Strong Coupling of the Multi-Partial-Wave Meson Isotriplet*

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The algebraic formulation of strong coupling is applied to the strong coupling of the SU_2-symmetric model in which all the partial waves of the π mesons are included. The strong-coupling group is a semidirect product of the $SU_2 \otimes SU_2$ internal symmetry group and an Abelian group which is generated by an infinite number of commuting generators corresponding to the vertices of the π mesons in different orbital angular momentum states. A physically interesting irreducible representation of the group is obtained which consists of a series of irreducible representations of the P-wave strong-coupling group. The Regge recurrences of isobars appear in this series. Each degenerate multiplet of isobars is specified by three quantum numbers—spin s, isospin i, and an additional quantum number v—which satisfy the angular momentum triangular relation. The following form of mass formula is obtained: $M(s,i,v)=M_0+m_0[x(x-1)s(s+1)+(1-x)i(i+1)+xv(v+1)]$.

I. INTRODUCTION

It has been shown[1] that the algebraic formulation of the static S- and P-wave strong-coupling theory reproduces many of the results obtained by the conventional field-theoretical strong-coupling theory.[2] The use of group-theoretical methods in the formulation has been found to be very powerful in the analysis of the complicated models, e.g., the P-wave SU_3-symmetric strong-coupling model.[3] In this paper we extend the formulation further to discuss the strong coupling of the model in which all the partial waves of the π mesons are included.

One of the shortcomings of the P-wave strong-coupling model is that the isobar band does not contain the isobars in a Regge recurrence. This is due to the static nature of the model which includes only P-wave meson interactions. Since the Regge recurrences of the baryons are empirically established,[4] it is desirable to modify the model so that the strong-coupling band includes the Regge recurrences. For this purpose we include all the partial waves of the mesons within a static model.[5] An appropriate framework to accomplish this is provided by Dyson,[6] who considers the scattering of an arbitrary partial-wave meson by an arbitrary-spin target instead of the conventional P-wave static model.

The derivation of the strong-coupling group from the Chew-Low equation discussed in Ref. 1 can still be applied in the present case to yield an infinite number of generators corresponding to the vertices of the pseudoscalar mesons in different orbital angular momentum states. The parity of the generators corresponding to the odd orbital angular momentum of the mesons is positive while the parity of the other generators is negative. Since these commuting generators include the P-wave generators ($l=1$), the new strong-coupling group contains the P-wave strong-coupling group as a subgroup. Since the isobar states of the strong coupling are described as a basis of a unitary irreducible representation of the strong-coupling group, and since the irreducible representation of a group is in general reducible with respect to its subgroup, the isobar band of the present model consists of a series of bands of the P-wave model. In this series one would expect Regge recurrences of isobars. Since both the positive- and the negative-parity interactions are included, one would obtain the isobars of negative parity as well as of positive parity. In this paper we confine ourselves only to the charge-independent interactions of π mesons.

In Sec. II, we follow Goebel's original discussion to derive the strong-coupling conditions from the Chew-Low equation by demanding the existence of its solutions in the strong-coupling limit. We also derive the mass condition as a sufficient condition for the solution of the Chew-Low equation in the strong-coupling limit.

In Sec. III, we discuss the nature of the strong-coupling group which is derived from the strong-coupling conditions with a consideration of the internal-symmetry group. An interesting representation of the group is obtained by using the method of induced representations. The physical content of the representation is then discussed.

In Sec. IV, we first discuss a general method by which mass operators of the isobars are obtained from the mass condition in case that the representation of the strong-coupling group is given. Then the method is applied to the present model using the representation obtained in Sec. III.

In Sec. V, we present some remarks.

The Appendix is devoted to obtaining the known mass formula of various models using the method developed in Sec. IV.

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[1] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).

[2] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); **14**, 633 (1941); W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942); S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **1**, 83 (1946); **1**, 109 (1946); **2**, 6 (1947).

[3] C. J. Goebel, Phys. Rev. Letters **16**, 1130 (1966).

[4] V. Barger and D. Cline, Phys. Rev. **155**, 1792 (1967).

[5] The first step in this direction has been taken by Bhasin and Pande, who considered the P- and D-wave neutral-meson model; V. S. Bhasin and L. K. Pande, Phys. Rev. **165**, 1927 (1968).

[6] F. J. Dyson, Phys. Rev. **100**, 344 (1955).

II. CHEW-LOW EQUATION AND ITS SOLUTIONS IN THE STRONG-COUPLING LIMIT

We shall consider the scattering of a meson by a heavy isobar. The meson state is specified by a capital letter, for example, by $A\equiv\alpha;\, l,\, m$, where α is the isospin component, l is the orbital angular momentum, and m is its z component. We denote the energy of the meson in A by ω_A. An isobar state is specified by a small letter, for example by $|a\rangle\equiv|i,i_z,s,s_z,\xi\rangle$, where i is the isospin, i_z is its z component, s is the spin, s_z is its z component, and ξ is the label to distinguish isobars with the same spin and isospin. The scattering amplitude of the reaction $\pi_A+N_a\rightarrow\pi_B+N_b$ is expressed as a matrix element of the operator $T_{\bar{B}A}(\omega_B)$ between the initial and the final isobar state, $\langle b|T_{\bar{B}A}(\omega_B)|a\rangle$, which is defined by $\langle^{(-)}b,B|V_A|a\rangle$. The meson source operator V_A in static models has the following general form:

$$V_A=[g_0/(2\omega_A)^{1/2}]u_A(\omega_A)O_A\,, \tag{2.1}$$

where u_A is a function of ω_A and its functional form depends only on l, whereas O_A is an energy-independent operator which in general depends on α, l, and m.

