the accuracy of the calculated eigenvalues by systematically enlarging our space of basis functions; that is, by keeping an increasing number of terms in the expansions (7.4) and (7.7). Since we expect the terms in the expansion (7.4) to become of decreasing importance as we go to increasing values of k, we made the index M in the sum (7.7) a function of k, observing at all times the inequality $M(k)>M(k+1)$. In practice we did not invert matrices larger than 54 by 54. For matrices of this size an accuracy of 1% and less was common for the case in which $m_1=m_2$, although an accuracy of $\approx 10\%$ could already be obtained for much smaller matrix sizes. This is to be compared with other calculations.[3,4] When making this comparison it is to be kept in mind, of course, that the presence of spin alone in our problem doubles the matrix sizes for a given choice of trial function.

Because of the orthogonality of the spherical harmonics it was evidently crucial to determine precisely the total number of terms which needed to be included in the expansion (7.4) if the computed eigenvalues were to be accurate to some specified amount. In practice it was found that we never had to include more than five terms in the expansion (7.4) if an accuracy of about 1% was desired. This suggests that our method of reducing the genuine two-dimensional partial differential equation in the variables R and $\cos\theta$ to an infinite set of coupled ordinary differential equations by expanding the solution in the form (7.4) has definite advantages over solving the partial differential equation directly, since it provides us with an approximation scheme in which we actually need to solve only a small set (in our case at most five) of coupled ordinary differential equations.

Tables II and III illustrate the above observations. In Table II we present a typical sequence of approximations for three different choices of the asymptotic behavior of the trial functions. The first column specifies the form of the trial function, the notation being

$$K;\ M(l),\ M(l+1),\ \cdots,\ M(K), \qquad (7.11)$$

where $M(k)$ is the maximum value of the summation index m in the expansion (7.7) as a function of the four-dimensional angular momentum k labeling the radial functions, and K is the maximum value of k considered in the partial-wave expansion (7.4). The table illustrates how the choice of the exponential parameter in (7.7) affected the convergence rate as well as the direction from which the computed eigenvalues approached the final value. We see that, independent of the choice of γ, the numbers do eventually converge to the same final value.

Table III illustrates our observation that special care had to be taken to include a sufficient number of terms in the expansion (7.4). Table IIIa shows the effect of improving only on the radial dependence; the numbers are seen to converge to the same final value independent of the choice of γ, as one would expect, since we may correct for a poor choice of the exponential parameter by including simply a larger number of terms in the expansion (7.7). Table IIIb shows, on the other hand, that by improving only on the "angular" dependence of the solutions we cannot correct a poor choice of γ, so that the numbers are seen to converge to different values for different choices of γ. However, in all cases the numbers are seen to converge to the wrong value, the correct value being 1.649 as seen from Table II. Table IIIb also shows that it was entirely sufficient for our purpose to include only the first five terms in the expansion (7.4) as we had already pointed out.

We have outlined here the general numerical procedure which was followed in the present calculation. We turn now to a discussion of certain difficulties inherent in our choice of variational principle.

B. Weakly Bound States with $m_2/m_1 \ll 1$

If we want to study the physically interesting case, that is, the bound states of the pions and nucleons with the experimentally measured masses, we run into difficulties with the Rayleigh-Ritz variational principle (7.2). The fact that we are dealing with weakly bound states, such as the "nucleon," for example, and with a very small mass ratio, $m_2/m_1=0.144$, implies a very asymmetric asymptotic behavior of the solution with respect to the forward and backward light cone, so that even in the bound state domain, the solution is actually exponentially rising in one part of the $(\cos\theta,E)$ plane, and decaying in another, although the product $\varphi^\dagger(-x)\varphi(x)$ is always exponentially decaying. These circumstances made it in general rather difficult to obtain accurate solutions for the case in which $m_2/m_1=0.144$, although the variational principle (7.2)

proved quite adequate for mass ratios $0.4 \leqslant m_2/m_1 \leqslant 1$, the lower bound being actually a function of the bound-state energy considered. There have been presented, of course, alternative ways of formulating the variational problem[3] which, in addition to being much more efficient, do overcome to some extent the above difficulties. However, instead of pursuing these alternatives, we will try to "patch up" the variational calculation as presented so far, by transferring some of the undesirable asymptotic properties of the solution to the potential. To this end we consider the transformation

$$\tilde{\varphi}(x) = \varphi(x) \exp(\tfrac{1}{2}\mu R \cos\theta), \qquad (7.12)$$

where μ is some parameter whose value we are free to choose. Thus, to consider an example, if we set $\mu = (m_1 - m_2)$, then $\tilde{\varphi}(x)$ will exhibit the asymptotic behavior $\tilde{\varphi}(x) \sim \exp[\tfrac{1}{2}(E - m_1 - m_2)R]$ along both the positive and negative time direction, replacing the original asymptotic behavior (see Fig. 3)

$$\varphi(x) \sim \exp[(\tfrac{1}{2}E - m_1)R]$$

for $\cos\theta = 1$, and $\varphi(x) \sim \exp[(\tfrac{1}{2}E - m_2)R]$ for $\cos\theta = -1$. We have thus achieved a complete symmetry with respect to the forward and backward time direction in the four-dimensional Euclidean space. Moreover, in the asymptotic domain, $\tilde{\varphi}(x)$ now decays exponentially everywhere in the bound-state region, and is in fact essentially independent of the relative time in the case of very weakly bound-states $[E/(m_1+m_2) \approx 1]$.[26]

