Inelastic Contributions and *D*-Function Poles*

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It is shown that the nonequivalence between the multichannel $\mathbf{ND}^{-1}$ method and the one-channel N/D method incorporating inelasticity can be advantageously exploited in a fully dynamical scheme. It is demonstrated that the D-function poles needed for equivalence can be generated out of the inelastic cuts and that the pole parameters may be calculated from models of multichannel scattering. The result is an effective one-channel N/D method, applicable to a class of problems wherein the incorporated inelastic effects are dynamically calculable.

I. INTRODUCTION

Several authors[1] have shown that the one-channel N/D equations with inelastic unitarity[2] and the multichannel $\mathbf{ND}^{-1}$ equations[3] can possess nonequivalent solutions. The effect is as follows: The multichannel method yields an amplitude T_{ij}, calculated from a given potential B_{ij}; an elasticity factor η can be extracted from this solution. An elastic amplitude T may be calculated from B_{11} and η, e.g., by the Frye-Warnock equations,[4] and it may happen that $T\neq T_{11}$. This phenomenon has been analyzed[5] and some criteria for the nonequivalence have been presented. In general, to achieve equivalence, the D function of the one-channel problem must be supplemented by poles to account for the possible corresponding zeros occurring on the physical sheet of T_{11}. This has been referred to as a resolution of the Castillejo-Dalitz-Dyson (CDD) ambiguity.[6]

In what follows, it is assumed that the complete, multichannel calculation of a scattering process is free of CDD ambiguities. However, any calculation based on a truncated unitarity condition can contain, in general, D-function poles. Further, the determination of the pole parameters cannot be made within the truncated problem, but rather by making contact with the multichannel problem. In other words, if a pole is needed in the one-channel D function to simulate inelastic contributions, then a multichannel calculation of the pole parameters must be possible and one can claim to have ascertained the dynamical origin of the D-function pole. It is the purpose of this paper to develop a one-channel N/D method which provides for the possible occurrence of D-function poles originating in inelastic contributions. The motion of these poles will be described by continuation in the inelastic couplings,[7] and it will be shown how the pole parameters can be calculated in terms of multichannel dynamical models. The reader may wonder what purpose it serves to derive such a single-channel parametrization if it is necessary to solve the multichannel problem first. The method presented here allows one to determine the D-function pole parameters in a gross approximation and to reserve the details of the elastic problem for more thorough treatment later.

It is quite clear that if one can establish the existence of a D-function pole, by one means or another, then its occurrence can be exploited in an often important way in a dynamical calculation. The very existence of a D-function pole is a strong dynamical statement. In particular, if the residue is of the correct sign, it can demand the existence of a bound state or a resonance. An immediate application occurs in P_{11} pion-nucleon scattering; such a pole is needed in a single-channel model to reproduce the zero of the phase around 175 MeV (pion lab kinetic energy).[8] P_{11} calculations have in fact been made,[9,10] but with CDD parameters determined from experimental data. However, such a procedure falls short of a fully dynamical scheme. In fact, one of the objectives of this paper is to close this gap.

II. EFFECTIVE N/D METHOD

We refer to the elastic channel as channel 1 and we take for our starting point the familiar multichannel construction[3] of the elastic partial-wave amplitude:

$$M_{11}=F/\mathfrak{D}, \tag{1}$$

in which

$$F=(\mathbf{N}\,\mathrm{adj}\mathbf{D})_{11} \tag{2}$$

and

$$\mathfrak{D}=\det\mathbf{D}. \tag{3}$$

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[6] L. Castillejo, R. Dalitz, and F. Dyson, Phys. Rev. 101, 453 (1956).

[7] The motion of D-function poles has also been described in terms of analytic continuation in the angular momentum. See, e.g., J. B. Hartle and C. E. Jones (Ref. 5) and Ann. Phys. (N. Y.) 38, 348 (1966).

[8] See, e.g., L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. 138, B190 (1965).

[9] D. Atkinson and M. B. Halpern, Phys. Rev. 150, 1377 (1966).

[10] B. I. Sheth and A. Tubis, Phys. Rev. 154, 1322 (1967).

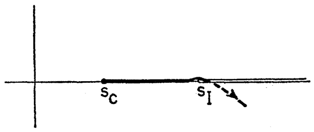

FIG. 1. The zero of F leaving the physical sheet as the inelastic effects are diminished.