The Chew-Low equation is given by

$$\langle b|T_{\bar{B}A}(\omega_B)|a\rangle=-\sum\left[\frac{\langle b|V_B{}^\dagger|n\rangle\langle n|V_A|a\rangle}{E_n-M_b-\omega_B-i\epsilon}+\frac{\langle b|V_A|n\rangle\langle n|V_B{}^\dagger|a\rangle}{E_n-M_a+\omega_B}\right], \tag{2.2}$$

where M_a is the mass of isobar a and the summation of the intermediate states is taken for all possible states. If we separate the intermediate states into one-isobar states, one-isobar plus one-meson states, etc., we can write the Chew-Low equation as a nonlinear equation of the scattering amplitude operator f in the isobar space, which is spanned by one-isobar states only:

$$\begin{aligned}f_{\bar{B}}{}^A(\omega_B)=g^2\Big[&A_B{}^\dagger\frac{1}{M-\overleftarrow{M}-\omega_B}A_A+A_A\frac{1}{M-\overrightarrow{M}+\omega_B}A_B{}^\dagger\Big]\\&+\sum_C\frac{1}{\pi}\int_\mu^\infty d\omega\,\rho(\omega)\Big[f_{\bar{C}B}{}^\dagger(\omega)\frac{1}{M-\overleftarrow{M}+\omega-\omega_B}\\&\times f_{\bar{C}A}(\omega)+f_{\bar{C}\bar{A}}{}^\dagger(\omega)\frac{1}{M-\overrightarrow{M}+\omega+\omega_B}f_{\bar{C}\bar{B}}(\omega)\Big]\\&+(\text{multimeson term})\,,\end{aligned} \tag{2.3}$$

where f, A, and M are operators defined in the isobar space and are given by

$$\langle b|T_{BA}(\omega_B)|a\rangle=-\frac{u_B{}^*(\omega_B)u_A(\omega_A)}{(2\omega_A2\omega_B)^{1/2}}\langle b|f_{\bar{B}A}(\omega_B)|a\rangle\,,$$

$$g_0\langle b|O_A|a\rangle=g\langle b|A_A|a\rangle\,,$$

$$M|a\rangle=M_a|a\rangle\,.$$

$\rho_A(\omega)$ is a combination of phase volume and the cutoff function u_A, and it is given by

$$\sum_{p_A}\frac{|u_A(\omega_A)|^2}{2\omega_A}=(1/\pi)\int d\omega_A\rho_A(\omega_A)\,.$$

In (2.3), we have used the following convention:

$$\langle a|A^\dagger(1/M-\overrightarrow{M}+\cdots)A|b\rangle=\sum_c\frac{\langle a|A^\dagger|c\rangle\langle c|A|b\rangle}{M_c-M_b+\cdots}.$$

As in Ref. 1, we assume that all isobars are degenerate in mass in the strong-coupling limit. The dependence of the coupling constant g in the mass operator M is assumed to be given by

$$M=M_0+\mathfrak{M}/g^2+O(1/g^4)\,, \tag{2.4}$$

where M_0 is a number times unit operator and $\mathfrak{M}$ is an operator. As in Ref. 1, we insert (2.4) into (2.3) and expand it as a power series of $1/g^2$. Since a partial-wave amplitude is bounded by unitarity, the strong-coupling condition

$$[A_A,A_B]=0 \tag{2.5}$$

must be satisfied in the strong-coupling limit. Assuming that the right-hand side of (2.5) approaches zero faster than $1/g^2$ and meson-production amplitudes are zero in the strong-coupling limit,[7] we obtain the following form for the Chew-Low equation[8] in the strong-coupling limit:

$$f_{\bar{B}A}(\omega)=-\frac{1}{\omega^2}\Lambda_{\bar{B}A}+\frac{1}{\pi}\int_\mu^\infty d\omega'\times\sum_c\left[\frac{f_{\bar{C}B}{}^\dagger(\omega')\rho_c(\omega')f_{\bar{C}A}(\omega')}{\omega'-\omega}+\frac{f_{\bar{C}\bar{A}}{}^\dagger\rho_cf_{\bar{C}\bar{B}}}{\omega'+\omega}\right], \tag{2.6}$$

where

$$\Lambda_{\bar{B}A}=[A_B{}^\dagger,[\mathfrak{M},A_A]]\,. \tag{2.7}$$

In order to obtain a solution of the equation, we require that all f's and Λ's commute among themselves. Since $\Lambda_{\bar{B}A}=\Lambda_{A\bar{B}}$, which can be shown by using (2.5), we also require the following symmetry property of the amplitude:

$$f_{\bar{B}A}(\omega)=f_{A\bar{B}}(\omega)\,. \tag{2.8}$$

The Chew-Low equation in the strong-coupling limit then becomes

$$f_{\bar{B}A}(\omega)=-\frac{1}{\omega^2}\Lambda_{\bar{B}A}+\frac{1}{\pi}\int_{\mu^2}^\infty\frac{d\omega'^2}{\omega'^2-\omega^2}\sum_C f_{\bar{C}B}{}^\dagger\rho_Cf_{\bar{C}A}\,. \tag{2.9}$$

We may regard f as a function in the complex ω^2 plane and it has a pole at the origin and a cut that goes from μ^2 to ∞.

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[7] Goebel has proved that the Born term of all production amplitudes approaches zero at least as fast as $1/g$ (private communication).