From a practical standpoint the transformation (7.12) complicates matters considerably, since we are dealing now with the potential $\tilde{V}(x) = V(x) \exp(\mu\tau)$, which is no longer rotationally invariant. However, we still need to compute only one-dimensional integrals. Thus, in order to evaluate the integral $\int d^4x\, \tilde{\phi}^\dagger(x) \tilde{V}(-x) \tilde{\phi}(x)$ in (7.2) we make use of the expansion

$$e^{-\mu R \cos\theta} = \sum_{k=0}^{\infty} (-)^k (k+1) \frac{I_{k+1}(\mu R)}{\mu R} C_k{}^1(\cos\theta) \qquad (7.13)$$

and of the reduction of the direct product of two four-dimensional spherical harmonics. In practice we needed only the reduction coefficient $C^l(nn'N)$ as given by the integral

$$C^l(nn'N) = N_n{}^l N_{n'}{}^l \int_0^\pi d\theta (\sin^2\theta)^{l+1} \times C_{n-l}{}^{l+1}(\cos\theta) C_{n'-l}{}^{l+1}(\cos\theta) C_N{}^1(\cos\theta). \qquad (7.14)$$

Here $C_n{}^l(\cos\theta)$ are the Gegenbauer polynomials and $N_n{}^l$ is the normalization coefficient defined in (A4). Note that $C^l(nn'0) = \delta_{nn'}$ and $C^0(nn'N) = 1$. With the aid of the orthonormality property

$$\int_{-1}^{1} dx (1-x^2)^{l+1/2} C_{n-l}{}^{l+1}(x) C_{n'-l}{}^{l+1}(x) = \delta_{nn'} (N_n{}^l)^{-2}$$

we deduce that

$$C^l(nn'N) = 0 \quad \text{unless} \quad |n-n'| \leqslant N \leqslant n+n'. \qquad (7.15)$$

It also follows from (7.14) and the property $C_n{}^l(-x) = (-)^n C_n{}^l(x)$ of the Gegenbauer polynomials that

$$C^l(nn'N) = (-)^{n+n'+N} C^l(nn'N), \qquad (7.16)$$

so that

$$C^l(nn'N) = 0 \quad \text{unless} \quad n+n'+N = \text{even integer}. \qquad (7.17)$$

Because of the properties (7.15) and (7.17) of the reduction coefficients the necessary potential integrals were expressable as a *finite* sum of one-dimensional integrals which could be computed accurately by use of Gaussian quadrature techniques. In practice the coefficients $C^l(nn'N)$ were computed numerically by expressing the integral (7.14) as a quadruple sum over products of Γ functions.

The effect of introducing the modification (7.12) is illustrated in Table IV, where we have listed for a sequence of increasing matrix sizes the computed coupling-constant values needed to give a bound state of the actual pion-nucleon system at the mass of the nucleon, $E = 1.0$. The potential was taken to be of the form (7.8b) with $R_c = 0.7$ and $M = m_1 = 1.0$. For the choice $\mu = 0$ in (7.12) the "convergence" of the calculation is seen to be *extremely* poor, whereas the choice $\mu = 0.856$ $(\mu = m_1 - m_2)$ is seen to lead to a satisfactory convergence of the numerical results. In practice the modification (7.12) provided us with one additional variational parameter which we were free to adjust so as to optimize the convergence properties of the calculation.

TABLE III. Computed values of E in the $J^P = \frac{3}{2}^+$, $I = \frac{3}{2}$ channel at two values of γ and at several matrix sizes, for $m_1 = m_2 = M = 1.0$ and the choice of potential (7.8a) with $a = b = 1.6$. Part (a) shows the effect of improving only on the radial dependence of the trial solution, and (b) shows the effect of improving only on the angular dependence of the trial solution. The correct value of E is 1.649.

Form of trial func.	($\gamma = 0.5$) E	($\gamma = 1.0$) E
	(a)	
2; 22	1.44970	1.66377
2; 33	1.55127	1.67021
2; 44	1.61259	1.67060
2; 55	1.64267	1.67119
2; 66	1.65673	1.67138
2; 77	1.66357	1.67138
	(b)	
3; 111	1.349512	1.644824
4; 1111	1.350293	1.645020
5; 11111	1.350098	1.644629
6; 111111	1.350098	1.644629
7; 1111111	1.350098	1.644629

[26] This does not represent an improvement, since our basis functions (7.7) place the time coordinate on equal footing with the space coordinates. This is of course the basic difficulty of dealing with weakly bound states in our approach.