In the complex s plane the elements of $\mathbf{N}$ have the left cuts of $\mathbf{M}$ on which

$$\mathrm{Im}N_{ij}=\sum_k \mathrm{Im}_L M_{ik}D_{kj}. \qquad (4)$$

The elements of $\mathbf{D}$ have the right cuts on which the relation

$$\mathrm{Im}D_{ij}=-\pi\rho_i N_{ij} \qquad (5)$$

is prescribed to guarantee the unitarity of $\mathbf{M}$. We consider only two-body channels; ρ_i is the phase-space factor for channel i occurring in the unitarity relation for $\mathbf{M}$:

$$\mathrm{Im}_R M_{ij}=\sum_k \pi M_{ik}\rho_k M_{kj}^*. \qquad (6)$$

We define the integral over the left cut by

$$B_{ij}=\frac{1}{\pi}\int_L \frac{ds'}{s'-s}\mathrm{Im}_L M_{ij}. \qquad (7)$$

Finally, as indicated in the Introduction, we assume that this $\mathbf{ND}^{-1}$ construction is free of any CDD ambiguity.[6]

We now wish to construct from the $\mathbf{ND}^{-1}$ solution a one-channel N/D method for $M=M_{11}$, the elastic amplitude. We write Eq. (1) in the form

$$M=\frac{N+(F-PN)/P}{D+C}, \qquad (8)$$

where

$$C=(\mathfrak{D}-PD)/P. \qquad (9)$$

We have introduced three new functions N, D, and P on which the formulation of the effective one-channel method is to be based. N is required to have the left cut of M (i.e., of $B=B_{11}$). We assume D to have the elastic right cut and on it we require

$$\mathrm{Im}D=-\pi\rho_1 N. \qquad (10)$$

We assume P to have the inelastic right cuts. Thus this method differs from that of Frye and Warnock[2] in that their N function incorporates the inelastic cuts.

If all of elastic unitarity is contained in D, then for $s<s_I$, the inelastic threshold, we must have $\mathrm{Im}C=0$. Since P is supposed to be real in this region, we have

$$\begin{aligned}0&=\mathrm{Im}\mathfrak{D}-P\,\mathrm{Im}D\\ &=-\pi\rho_1(F-PN), \quad s_1<s<s_I,\end{aligned} \qquad (11)$$

where s_1 is the elastic threshold.

It follows that

$$F=PN \qquad (12)$$

throughout the cut plane, so that Eq. (8) becomes simply

$$M=N/(D+C). \qquad (13)$$

Thus we require on the left cut

$$\mathrm{Im}N=\mathrm{Im}_L M(D+C). \qquad (14)$$

The function C contains the inelastic effects and, as such, is the key to exploiting equivalence. To construct C we need its discontinuity; for $s>s_I$,

$$\begin{aligned}\mathrm{Im}C^{-1}&=[\mathrm{Im}(P\mathfrak{D}^*)+|P|^2\,\mathrm{Im}D]/|\mathfrak{D}-PD|^2\\ &=\left|\frac{\mathfrak{D}}{\mathfrak{D}-PD}\right|^2\frac{\mathrm{Im}M}{N}+\left|\frac{P}{\mathfrak{D}-PD}\right|^2\mathrm{Im}D. \end{aligned} \qquad (15)$$

Thus, for $s>s_I$ we have

$$\begin{aligned}\mathrm{Im}C&=-\mathrm{Im}D-|\mathfrak{D}/P|^2(\mathrm{Im}M)/N\\ &=\pi\rho_1 N-N\,\mathrm{Im}M/|M|^2\\ &=-N\sum_{i>1}\pi\rho_i|M_i'/M|^2. \end{aligned} \qquad (16)$$

In Eq. (16), M_i' is the production amplitude

$$M_i'=F_i'/\mathfrak{D}, \qquad (17)$$

where

$$F_i'=(\mathbf{N}\,\mathrm{adj}\mathbf{D})_{1i}=(\mathbf{N}\,\mathrm{adj}\mathbf{D})_{i1}. \qquad (18)$$

Therefore we can write

$$\mathrm{Im}C=-N\sum_{i>1}\pi\rho_i|F_i'/F|^2, \quad s>s_I, \qquad (19)$$

or, alternatively, in terms of the elasticity η,

$$\mathrm{Im}C=-\frac{N}{|M|^2}\frac{1-\eta^2}{4\pi\rho_1}, \qquad (20)$$

where $\eta e^{2i\delta}=1+2\pi i\rho_1 M$, and δ is the real part of the phase shift.