[8] T. Cook, Ph.D. thesis, University of Wisconsin, 1967 (unpublished).

A formal solution of (2.9) can be obtained by the N/D method[9];

$$f_{\bar{B}A}(\omega)=-\frac{1}{\omega^2}\sum_C \Lambda_{\bar{B}C}[1+K(\omega^2)\Lambda]^{-1}{}_{\bar{C}A}, \quad (2.10)$$

where $K(\omega^2)$ is the diagonal matrix

$$K_C(\omega^2)=\frac{\omega^2}{\pi}\int_{\mu^2}^{\infty}\frac{\rho_C(\omega'^2)d\omega'^2}{\omega'^4(\omega'^2-\omega^2)}. \quad (2.11)$$

To be a true solution of (2.9) the function $[1+K(\omega^2)\Lambda]^{-1}$ should not produce new singularities in the complex ω^2 plane except at ∞.[10]

A sufficient condition for this is provided by the mass condition

$$\sum_C \Lambda_{\bar{A}C}\kappa_C\Lambda_{\bar{C}B}=\Lambda_{\bar{A}B}, \quad (2.12)$$

where

$$\kappa_C=-K(-\infty)=\frac{1}{\pi}\int_{\mu^2}^{\infty}d\omega^2\frac{\rho_C(\omega^2)}{\omega^4}. \quad (2.13)$$

Proof: For this purpose it is sufficient to prove that the equation

$$(1+K\Lambda)\psi=0 \quad (2.14)$$

has no solutions except $\psi=0$ if the condition (2.12) is imposed. Here we examine it only for a negative real ω^2 since one can prove it similarly for other ω^2 with a slight modification. If we multiply Eq. (2.14) by $(\Lambda\psi)^\dagger$ from the left and use (2.12), we obtain

$$\sum_C [\kappa_C+K_C(\omega^2)]|\sum_A \Lambda_{\bar{C}A}\psi_A|^2=0.$$

Since $K_C(\omega^2)$ is a monotonic function of ω^2, $\kappa_C+K_C(\omega^2)>0$ for finite negative ω^2. Thus $\Lambda\psi=0$, which leads to $\psi=0$ by virtue of Eq. (2.14).

III. STRONG-COUPLING GROUP AND REPRESENTATIONS

The commuting operators A_B obtained in the previous section generate an Abelian group of unitary transformations in the isobar space. We denote the group by T. If there is an internal symmetry group K which is assumed to be compact, the isobar states form a space of unitary representation (in general reducible) of K. Since meson states form a basis of representation of K, the operators A must be tensor operators of K in the isobar space in order that the theory be invariant under K. Thus, the isobar space is a space of unitary representations of G which is a semidirect product of K and T; $G=K\cdot T$. We call G the strong-coupling group. In this paper we confine ourselves only to the case of spin and isospin internal symmetry; i.e., $K\equiv SU_2\otimes SU_2$.

In order to define the group G by commutation relations of its generators, we denote the index of A in terms of the spherical basis α; l, m and use the standard phase convention[11];

$$A^\dagger{}_{\alpha lm}=(-)^{(\alpha+m)}A_{-\alpha l-m},$$

where the dagger denotes the Hermitian conjugate. Then, (2.5) becomes

$$[A_{\alpha lm},A_{\alpha' l' m'}]=0. \quad (3.1)$$

Commutation relations among generators of spin (J_i), isospin (I_α), and A_s are given by

$$[J_i,A_{\alpha lm}]=[l(l+1)(2l+1)]^{1/2}\sum_{m'}\begin{pmatrix}1 & l & m'\\ i & m & l\end{pmatrix}A_{\alpha lm'}, \quad (3.2)$$

$$[I_\alpha,A_{\beta lm}]=(\surd 6)\sum_\gamma\begin{pmatrix}1 & 1 & \gamma\\ \alpha & \beta & 1\end{pmatrix}A_{\gamma lm}, \quad (3.3)$$

$$[J_i,J_j]=(\surd 6)\sum_k\begin{pmatrix}1 & 1 & k\\ i & j & 1\end{pmatrix}J_k, \quad (3.4)$$

$$[I_\alpha,I_\beta]=(\surd 6)\sum_\gamma\begin{pmatrix}1 & 1 & \gamma\\ \alpha & \beta & 1\end{pmatrix}I_\gamma, \quad (3.5)$$

$$[I_\alpha,J_i]=0, \quad (3.6)$$

where the coefficients are Wigner's 3-j symbols, with the convention

$$\begin{pmatrix}j_1 & j_2 & m_3\\ m_1 & m_2 & j_3\end{pmatrix}=(-)^{j_3-m_3}\begin{pmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & -m_3\end{pmatrix}.$$

Notice also that the spherical basis of the SU_2 generators (J_i,I_α) is different from the conventional one.[11] The strong-coupling group G is defined as a group generated by J, I, and A, which satisfy the commutation relations (3.1) to (3.6).

In Ref. 12, we have discussed irreducible representations of the strong-coupling groups by using the method of induced representation. The general method developed there can be applied to the present case.

Since the operators A commute with each other, they can be simultaneously diagonalized. A set of eigenvalues of all the operators A specifies an eigenvector. Let N be the number of A's [in the present model $N=\infty$]. The eigenvector is then specified by a point in the N-dimensional space Ω. Since the A's are the tensor

[9] We thank Professor R. Warnock for providing us this solution. Later I was informed by Professor Goebel that he also knew the solution.