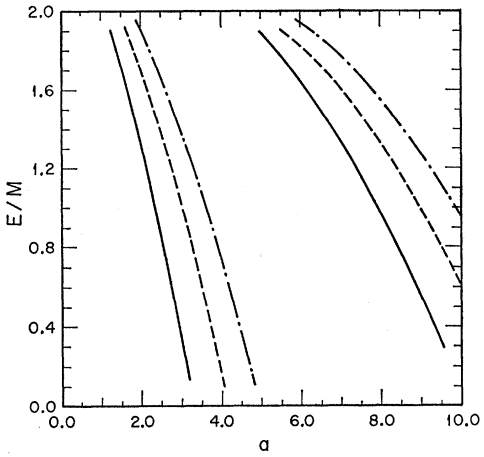

FIG. 5. Bound-state energies versus a for $J^P=\frac{3}{2}^+$, $I=\frac{3}{2}$, $m_1=m_2=M=1.0$, and the choice of potential (7.8a) as computed at three values of the parameter b: $b=1.6$ (———), $b=1.4$ (- - - -), and $b=1.2$ (— — —).

VIII. NUMERICAL RESULTS

In the present calculation we devote our attention exclusively to the $J=\frac{3}{2}$ and $J=\frac{1}{2}$ channels. For convenience we divide this discussion into several parts corresponding to the different values of the total angular momentum J, isotopic spin I, and parity that were considered. To a large extent we are concerned only with the general features of the eigenvalue spectrum of Eq. (3.1), which are not expected to depend critically on the ratio m_2/m_1. Because of the technical difficulties encountered when dealing with a ratio $m_2/m_1 \ll 1$ (as in the pion-nucleon case), we have placed somewhat greater emphasis on the case in which $m_1=m_2$. The numerical results are summarized in Figs. 5–8, and in Tables V and VI. (For a listing of the computed eigenvalues to several significant figures, see Ref. 21.)

A. $N^*(J^P=\frac{3}{2}^+, I=\frac{3}{2})$ Channel

In this channel the coupling constant λ is positive [recall that λ is related to the pseudoscalar coupling constant G as in (2.8)]; the parameters a and b in (7.8a) were taken to be positive, as well. In Fig. 5 we have summarized our results for the choice of potential (7.8a) with $m_1=m_2=M=1.0$. The energy spectrum was computed for several values of a and b. We see that both the singular and nonsingular components of the potential (7.8a) "act" attractively[27] in this channel. Moreover, for the range of coupling-constant values considered here, there exist no excited states in the energy spectrum.[28] We also note that for the undamped nucleon-exchange potential (3.2) we obtain a bound state of mass $M_B \approx 1.65$ (in our units) for $\lambda=1.6$ ($G^2/4\pi=10.0$).

In Fig. 6 we have summarized the corresponding results for the correct pion-nucleon kinematics, that is, for $m_1=M=1.0$ and $m_2=0.144$, the pion mass. Again we studied the energy spectrum as a function of the parameter a in (7.8a), with $b=1.6$. We note that for the exchange potential (3.2), that is, for $a=b$ in (7.8a), we obtain a bound state of mass $M_B \approx 1.04$ at a coupling constant $\lambda=1.6$ ($G^2/4\pi=10$). It is quite reasonable to interpret this bound state as the $N^*(1236)$, since, for the singular force (3.2), the N^* could emerge in our calculation as a weakly bound state, while a more realistic treatment of the short-range part of the force may "predict" it as a resonance (for the same value of the

TABLE IV. Values of the coupling constant λ required to give a bound state with $E=1.0$ in the $J^P=\frac{1}{2}^+$, $I=\frac{1}{2}$ channel, as computed for a sequence of increasing matrix sizes and for two values of the parameter μ in (7.12). The potential was taken to be of the form (7.8b) with $R_c=0.7$, and $m_1=M=1.0$, $m_2=0.144$ (the pion mass). The "convergence" of the numerical results is seen to be *extremely* poor for the choice $\mu=0$, but is considerably better for the choice $\mu=0.856$.

Form of trial func.	($\mu=0.0$) λ	($\mu=0.856$) λ
2; 10	−0.9937	−0.9546
4; 3200	−0.9612	−0.8636
5; 54200	−0.7266	−0.8436
5; 76410	−1.0129	−0.8361

TABLE V. Energy spectrum in the $J^P=\frac{3}{2}^+$, $I=\frac{3}{2}$ channel as computed at several values of the exchanged mass M for $m_1=m_2=1.0$ and the choice of potential (7.8a) with $a=b=1.6$.