The following is the representation for C:

$$C=\frac{1}{\pi}\int_I \frac{ds'}{s'-s}\mathrm{Im}C, \qquad (21)$$

in which I denotes a contour on top of the inelastic cuts and where $\mathrm{Im}C$ is given by (19) or (20). Equation (21) exhibits C to be a function which vanishes as the inelastic effects are turned off; when this happens, Eq. (13) reverts to $M=N/D$. Strictly speaking, the right side of (21) should also contain a constant C_∞ which goes to zero as the inelastic effects are turned off. This constant can be consistently absorbed in the normalization of D to 1 at infinity, and so we have left it out to begin with. Thus we have the following set of equations:

$$N=\frac{1}{\pi}\int_L \frac{ds'}{s'-s}\mathrm{Im}B(D+C), \qquad (22)$$

$$D=1-\int_R \frac{ds'}{s'-s}\rho_1 N, \qquad (23)$$

$$C=-\frac{1}{\pi}\int_I \frac{ds'}{s'-s}\frac{1-\eta^2}{4\pi\rho_1}\frac{N}{|M|^2}, \tag{24}$$

together with Eq. (13). For given ImB and η, these equations comprise the analog, in the method presented here, of the Frye-Warnock equations. Such a system does not appear to be very tractable to solution, and it would seem to have served little purpose to set up this machinery for N, D, and C. To implement these equations and extract a useful approximation scheme from them, we now consider the possibility of using an approximate multichannel solution for F and F_i', inserted into Eqs. (19) and (21) for C. In particular, if the most important contribution to C can be extracted by means of a gross treatment of the multichannel problem, then some progress will have been made by this procedure. Further, multichannel problems can often be handled only in terms of gross approximations (such as in pole models). We can attempt to apply such methods to obtain C in a rough way and then treat the elastic forces (i.e., B) in as much detail as we wish when we turn to solving for N and D. A procedure such as this should provide a better over-all method of calculating M then would a gross approximation to all of $\mathbf{ND}^{-1}$. This effective N/D method will be feasible when C can be shown to have a pole on the physical sheet. We will then have fully exploited the equivalence issue, discussed, e.g., by Bander, Coulter, and Shaw,[1] when this is the case because the parameters in the pole approximation to C can be obtained in a crude calculation of the coupled-channel problem. We will discuss some pole-parameter models in Sec. III. At this point we will show how the pole approximation to C is extracted for those problems where a pole can occur.

In Sec. III we shall demonstrate, in terms of multichannel models, how the function F can have a real zero. The property of F which we assume to apply here and which we shall exhibit in Sec. III is that there exists a point s_c, where $F=0$, which migrates toward the right cut of F as the inelastic couplings are reduced. Given that F has a real zero which behaves in this way (for a given problem), we can show how to extract the pole approximation to C.

F is unlikely to have a real zero for $s>s_I$ since this would require Re$F=0$ and Im$F=0$ simultaneously. Therefore if we have a real zero for $s<s_I$ which migrates to the right, it must leave the physical sheet through the lowest inelastic cut and end up on sheet II,[11] as shown in Fig. 1. We can continue F into sheet II and, introducing the channel-2 phase-space factor, we can write

$$F^{\mathrm{II}}=F+2\pi i\rho_2\bar{F}, \tag{25}$$

where $\bar{F}$ is defined in terms of the analytic continuation of the discontinuity of F across the first inelastic cut below the second inelastic threshold. (For example, in a two-channel problem $\bar{F}=\det\mathbf{N}$.) All we need to know about $\bar{F}$ is that it is real on the right up to the second inelastic threshold. For $s<s_I$, F^{II} is real;

$$F^{\mathrm{II}}=F-2\pi|\rho_2|\bar{F}. \tag{26}$$

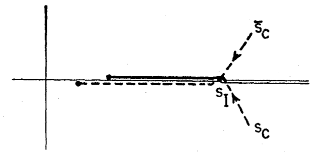

FIG. 2. Motion of the zeros of F and F^{II} as the inelastic effects are increased. Dashed lines represent motion on sheet II.