[10] Since we took the strong-coupling limit we expect states at infinitely high energy which are generated from some spin states of the bare nucleon.

[11] See, for example, A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957).

[12] T. Cook and B. Sakita, J. Math. Phys. 8, 708 (1967); see also C. J. Goebel, *Non-Compact Groups in Particle Physics* (W. A. Benjamin, Inc., New York, 1966).

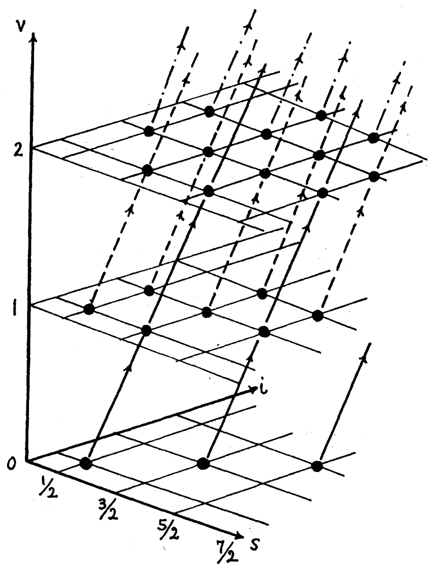

FIG. 1. Isobar states. The black dots correspond to the isobar states in the strong-coupling band. The Regge recurrences are specified by the arrows.

operators of K, a group element of K can be realized as a transformation in Ω. Thus, one can consider the orbits in Ω such that any points in an orbit can be connected by at least one of the transformations induced by K. If a set of eigenvalues $\{A^{(0)}\}$ of A is given, the orbit on which the point $\{A^{(0)}\}$ lies is specified. The subgroup of K whose elements transform the point $\{A^{(0)}\}$ to itself is called the little group associated with $\{A^{(0)}\}$, and is denoted by K_0.

Components of an irreducible representation of K can be specified by a set of irreducible representations of the groups in a chain $K\supset K_1\supset K_2\cdots\supset K_0$. Thus, we may specify the component by (ξL), where L is an irreducible representation of K_0 and ξ stands for the representations of the groups in the chain.

Here we quote two important results of Ref. 12:

(i) An irreducible unitary representation of the group $G=K\cdot T$ is specified by an orbit (determined by $\{A^{(0)}\}$) and an irreducible representation of K_0 (denoted by L).

(ii) The irreducible representation $(A^{(0)}L)$ of G contains those representations of K which have a component (ξL) with multiplicity given by the number of different values that ξ may take on the same L.

Thus, one can obtain an irreducible representation of G and its reduction under K by the following procedure: (a) Fix the form of $A^{(0)}$, which specifies the orbit. (b) Find the little group K_0 and obtain an irreducible representation L of K_0. (c) Find a chain of groups $K\supset\cdots\supset K_0$, and find all the representations of K which contain L in the reduction based on this chain. Then the representation of G is given by $(A^{(0)}L)$ and its reduction under K is as stated in (ii).

Following this procedure we first set

$$A_{\alpha lm}{}^{(0)}=\begin{pmatrix}1 & l & l-1\\ \alpha & m & 0\end{pmatrix}a_l, \tag{3.7}$$

where a_l is a parameter.[13] The little group K_0 in this case is the one-parameter group generated by V_z, where

$$\mathbf{V}=\mathbf{I}+\mathbf{J}. \tag{3.8}$$

Therefore, the representation determined by the orbit (3.7) is classified by the eigenvalue v_z of V_z. Since $\mathbf{V}$ generates an SU_2 group and since it is the little group for the P-wave strong-coupling model,[12] the following chain is useful to specify a basis of a representation in terms of basis of representations for the P-wave strong-coupling group:

$$(SU_2)_J\otimes(SU_2)_I\supset(SU_2)_V\supset(U_L)_{V_z}.$$

Let $v(v+1)$ be the eigenvalue of $\mathbf{V}^2$. Then v can be used for ξ. Since $v\geq v_z$, the representation (v_z) of the group contains all the P-wave strong-coupling representations (v) of $v\geq v_z$. The basis of the representation is therefore labelled by i, i_z, s, s_z, v, and it consists of all (i,s,v) which satisfy the triangular relation $\Delta(i,s,v)$ and $v\geq v_z$.

The representation $(v_z=0)$ is of special interest because only this representation includes the P-wave strong-coupling representation $(v=0)$. If the mass operator is diagonal on the i, s, v basis, isobar states are specified by i, s, v, with the triangular relation. Each isobar state is then represented by a three-dimensional lattice point as shown in Fig. 1. The matrix element of the operator $A_{\alpha lm}$ can be computed by using Eq. (32) of Ref. 12:

$$\langle i'i_z's's_z'v'|A_{\alpha lm}|ii_zss_zv\rangle =\begin{pmatrix}i_z' & 1 & i\\ i' & \alpha & i_z\end{pmatrix}\begin{pmatrix}s_z' & l & s\\ s' & m & s_z\end{pmatrix}\langle i's'v'\|A_l\|isv\rangle,$$

$$\langle i's'v\|A_l\|isv\rangle =(-)^v[(2i+1)(2i'+1)(2s+1)\times(2s'+1)(2v+1)(2v'+1)]^{1/2}\times\begin{pmatrix}v' & l-1 & v\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}v & i & s\\ l-1 & 1 & l\\ v' & i' & s'\end{Bmatrix}a_l, \tag{3.9}$$

where

$$\begin{Bmatrix}\cdots\\ \cdots\\ \cdots\end{Bmatrix}$$

is the 9-j symbol.[11]

[13] One can add the additional term

$$\begin{pmatrix}1 & l & l+1\\ \alpha & m & 0\end{pmatrix}b_l,$$

which does not change the orbit, the little group, and the parity considerations.