M	E
1.0	1.649
0.8	1.702
0.6	1.783
0.5	1.832
0.4	1.890

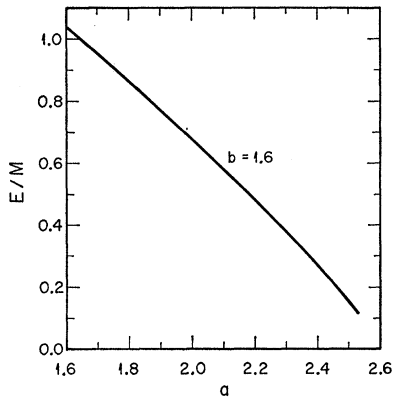

FIG. 6. Bound-state energies versus a for $J^P=\frac{3}{2}^+$, $I=\frac{3}{2}$, $m_1=M=1.0$, $m_2=0.144$ (the pion mass), and the choice of potential (7.8a) with $b=1.6$.

[27] By "act" attractively (repulsively) we mean that the binding energy of the bound state increases (decreases) as we increase the magnitude of the coupling constant. We will call a force attractive whenever it is capable of giving a bound state.

[28] Note that the terms "ground state" and "excited state" refer here to a given quantum number n, and in particular to $n=l$ for the l_+ solution, and to $n=l-1$ for the l_- solution.

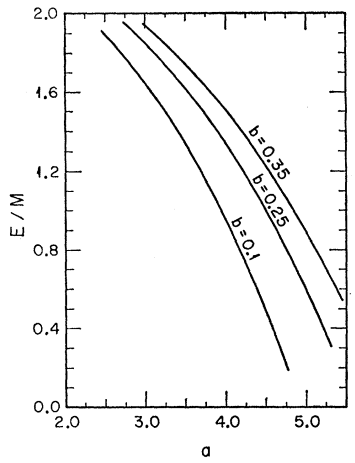

FIG. 7. Bound-state energies versus a for $J^P=\frac{1}{2}^+$, $I=\frac{3}{2}$, $m_1=m_2=M=1.0$, and the choice of potential (7.8a) as computed at three values of the parameter b.

coupling constant). The above value for the mass of the N^* should be compared with the value calculated by Abers and Zemach,[6] who obtained the N^* in their N/D calculation as a bound state with a mass $m(N^*)=1.13$ (in our units) at the experimentally measured value of the coupling constant, and for a force input corresponding to the exchange of an elementary nucleon.

In Table V we give the bound-state energies as computed for several values of the mass M of the exchanged nucleon, for $m_1=m_2=1.0$ and the potential (3.2) with $\lambda=1.6$. We find that for increasing values of M (with a corresponding decrease in the range of the force) the two-body system becomes more deeply bound, which seems at first to contradict our intuition. A glance at the potential (3.2) shows, however, that the mass of the exchanged nucleon controls not only the "range" of the force, but also the effective coupling-constant strength.

B. $J^P=\frac{1}{2}^+$, $I=\frac{3}{2}$ Channel

Again only positive values of λ are to be considered here. The parameters a and b in (7.8a) were taken to be positive, as well. No stable particle with the above quantum numbers has been observed to date. Because of the very restricted range of coupling-constant values that could be considered here in the singular case (see Table I), we were unable to obtain any bound states unless we considered the more general forms (7.8) for the potential, with $a\neq b$ and $R_c\neq 0$. This was the case in all $J=\frac{1}{2}$ channels, independent of the isotopic spin and parity, and was of course the reason for considering at all the modifications (7.8). In Fig. 7 we give a plot of the computed bound-state energies as a function of the parameters a and b in (7.8a), with $m_1=m_2=M=1.0$. Qualitatively the situation is very similar to that represented in Fig. 5, except that now the singular component of the potential (7.8a) acts repulsively, whereas it acted attractively in the N^* channel. The nonsingular component acts attractively, as before. One might expect that the situation would be reversed if we considered negative values of the parameters a and b. In particular, since the overall force was found to be attractive for a, $b>0$, we might expect it to be repulsive for a, $b<0$. This is not the case, bound-state solutions having been obtained in both cases, as the following discussion will show.

TABLE VI. Values of the coupling constant λ required to obtain a bound state at $E=1.0$ in the N^* and nulceon channel, as computed for several values of the cutoff parameter R_c in (7.8b), with $m_1=M=1.0$. and $m_2=0.144$ (the pion mass). (We present only the smallest value of $|\lambda|$ in the eigenvalue spectrum.)

R_c	(N^* channel) λ	(N channel) λ
0.3	2.11	−0.69
0.5	2.25	−0.77
0.7	2.38	−0.83
1.0	2.53	−0.90

C. Nucleon ($J^P=\frac{1}{2}^+$, $I=\frac{1}{2}$) Channel

In this channel $\lambda<0$. The parameters a and b in (7.8a) were taken to be negative, as well. In Fig. 8 we have plotted the computed bound-state energies as a function of the coupling constant λ, for the choice of potential (7.8b) with $R_c=1.0$ and $m_1=m_2=M=1.0$. We have found it instructive this time to study explicitly both the positive and negative energy spectrum. As we have pointed out already, our eigenvalue problem is