Thus if $F=0$ at a point s_c on sheet I, then F^{II} will have a real zero at a nearby point $\bar{s}_c$ on sheet II. Since $\rho_2\to 0$ at s_I, $\bar{s}_c$ must migrate to the right and arrive at s_I when s_c does. When s_c passes into sheet II, $\bar{s}_c=s_c^*$. These observations are summarized in Fig. 2.

We have already seen that C is a function which arises by virtue of the inelastic effects. We now assume that we can continue in the inelastic couplings and that, to begin with, they are such that the zero of F is on sheet II. We can rewrite (19) as

$$\mathrm{Im}C=-\pi N\rho_2|F_2'|^2/FF^{\mathrm{II}}, \tag{27}$$

a formula valid for $s\gtrsim s_I$. We see that the analytic continuation of ImC has a pole at s_c which deforms the contour I in the representation (21) as the inelastic coupling is increased. Note that $\bar{s}_c$ does not deform the contour. Therefore, if we take the point of view that C is definable by continuation in the inelastic couplings and if F has a zero which can be brought out of sheet II, then we have succeeded in defining the desired pole approximation to (21):

$$C=-\int_{I_c}\frac{ds'}{s'-s}\frac{N\rho_2|F_2'|^2}{F^{\mathrm{II}}R(s'-s_c)}, \tag{28}$$

where the contour I_c is a small clockwise circle around s_c and where

$$R=(dF/ds)_{s=s_c}. \tag{29}$$

We have discarded the background integral over I in writing (28); the result is

$$C=-\frac{1}{R(s-s_c)}\left[\frac{N|F_2'|^2}{\bar{F}}\right]_{s=s_c}. \tag{30}$$

It follows that

$$C=N(s_c)\mathfrak{D}(s_c)/R(s-s_c), \tag{31}$$

since the relation $|F_2'|^2=-\bar{F}\mathfrak{D}$ holds where $F=0$. We also recognize that Eq. (31) must hold if Eqs. (9) and (12) are implemented by the knowledge that F has a zero.[12] When this is the case, the effective N/D method is

[11] Hartle and Jones (Ref. 7) analyze this sort of migration but in terms of continuation in the angular momentum.

[12] Note that even for s_c above the elastic threshold $\mathfrak{D}(s_c)$ is real since Im$\mathfrak{D}$ is proportional to F.

defined by (13) and (31) together with the relations N and D, Eqs. (22) and (23).

Note that the pole in C, in the integrand of N, cannot reach the left cut. This is because the zero of F, which causes the pole, is bounded away from the left branch point where F has a logarithmic singularity. Conceivably, the pole could go into the complex plane and drop down on the cut, but only if there were a partner pole which followed a complex conjugate path in the opposite half-plane.

Another route to the effective N/D solution is possible when we use an approximate multichannel solution leading to the pole approximation for C. To see this, we rewrite Eq. (22) as

$$N=\frac{1}{\pi}\int_L \frac{ds'}{s'-s}\,\mathrm{Im}B\,\mathfrak{D}/P\,, \tag{32}$$

in which we can write

$$\frac{1}{P}=1-\frac{1}{\pi}\int_I \frac{ds'}{s'-s}\frac{\mathrm{Im}F}{|F|^2}N.$$

For $1/P$ we adopt the same approach as for C and introduce an approximate $\mathbf{ND}^{-1}$ solution to obtain F. Again, if the F so obtained has a real zero, then $1/P$ can be approximated by a pole:

$$1/P=1+N(s_c)/R(s-s_c)\,. \tag{33}$$

If this expression is returned to Eq. (32), we get

$$N=\beta+N(s_c)[\beta-\beta(s_c)]/R(s-s_c)\,, \tag{34}$$

in which

$$\beta=\frac{1}{\pi}\int_L \frac{ds'}{s'-s}\,\mathrm{Im}B\,\mathfrak{D}$$

and

$$N(s_c)=\beta(s_c)/[1-\beta'(s_c)/R]. \tag{35}$$

From this result D and C can be calculated. Thus by taking this route we see that the assumed approximate multichannel solution, leading to a real zero of F, has succeeded in generating the solution to the effective one-channel problem.

The two routes to $N/(D+C)$ are different, corresponding to two different ways of implementing the approximate $\mathbf{ND}^{-1}$ solution and of exploiting the D-function pole. The choice of which method to follow should depend on the particular problem under investigation.