Since the parity of the operator $A_{\alpha lm}$ is given by $(-)^{l-1}$, and the $3j$ symbol

$$\begin{pmatrix} v' & l-1 & v \\ 0 & 0 & 0 \end{pmatrix}$$

is zero if $v+v'+l-1$ is odd, the parity of isobars in the band can be specified by v and given by $\epsilon(-)^v$, where ϵ is the parity of the $v=0$ band. This parity assignment is a specific nature of the representation $(v_v=0)$. For the other representations we must double the representation in general because of the parity considerations. This is another interesting aspect of the $v_v=0$ representation.

The triangular relation can be written as

$$s=|v-i|,\ |v-i|+1,\ \cdots,\ v+i \qquad (3.10)$$

for a given isospin i. Each term in (3.10) corresponds to the Regge recurrence[14] which is designated by the arrows in Fig. 1. For example, there are two Regge recurrences for $i=\frac{1}{2}$, i.e., $S^P=\frac{1}{2}^+,\ \frac{3}{2}^-,\ \frac{5}{2}^+,\ \cdots$ corresponding to $s=v+\frac{1}{2}$, and $S^P=\frac{1}{2}^-,\ \frac{3}{2}^+,\ \frac{5}{2}^-,\ \cdots$ corresponding to $s=v-\frac{1}{2}$ $(v\geq 1)$.

IV. MASS FORMULA

In the previous section we obtained an interesting representation of the strong-coupling group. In this section we shall obtain a mass formula which is compatible with this representation and the mass conditions (2.7) and (2.12).

Let us first discuss a general method to obtain mass operators for a given representation of $G=K\cdot T$. Let J_i and A_α be generators of K and T, respectively. The mass conditions (2.7) and (2.12) can be written in general as

$$\Lambda_{\alpha\beta}=[A_\alpha,[\mathfrak{M},A_\beta]], \qquad (4.1)$$

$$\sum \Lambda_{\alpha\gamma}\tau^{\gamma\delta}\Lambda_{\delta\beta}=\Lambda_{\alpha\beta}. \qquad (4.2)$$

In order to satisfy the nonlinear equation (4.2), the mass operator should be quadratic in J:

$$\mathfrak{M}=\sum_{ij} J_i\Omega^{ij}J_j, \qquad (4.3)$$

where Ω^{ij} is symmetric and assumed to satisfy

$$[A_\alpha,\Omega^{ij}]=0.$$

Then,

$$\Lambda_{\alpha\beta}=-2\sum_{ij}[J_i,A_\alpha]\Omega^{ij}[J_j,A_\beta].$$

If we define ω_{ij} by

$$\omega_{ij}=\sum_{\alpha\beta}[J_i,A_\alpha]\tau^{\alpha\beta}[J_j,A_\beta], \qquad (4.4)$$

then

$$(\Lambda\tau\Lambda)_{\alpha\beta}=4\sum_{ijkl}[J_i,A_\alpha]\Omega^{ij}\omega_{jk}\Omega^{kl}[J_l,A_\beta].$$

In order to satisfy (4.2) the following condition for Ω is sufficient:

$$\sum_{jk}\Omega^{ij}\omega_{jk}\Omega^{kl}=-\tfrac{1}{2}\Omega^{il}. \qquad (4.5)$$

The problem then becomes to find those Ω's which satisfy (4.5). For a given induced representation of G, the matrix ω can be evaluated at a fixed point of the orbit (replacing A by $A^{(0)}$), which we denote by P. Ω may be solved at P using (4.5), and we denote this solution by $\Omega^{(0)}$. Since $\mathfrak{M}$ is invariant under K, Ω^{ij} must be a tensor operator of K (covariance condition). From the covariance, the general form of Ω at any point on the orbit as a function of A's can be obtained. The application of this method for the known already solved problems is given in the Appendix.

For calculational purposes, we introduce contragradient tensors with upper indices which are related to ordinary tensors by the following equations:

$$A^\alpha{}_{lm}=(-)^{1-\alpha}A_{-\alpha lm},$$
$$A_\alpha{}^{lm}=(-)^{l-m}A_{\alpha l-m}.$$

For the present problem Eqs. (2.7) and (2.12) then become

$$\Lambda^{\alpha'l'm'}{}_{\alpha lm}=[A^{\alpha'l'm'},[\mathfrak{M},A_{\alpha lm}]], \qquad (4.6)$$

$$\sum_{\beta jn}\Lambda^{\alpha'l'm'}{}_{\beta jn}\tau_j\Lambda^{\beta jn}{}_{\alpha lm}=\Lambda^{\alpha'l'm'}{}_{\alpha lm}, \qquad (4.7)$$

where

$$\tau_j=(-)^{j+1}\kappa_j. \qquad (4.8)$$

Since the internal symmetry group is assumed to be $SU_2\otimes SU_2$, ω defined by (4.4) is a 6×6 matrix, which we write

$$\omega=\begin{pmatrix} o^\alpha{}_\beta & r^\alpha{}_j \\ r^i{}_\beta & q^i{}_j \end{pmatrix}, \qquad (4.9)$$

where the 3×3 matrices o, r, and q are given by

$$\begin{aligned} o^\alpha{}_\beta&=\sum_{\gamma lm}[I^\alpha,A_{\gamma lm}]\tau_l[I_\beta,A^{\gamma lm}],\\ r^\alpha{}_j&=\sum_{\gamma lm}[I^\alpha,A_{\gamma lm}]\tau_l[J_j,A^{\gamma lm}],\\ q^i{}_j&=\sum_{\gamma lm}[J^i,A_{\gamma lm}]\tau_l[J_j,A^{\gamma lm}]. \end{aligned} \qquad (4.10)$$