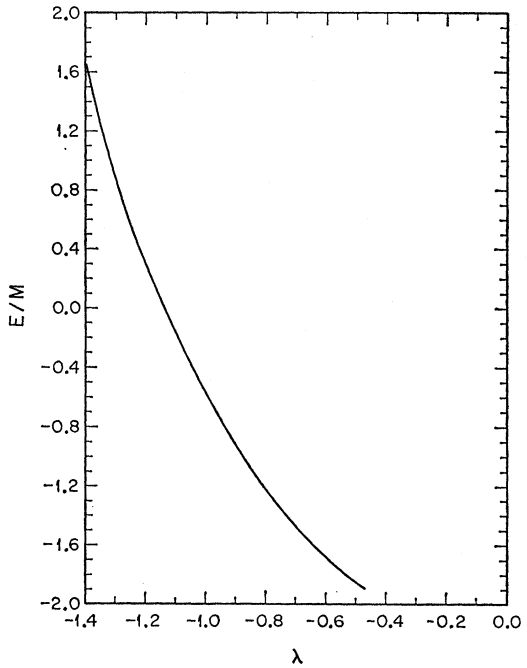

FIG. 8. Bound-state energies versus λ for $J^P=\frac{1}{2}^+$, $I=\frac{1}{2}$, $m_1=m_2=M=1.0$, and the choice of potential (7.8b) with $R_c=1.0$.

just as well defined in the negative-energy domain as it is in the positive one, provided that $-(m_1+m_2)<E<(m_1+m_2)$. The results presented in Fig. 8 indicate a somewhat abnormal situation, the computed binding energies bearing an inverse relationship to the magnitude of the coupling constant. However, in view of the MacDowell symmetry as stated in the form (6.8), this abnormal situation in the 1_- channel (we use the notation $l_\pm$) is a direct consequence of the existence of a normal situation in the 0_+ channel. This has been explicitly verified numerically. Stated more generally, given a normal situation in the $l_\pm$ channel, we will have a corresponding abnormal situation in the $(l\pm1)_\mp$ channel. It is clearly sufficient to compute the positive and negative energy spectrum in all angular momentum channels of a given parity, since the energy spectrum in the remaining channels of opposite parity may be obtained from here with the aid of (6.8). We note that the MacDowell symmetry has been shown[17] to be a consequence of extended Lorentz invariance (invariance under complex Lorentz transformations), which is itself a consequence of the invariance of the S matrix under real Lorentz transformations. Hence, the usual requirement of Lorentz invariance provides the physical reason for the existence of the abnormal situation as observed above.

Finally, it is interesting to compare the strength of the force in the N^* channel with that in the nucleon channel. Thus, fixing the bound-state energy at the nucleon mass ($E=1.0$), we have computed in these two channels the eigenvalues λ as a function of the cutoff parameter R_c for the choice of potential (7.8b), with $m_1=M=1.0$ and $m_2=0.144$, the pion mass. The results are summarized in Table VI. We note that the computed values of $G^2/4\pi$ [see (2.8)] needed to give a bound state of the πN system at the nucleon mass are larger in the N^* channel than in the nucleon channel. (Recall that an estimate based on the Born term alone would predict the nucleon as contributing the dominant force in the N^* channel, not in the nucleon channel.) We observe, however, that this statement is misleading, since the situation in the N^* channel is a normal one, whereas in the nucleon channel we are dealing with an abnormal situation.

We should like to conclude this discussion on a somewhat pessimistic note. Our results have shown that any estimate of the relative strength of the forces in the various channels as based on the Born diagram of Fig. 2 will fail completely in the bound-state problem, as of course was to be expected. Moreover, our calculation has also shown that, for the bound-state problem, the sign of the coupling constant does not in general provide a criterion for distinguishing between attractive and repulsive forces (unless the operators involved are either positive or negative definite, as is the case in a ϕ^3 theory with equal-mass particles). This implies in particular that the study of crossing matrices alone cannot provide in general an estimate of the relative sign and strength of the forces in a bound-state calculation, except when this study is made in the context of a particular dynamical model.

IX. SUMMARY

Our calculation has demonstrated that even marginally singular Bethe-Salpeter equations may be solved by standard numerical techniques, provided that proper care is taken of the boundary conditions at the origin (of the Wick-rotated space-time). From the computational point of view, the presence of spin in the problem did not in any fundamental way affect our ability to solve the equation, although the algebraic aspect of the problem was considerably more involved than in the absence of spin, the added complexity being principally due to the particular approach taken in this paper. Thus we could have chosen to solve the BS equation directly in the form of a two-dimensional differential or integral equation. Other[4] calculations have shown, however, that such an approach would require the inversion of typically 100-by-100-dimensional matrices in order to compute the desired numbers to an accuracy of a few percent. This is to be compared with our calculation in which twenty-seven was the maximum *effective* matrix size ever considered, the corresponding accuracy of the computed eigenvalues being frequently better than 1%. In fact, in most of the cases the approximation of the trial solution by the first one or two terms in the expansions (7.4) and (7.7) was sufficient to obtain an accuracy better than 10%. Our particular treatment offers additional advantages as well. Thus we recall that in practice we needed to include at most the first five terms in the expansion (7.4) of the solution, in order to achieve a 0.1% accuracy (and better). Moreover, our approach allows us to give a proper treatment of the boundary conditions at the origin. In fact, the critical dependence of these boundary conditions on the coupling-constant strength in marginally singular equations such as considered in this paper suggests, that our approach may well be *the* approach to the singular problem.

As for the physical content of our calculation, we do not know the range of validity of the ladder approximation, nor is it generally believed that the forces in nature are as singular as the ones we have considered (in the absence of a cutoff). In addition, a more realistic treatment would have to include the forces arising from the exchange of higher spin particles, such as the N^*, for example. We have thus strongly emphasized the mathematical rather than the physical aspect of this problem.

ACKNOWLEDGMENTS

I am pleased to thank Professor Charles Zemach for suggesting this problem and for very helpful discussions, and Professor Charles Schwartz for his guidance and advice during the progress of this work. I should also

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APPENDIX A: FOUR-DIMENSIONAL SPINOR SPHERICAL HARMONICS

We introduce first the notation and conventions which are followed throughout our work. A point $(x_\alpha)=(x_1,x_2,x_3,x_4)=(\mathbf{r},\tau)$ in the four-dimensional Euclidean space can also be represented in terms of the length $R=(\mathbf{r}^2+\tau^2)^{1/2}$ of the four-vector, the polar and azimuthal angles ϑ and ϕ in the (x_1,x_2,x_3) plane, and the angle θ between the four-vector and the four-axis:

$$(x_\alpha)=(R\sin\theta\sin\vartheta\sin\phi,\ R\sin\theta\sin\vartheta\cos\phi,\ R\sin\theta\cos\vartheta,\ R\cos\theta). \quad \text{(A1)}$$

We use the notation $\hat{R}=(x_\alpha/R)$ to denote a four-vector of unit length. The differential element of solid angle $d\Omega_R$ and the differential volume element dV are given by

$$d\Omega_R=\sin^2\theta\sin\vartheta d\theta d\vartheta d\phi,\qquad dV=R^3dRd\Omega_R. \quad \text{(A2)}$$

The four-dimensional spinor-spherical harmonics are defined by

$$Y_n{}^{l\pm M}(\hat{R})_\alpha=N_n{}^l\mathcal{Y}^{l\pm M}(\hat{r})_\alpha\sin^l\theta C_{n-l}{}^{l+1}(\cos\theta). \quad \text{(A3)}$$

Here $C_n{}^m(\cos\theta)$ are the Gegenbauer polynomials, $N_n{}^l$ is the normalization constant

$$N_n{}^l=\left[\frac{2^{2l+1}(n+1)(n-l)!(l!)^2}{\pi(n+l+1)!}\right]^{1/2}, \quad \text{(A4)}$$

and $\mathcal{Y}^{l\pm M}(\hat{r})_\alpha$ are the conventional three-dimensional spinor spherical harmonics

$$\mathcal{Y}^{l\pm M}(\hat{r})_\alpha=\sum_m i^lC(l,\tfrac{1}{2},l\pm\tfrac{1}{2};m\alpha M)Y_{lm}(\hat{r}), \quad \text{(A5)}$$

where $C(lsJ;m\alpha M)$ are the Clebsch-Gordan coefficients as defined by Rose,[29] and $Y_{lm}(\hat{r})$ are the usual spin-zero spherical harmonics, with $\hat{r}=\mathbf{r}/r$. The functions (A5) are orthonormal and satisfy the useful relation (we omit the magnetic quantum number M)

$$(\boldsymbol{\sigma}\cdot\hat{r})\mathcal{Y}^{l\pm}(\hat{r})=\pm i\mathcal{Y}^{(l\pm1)\mp}(\hat{r}). \quad \text{(A6)}$$

Similarly, the functions (A3) are orthonormal

$$\sum_\alpha\int d\Omega_RY_n{}^{l\pm M}(\hat{R})_\alpha{}^*Y_{n'}{}^{l\pm'M'}(\hat{R})_\alpha=\delta_{l_\pm,l_\pm'}\delta_{MM'}\delta_{nn'}, \quad \text{(A7)}$$

and satisfy the completeness relation

$$\sum_{l_\pm,M,n}Y_n{}^{l\pm M}(\hat{R})_\alpha{}^*Y_n{}^{l\pm M}(\hat{R}')_\beta=\delta_{\alpha\beta}\delta(\Omega_{R'}-\Omega_R), \quad \text{(A8)}$$

as well as the addition theorem

$$\sum_{l\pm M}Y_n{}^{l\pm M}(\hat{R})_\alpha{}^*Y_n{}^{l\pm M}(\hat{R}')_\beta=[(n+1)/2\pi^2]\delta_{\alpha\beta}C_n{}^1(\cos\gamma), \quad \text{(A9)}$$

where

$$\cos\gamma=\hat{R}\cdot\hat{R}'=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\omega,\qquad \cos\omega=\cos\vartheta\cos\vartheta'+\sin\vartheta\sin\vartheta'\cos(\phi-\phi'). \quad \text{(A10)}$$