Evaluating these matrix elements at P by using (3.2), (3.3), and (3.7), we obtain the following expressions after a long tedious calculation:

$$\begin{aligned} o^{(0)\alpha}{}_\beta&=\delta^\alpha{}_\beta o_0+\Delta^\alpha{}_\beta o_2,\\ r^{(0)\alpha}{}_j&=\delta^\alpha{}_j r_0+\Delta^\alpha{}_j r_2,\\ q^{(0)i}{}_j&=\delta^i{}_j q_0+\Delta^i{}_j q_2, \end{aligned} \qquad (4.11)$$

where

$$\Delta^\alpha{}_\beta=(\sqrt{30})\begin{pmatrix} \alpha & 1 & 2 \\ 1 & \beta & 0 \end{pmatrix}, \qquad (4.12)$$

[14] By a Regge recurrence we mean a spin recurrence for a fixed isospin with alternating parity.

$$o_0=\tfrac{2}{3}\sum_{l=1}^{\infty}\frac{1}{2l-1}\kappa_l|a_l|^2,$$
$$r_0=-\tfrac{1}{3}\sum_{l=1}^{\infty}\frac{l+1}{2l+1}\kappa_l|a_l|^2,$$
$$q_0=\tfrac{1}{6}\sum_{\rho=1}^{\infty}\frac{(l+1)^2(5l+2)}{(2l-1)(2l+1)},$$
$$o_2=\tfrac{1}{6}\sum_{l=2}^{\infty}\frac{l-1}{(2l-1)(2l+1)}\kappa_l|a_l|^2, \quad (4.13)$$
$$r_2=\tfrac{1}{3}\sum_{l=2}^{\infty}\frac{(l-1)(l+1)}{(2l-1)(2l+1)}\kappa_l|a_l|^2,$$
$$q_2=\tfrac{1}{12}\sum_{l=2}^{\infty}\frac{(l-1)(l+1)(5l+6)}{(2l-1)(2l+1)}\kappa_l|a_l|^2.$$

We can write $\omega^{(0)}$ in the following compact form:

$$\omega^{(0)}=1\otimes\omega_0+\Delta\otimes\omega_2, \quad (4.14)$$

where ω_0 and ω_2 are the 2×2 matrices

$$\omega_i=\begin{pmatrix} o_i & r_i \\ r_i & q_i\end{pmatrix},\quad i=0,2 \quad (4.15)$$

and 1 and Δ are, respectively, 3×3 unit matrix and the matrix (4.12).

Since

$$\Delta^2=2\times1-\Delta,$$

we may set

$$\Omega^{(0)}=1\otimes\Omega_0+\Delta\otimes\Omega_2, \quad (4.16)$$

where Ω_0 and Ω_2 are 2×2 symmetric matrices. The first term has the form which appears in the SU_2-symmetric P-wave strong-coupling problem, $[SU_2\otimes SU_2]\times T_9$. Thus, the mass operator due to this term is diagonal with respect to quantum number v. On the other hand, the second term produces in general off-diagonal matrix elements. In this paper we look for the possibility of $\Omega_2=0$ only, so that all physical isobar states are characterized by i, s, and v.

Inserting (4.16) into (4.5), we obtain

$$\Omega_0\omega_0\Omega_0=-\tfrac{1}{2}\Omega_0, \quad (4.17)$$

$$\Omega_0\omega_2\Omega_0=0, \quad (4.18)$$

where we set $\Omega_2=0$. Since $\omega_2\neq0$, as can be seen from (4.13) unless $a_l=0$ for $l\geq2$, (4.18) implies $\det\Omega_0=0$. Thus, the general form of Ω_0 is

$$\Omega_0=-\begin{pmatrix}\alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2\end{pmatrix}=-\xi\xi^T, \quad (4.19)$$

where ξ is the vector with components α and β. Inserting this into (4.17) and (4.18), we obtain

$$\begin{aligned}\alpha^2o_0+\beta^2q_0+2\alpha\beta r_0&=+\tfrac{1}{2},\\ \alpha^2o_2+\beta^2q_2+2\alpha\beta r_2&=0.\end{aligned} \quad (4.20)$$

Because of the second equation and the fact that $o_2>0$, $r_2<0$, and $q_2>0$ due to (4.12), the sign of α and β must be the same. Since these equations have a wide variety of solutions for α and β, one can get almost any positive value for α and β at will if one changes the parameters a_l continuously. Thus, we may regard α and β, being positive parameters, as being adjusted.