APPENDIX B: FOUR-DIMENSIONAL ROTATION GROUP

In this Appendix we construct the projection operators $P_{n\pm}$ and $\tilde{P}_{n\pm}$ for the irreducible spaces of the representations $(\frac{1}{2}n\pm\frac{1}{2},\frac{1}{2}n)$ and $(\frac{1}{2}n,\frac{1}{2}n\pm\frac{1}{2})$ of O_4, the four-dimensional rotation group.[30] It is well known[31] that the group O_4 is characterized by the six Hermitian generators of infinitesimal rotations A_i and B_i $(i=1,2,3)$ with the familiar angular momentum commutation relations

$$[A_i,A_j]=i\epsilon_{ijk}A_k,\quad [B_i,B_j]=i\epsilon_{ijk}B_k.$$

Moreover, $[A_i,B_j]=0$; $\mathbf{A}^2$ and $\mathbf{B}^2$ are the two Casimir operators of the group; in a particular irreducible representation labeled by the pair of indices (a,b), $\mathbf{A}^2=a(a+1)I$, $\mathbf{B}^2=b(b+1)I$; the representations (a,b) are unitary and of dimensionality $(2a+1)(2b+1)$. To give a specific example we note that on the space of scalar fields, A_i and B_i are just the differential operators

$$A_i=\tfrac{1}{2}[L_i-i(x_i\partial_4-x_4\partial_i)],\qquad B_i=\tfrac{1}{2}[L_i+i(x_i\partial_4-x_4\partial_i)], \quad \text{(B1)}$$

where $L=-i(\mathbf{r}\times\boldsymbol{\nabla})$ are the usual generators of infinitesimal rotations in three dimensions. We have the following reduction of the direct product space

$$(\tfrac{1}{2}n,\tfrac{1}{2}n)\otimes(\tfrac{1}{2},0)\simeq(\tfrac{1}{2}n+\tfrac{1}{2},\tfrac{1}{2}n)\oplus(\tfrac{1}{2}n-\tfrac{1}{2},\tfrac{1}{2}n)$$
$$(\tfrac{1}{2}n,\tfrac{1}{2}n)\otimes(0,\tfrac{1}{2})\simeq(\tfrac{1}{2}n,\tfrac{1}{2}n+\tfrac{1}{2})\oplus(\tfrac{1}{2}n,\tfrac{1}{2}n-\tfrac{1}{2}).$$

The combined set of functions

$$Y_n{}^{l+M}(\hat{R})\quad\text{and}\quad Y_n{}^{(l+1)-,M}(\hat{R})$$

defined in (A3) (with the indices l and M taking on all allowed values for a given value of n) form a basis for either of these two $2(n+1)^2$-dimensional direct product spaces. In Sec. IV we make use of precisely this fact, when we construct the basis functions for the irreducible spaces $(\frac{1}{2}n\pm\frac{1}{2},\frac{1}{2}n)$ and $(\frac{1}{2}n,\frac{1}{2}n\pm\frac{1}{2})$ contained in the above reduction of the direct product space.

We construct next the projection operators for the irreducible spaces $(\frac{1}{2}n\pm\frac{1}{2},\frac{1}{2}n)$ and $(\frac{1}{2}n,\frac{1}{2}n\pm\frac{1}{2})$ which we denote by $P_{n\pm}$ and $\tilde{P}_{n\pm}$, respectively. To this end we define the generators of infinitesimal rotations in the above direct product spaces [the direct product with

[29] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

[30] I am indebted to Professor C. Zemach for very informative discussions on the four-dimensional rotation group.

[31] Paul Roman, *Theory of Elementary Particles* (Interscience Publishers, Inc., New York, 1960).

the identity is understood, so that $A_i=A_i(\frac{1}{2}n,\frac{1}{2}n)\otimes I(\frac{1}{2},0)$, etc.]:

$$\mathfrak{A}_i=A_i+\tfrac{1}{2}\sigma_i, \quad \mathfrak{B}_i=B_i \tag{B2}$$

in the direct product space $(\frac{1}{2}n,\frac{1}{2}n)\otimes(\frac{1}{2},0)$, and

$$\bar{\mathfrak{A}}_i=A_i, \quad \bar{\mathfrak{B}}_i=B_i+\tfrac{1}{2}\sigma_i, \tag{B3}$$

in the direct product space $(\frac{1}{2}n,\frac{1}{2}n)\otimes(0,\frac{1}{2})$.