Inserting (4.18) into (4.16) and setting $\Omega_2=0$, we obtain the following 6×6 $\Omega^{(0)}$ matrix:

$$\Omega^{(0)}=-\alpha(\alpha-\beta)\begin{pmatrix}1&0\\0&0\end{pmatrix}-\beta(\beta-\alpha)\begin{pmatrix}0&0\\0&1\end{pmatrix}-\alpha\beta\begin{pmatrix}1&1\\1&1\end{pmatrix}. \quad (4.21)$$

From covariance, we can write down the general expression of Ω which takes value (4.12) on P. In order to obtain the mass formula of isobars, however, it is sufficient to note that the first, second, and third terms in (4.21) [when they are inserted into (4.3)] give, respectively, $\mathbf{J}^2$, $\mathbf{I}^2$, and $\mathbf{V}^2$ which are diagonal for the physical states specified by i, s, and v. Therefore, we can easily write down the mass formula of the isobars:

$$M(i,s,v)=M_0+m_0[x(x-1)s(s+1)+(1-x)i(i+1)+xv(v+1)], \quad (4.22)$$

where m_0 is a parameter proportional to $1/g^2$, while x is a positive number.

V. REMARKS

It is obvious that the strong-coupling condition (2.5) is a necessary condition for the strong-coupling theory, so that the isobar states must form a space of representations of the strong-coupling group. It is not obvious, however, whether or not the mass condition (2.12) is a necessary condition for the solution of the Chew-Low equation in the strong-coupling limit. In the P-wave strong-coupling model, it is not a necessary condition although it is, of course, a sufficient condition. Therefore, it is likely that the mass condition is only a sufficient condition also in the present model. This point needs further investigation.

The representation obtained in Sec. III may not be the only interesting representation. A systematic investigation of representations of the group is desirable.

The mass formula obtained in Sec. IV is not unique even if we assume the mass condition, since we set $\Omega_2=0$. Derivation of a general mass formula is also desired.

The final question is whether or not this model can be extended to the SU_3-symmetry scheme. The representation of the group is relatively easy to obtain,[15] but mass formulas are difficult, as in the case of the P-wave unitary symmetric model.

Note added in manuscript: C. Goebel has pointed out to me that the condition $\Omega_2=0$ is not necessary for the diagonalization of the mass operator in the i, s, v basis. For the $(v_z=0)$ representation the form

$$\Omega_2=\begin{pmatrix} 2\mu & \mu+\nu \\ \mu+\nu & 2\nu \end{pmatrix}$$

is sufficient for this purpose. The mass formula based on this restriction has the form $M(i,s,v)=M_0+as(s+1)$ $bi(i+1)+cv(v+1)$, but $a<0$, $b>0$, $c>0$. The author thanks Professor Goebel for valuable communications.

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APPENDIX

In this Appendix we apply the method described in Sec. IV to obtain the known mass operators of various models,[1,2] i.e., (i) $SU_2\times T_3$, (ii) $[SU_2\otimes SU_2]\times T_9$, and (iii) $[SU_2\otimes SU_3]\times T_{27}$.

First let us notice the following simplification. If one finds $\tilde{\Omega}$ which satisfy

$$\sum_j \tilde{\Omega}^{(0)ij}[A_\alpha,J_j]=\sum_i [A_\alpha,J_i]\tilde{\Omega}^{(0)ij}=0, \quad \text{(A1)}$$

then

$$[\textstyle\sum_{ij}J_i\tilde{\Omega}^{ij}J_j,A_\alpha]=0 \quad \text{(A2)}$$

and

$$\tilde{\Omega}\omega=\omega\tilde{\Omega}=0.$$

Therefore, Eq. (4.5) becomes

$$\Omega'\omega\Omega'=-\tfrac{1}{2}(\Omega'+\alpha\tilde{\Omega}), \quad \text{(A3)}$$

where

$$\Omega=\Omega'+\alpha\tilde{\Omega} \quad \text{(A4)}$$

and α is an arbitrary parameter. Because of (A2), the second term of (A4) when inserted into (4.3) does not contribute to the mass differences of isobars. Thus, the mass operator is given by

$$\mathfrak{M}=\sum_{ij} J_i\Omega'^{ij}J_j. \quad \text{(A5)}$$

[15] A representation of the group has been obtained, which contains the representation of Sec. III and Goebel's representation of the SU_3-symmetric P-wave strong-coupling group (Ref. 3). It will be published elsewhere together with a mass formula.

(i) $SU_2\times T_3\,[J_i,A_i,\,i=1,2,3]$.
In this case, ω is given by

$$\omega_{ij}=-\delta_{ij}A^2+A_iA_j.$$

Since $\sum_i A_i[J_i,A_j]=0$, one can set

$$\tilde{\Omega}_{ij}=A_iA_j.$$

Thus, $\Omega'_{ij}=(2/A^2)\delta_{ij}$ is the solution of (A4), which leads to $m=(2/A^2)J^2$.

(ii) $[SU_2\otimes SU_2]\times T_9\,[J_i\,(i=1,2,3), I_\alpha\,(\alpha=1,2,3), A_{i\alpha}]$. The 6×6 matrix ω is given by

$$\omega=\begin{pmatrix} \omega_{ij} & \omega_{i\beta} \\ \omega_{\alpha i} & \omega_{\alpha\beta} \end{pmatrix},$$

where

$$\omega_{ij}=-\delta_{ij}A^2+\sum_\alpha A_{i\alpha}A_{j\alpha},$$

$$\omega_{\alpha\beta}=-\delta_{\alpha\beta}A^2+\sum_i A_{i\alpha}A_{i\beta},$$

$$\omega_{\alpha j}=-\sum_{\beta\gamma il}\epsilon_{\alpha\beta\gamma}A_{i\gamma}\epsilon_{jil}A_{l\beta}.$$

Evaluating it on the fixed point determined by

$$A^{(0)}{}_{i\alpha}=(\tfrac{1}{3}A^2)^{1/2}\delta_{i\alpha},$$

we obtain

$$\omega^{(0)}=-\tfrac{2}{3}A^2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

where 1 is 3×3 unit matrix. Since $\tilde{\Omega}^{(0)}$ is given by

$$\tilde{\Omega}^{(0)}=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

we can set

$$\Omega'^{(0)}=\begin{pmatrix} a\times1 & 0 \\ 0 & b\times1 \end{pmatrix}.$$

Using (A4), we obtain

$$a^2+ba=\kappa a,$$
$$b^2+ba=\kappa b,\quad \kappa=3/4A^2.$$

So,

$$a=n=0 \quad \text{(A6)}$$

and

$$a+b=\kappa \quad \text{(A7)}$$

are solutions. The former solution (A6) is a trivial one and the latter solution (A7) leads to the mass formula of Ref. 1.