Now, in the $(\frac{1}{2}n+\frac{1}{2},\frac{1}{2}n)$ representation,

$$\mathfrak{A}^2=\tfrac{1}{4}(n+1)(n+3), \quad \mathfrak{B}^2=\tfrac{1}{4}n(n+2),$$

so that we obtain with the aid (B2),

$$\boldsymbol{\sigma}\cdot\mathbf{A}=\tfrac{1}{2}n.$$

Similarly, in the $(\frac{1}{2}n-\frac{1}{2},\frac{1}{2}n)$ representation,

$$\mathfrak{A}^2=\tfrac{1}{4}(n^2-1), \quad \mathfrak{B}^2=\tfrac{1}{4}n(n+2)$$

so that

$$\boldsymbol{\sigma}\cdot\mathbf{A}=-\tfrac{1}{2}(n+2).$$

In exactly the same manner we conclude that

$\boldsymbol{\sigma}\cdot\mathbf{B}=\frac{1}{2}n$ in the $(\frac{1}{2}n,\frac{1}{2}n+\frac{1}{2})$ representation,
$\boldsymbol{\sigma}\cdot\mathbf{B}=-\frac{1}{2}(n+2)$ in the $(\frac{1}{2}n,\frac{1}{2}n-\frac{1}{2})$ representation.

The desired projection operators are thus given by

$$\begin{aligned} P_{n+}&=\frac{\frac{1}{2}(n+2)+\boldsymbol{\sigma}\cdot\mathbf{A}}{n+1}, & P_{n-}&=\frac{\frac{1}{2}n-\boldsymbol{\sigma}\cdot\mathbf{A}}{n+1}, \\ \tilde{P}_{n+}&=\frac{\frac{1}{2}(n+2)+\boldsymbol{\sigma}\cdot\mathbf{B}}{n+1}, & \tilde{P}_{n-}&=\frac{\frac{1}{2}n-\boldsymbol{\sigma}\cdot\mathbf{B}}{n+1}, \end{aligned} \tag{B4}$$

We easily verify that

$$P_{n\pm}{}^2=P_{n\pm}, \quad P_{n\pm}P_{n\mp}=0, \quad P_{n+}+P_{n-}=1,$$

with $\tilde{P}_{n\pm}$ satisfying the same relations.

APPENDIX C: PROOF OF ELASTIC UNITARITY

In this Appendix we show that the scattering amplitude $f(\mathbf{k}',\mathbf{k})$ defined in (2.18) satisfies elastic unitarity. For the purpose of this discussion it is convenient to rotate the t-integration contour in (2.18) to the imaginary axis. This can be done for the c.m. energy E below the first inelastic threshold, the result being

$$f(\mathbf{k}',\mathbf{k})=\frac{\lambda m_1}{4\pi E}\bar{U}(\mathbf{k}')\int d^4x e^{-i\mathbf{k}'\cdot\mathbf{r}+\frac{1}{2}\omega_{12}\tau}V(x)\phi_{\mathbf{k}}(-x), \tag{C1}$$

where $\phi_{\mathbf{k}}(x)$ is the solution to Eq. (3.6). We easily show that

$$\gamma^0\phi_{\mathbf{k}}(-\mathbf{r},\tau)=\phi_{-\mathbf{k}}(\mathbf{r},\tau). \tag{C2}$$

It follows then from (C1) and (C2) that

$$f^\dagger(\mathbf{k},\mathbf{k}')=\frac{\lambda m_1}{4\pi E}\int d^4x\phi_{-\mathbf{k}'}{}^\dagger(x)V(-x)U(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-\frac{1}{2}\omega_{12}\tau}, \tag{C3}$$

where the "dagger" denotes the Hermitian conjugate in the spin space only. We note here parenthetically that with the aid of

$$CU^*(-\mathbf{k})C^{-1}=U(\mathbf{k}), \quad CV^*(x)C^{-1}=V(x), \tag{C4}$$

where C is the usual charge-conjugation matrix as defined in (7.6), we easily show that

$$f(\mathbf{k}',\mathbf{k})=f(-\mathbf{k}',-\mathbf{k}) \tag{C5}$$

and

$$f^\dagger(\mathbf{k},\mathbf{k}')=Cf^*(-\mathbf{k}',-\mathbf{k})C^{-1}. \tag{C6}$$

Statements (C5) and (C6) follow, of course, directly from the invariance of the theory under space inversion and time reflection. Finally, noting that

$$H(x)-H^\dagger(x)=\frac{im_1|\mathbf{k}|}{2\pi E}\int\frac{d\Omega_k}{4\pi^2}\times\sum_\sigma U(\mathbf{k},\sigma)\bar{U}(\mathbf{k},\sigma)e^{i\mathbf{k}\cdot\mathbf{r}-\frac{1}{2}\omega_{12}\tau} \tag{C7}$$

we obtain with the aid of (C1), (C3), and Eq. (3.6),

$$i[f(\mathbf{k}',\mathbf{k})-f^\dagger(\mathbf{k},\mathbf{k}')]=\frac{|\mathbf{k}|}{2\pi}\int d\Omega_q\, f(\mathbf{k}',\mathbf{q})f^\dagger(\mathbf{k},\mathbf{q}), \tag{C8}$$

which is the statement of elastic unitarity.