(iii) $[SU_2\otimes SU_3]\times T_{27}\ [J_i\,(i=1,\ 2,\ 3),\ F_\alpha\,(\alpha=1,$ $2,\cdots,8),\ A_{i\alpha}]$. The 11×11 ω matrix evaluated on the point determined by

$$A_{i\alpha}=(\tfrac{1}{3}A^2)^{1/2}\delta_{i\alpha},\quad \text{for}\quad \alpha=1,2,3$$
$$A_{i\alpha}=0,\qquad \text{for}\quad \alpha>3$$

is

$$\omega=-\tfrac{2}{3}A^2\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & \tfrac{3}{8} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{matrix} i\\ \alpha=1,2,3\\ \alpha=4,5,6,7\\ \alpha=8\end{matrix}$$

So,

$$\tilde{\Omega}^{(0)}=\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \beta\end{bmatrix},$$

where β is an arbitrary parameter. Setting

$$\Omega'^{(0)}=\begin{bmatrix}a^2 & -ab & 0 & 0\\ -ab & b^2 & 0 & 0\\ 0 & 0 & \tfrac{3}{8}c^2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

we obtain

$$a^2=\kappa(a+\alpha),$$
$$-ab=\kappa\alpha,$$
$$b^2=\kappa(b+\alpha),$$
$$\tfrac{3}{8}c^2=\kappa c,$$
$$\alpha\beta=-c,\quad \kappa=3/4A^2.$$

The solution with the condition $a\neq 0$, $b\neq 0$, $c\neq 0$ is

$$a+b=\kappa,\ c=8/3,$$

which leads to Goebel's mass operator[3,7]

$$\mathfrak{M}=aJ^2+(b-(8/3)\kappa)(3/A^2)A_{i\alpha}A_{i\beta}F_\alpha F_\beta+(8/3)\kappa F_\alpha F_\alpha.$$

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Modified Two-Pion-Exchange Model for Double-Charge-Exchange Processes

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The eikonal method, applied to the problem of coupled channels of different mass in potential theory, is used to derive an expansion for the scattering amplitudes in powers of the potentials between different channels. Just as single-exchange processes are given approximately by the first-order term in the Born series with absorptive modifications, double-exchange processes are given approximately by the modified imaginary part of the second-order term. This result is taken over to a relativistic theory, with one-pion-exchange "potentials" connecting different channels, and used to estimate the differential cross section for the double-charge-exchange process $\pi^-+p\rightarrow\pi^++\Delta^-$.

I. INTRODUCTION

THE modified one-particle-exchange model[1–4] has had considerable success in explaining high-energy inelastic scattering. This has been for the most part limited to those cases where the exchanged particle is a pion, presumably because of the long range of the one-pion-exchange force.[5] In this paper we wish to suggest an extension of this method to a particular class of two-pion-exchange processes—those in which the two exchanged pions carry two units of charge.[6] We make this restriction because the two-pion-exchange amplitudes will in general be very small at high energy, and could hardly be seen above a background due to one-pion exchange if this could lead to the same final state.

Our original expectation was that the amplitude for double-charge exchange should be closely related to the product of two single-charge-exchange amplitudes. This we have found to be true in coupled-channel potential theory using the eikonal method,[7] but only after several approximations have been made. These approximations should be reasonably good at high energy,[8] but because of them we can expect only qualitative agreement between our calculations and experiment even if our general line of reasoning is correct.

We begin Sec. II by introducing the eikonal method in potential theory for coupled channels of different mass. To a large extent this merely reproduces results obtained by Durand and Chiu[9] using the WKB approximation; it is included here to make clear our nota-

[1] For a recent review, with many additional references, see A. C. Hearn and S. D. Drell, in *High-Energy Physics* (Academic Press Inc., New York, 1967), Vol. II, p. 219.
[2] N. J. Sopkovich, Nuovo Cimento **26**, 186 (1962).
[3] J. D. Jackson, Rev. Mod. Phys. **37**, 484 (1965).
[4] J. D. Jackson, J. T. Donohue, K. Gottfried, R. Keyser, and B. E. Y. Svensson, Phys. Rev. **139**, B428 (1965).
[5] A clear discussion of this point is contained in L. Durand, Phys. Rev. Letters **19**, 1345 (1967).
[6] A different treatment of two-pion exchange without this restriction is contained in P. Smrz and H. C. von Baeyer, Nuovo Cimento **51**, 889 (1967).
[7] R. J. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1959), Vol. I, p. 315.
[8] In potential theory these approximations could be checked by exact numerical calculations similar to those of D. B. Lichtenberg, J. Will, and D. Ellis, Phys. Rev. **143**, 1375 (1965).
[9] L. Durand and Y. T. Chiu, Phys. Rev. **139**, B646 (1965).