III. MODELS FOR D-FUNCTION POLES

In this section we must substantiate the claim that F can have a real zero which moves out of sheet II as the inelasticity is turned on. We will do this in terms of a two-channel illustration based on simple pole models of $\mathbf{ND}^{-1}$.

If we define

$$(\mathrm{Im}B_{ij})=-\pi\begin{pmatrix} a\delta(s-z_0) & b\delta(s-z_1)\\ b\delta(s-z_1) & c\delta(s-z_2)\end{pmatrix}, \tag{36}$$

then a little effort yields $\mathfrak{D}$ and $F=N_{11}D_{22}-N_{12}D_{21}$.

$$\begin{aligned}F=\Delta^{-1}\{&[a/(s-z_0)][1-c(s-z_2)Q_{22}^{(2)}]+b^2[Q_{11}^{(2)}\\&+c(z_2-z_1)(Q_{11}^{(2)}Q_{221}^{(2)}-Q_{21}^{(2)}Q_{112}^{(2)})]\\&\times[(a/(s-z_0))(z_1-z_0)(s-z_1)Q_{110}^{(1)}\\&\qquad+1-a(z_1-z_0)Q_{001}^{(1)}]\}\,,\end{aligned} \tag{37}$$

$$\begin{aligned}\mathfrak{D}=\Delta^{-1}\{&[1-a(s-z_0)Q_{00}^{(1)}][1-c(s-z_2)Q_{22}^{(2)}]\\&-b^2(s-z_1)^2[Q_{11}^{(1)}+a(z_0-z_1)\\&\times(Q_{11}^{(1)}Q_{001}^{(1)}-Q_{01}^{(1)}Q_{110}^{(1)})][Q_{11}^{(2)}+c(z_2-z_1)\\&\times(Q_{11}^{(2)}Q_{221}^{(2)}-Q_{21}^{(2)}Q_{112}^{(2)})]\}\,.\end{aligned} \tag{38}$$

The quantity Δ is an irrelevant constant related to the normalization $\mathfrak{D}(\infty)=1$:

$$\begin{aligned}\Delta=(1+aI_{00}^{(1)})(1+cI_{22}^{(2)})-b^2[&I_{11}^{(1)}(1+aI_{00}^{(1)})\\&-aI_{10}^{(1)2}][I_{11}^{(2)}(1+cI_{22}^{(2)})-cI_{12}^{(2)2}].\end{aligned} \tag{39}$$

In (37)–(39) the I's and Q's are the dispersion integrals

$$I_{ij}^{(\alpha)}=\int ds'\rho_\alpha/(s'-z_i)(s'-z_j)\,, \tag{40}$$

$$Q_{ij}^{(\alpha)}=\int ds'\rho_\alpha/(s'-s)(s'-z_i)(s'-z_j)\,, \tag{41}$$

and

$$Q_{ijk}^{(\alpha)}=\int ds'\rho_\alpha/(s'-z_i)(s'-z_j)(s'-z_k)\,. \tag{42}$$

It helps to recognize the following elementary pole-model D functions:

$$\begin{aligned}D^{(1)}&=1-a(s-z_0)Q_{00}^{(1)}\,,\\ D^{(2)}&=1-c(s-z_2)Q_{22}^{(2)}\,,\\ D^{(12)}&=1-b^2(s-z_1)^2Q_{11}^{(1)}Q_{11}^{(2)}\,.\end{aligned} \tag{43}$$

$D^{(1)}$ and $D^{(2)}$ occur in uncoupled models of elastic scattering in channels 1 and 2, respectively. $D^{(12)}$ is the two-channel D function driven by a purely off-diagonal force.[13]

A simple case can be considered first in which we set $z_0=z_1=z_2$. The results are

$$\begin{aligned}F\Delta&=[a/(s-z_0)]D^{(2)}+b^2Q_{00}^{(2)}\,,\\ \mathfrak{D}\Delta&=D^{(1)}D^{(2)}-b^2(s-z_0)^2Q_{00}^{(1)}Q_{00}^{(2)}\,.\end{aligned} \tag{44}$$

Let a be fixed and positive. If b is fixed and c is positive, we can think of increasing c until a channel-2 bound state occurs. As c is further increased, a zero of F comes out of the inelastic cut. Note that at the zero, s_c, $\mathfrak{D}\Delta=D^{(2)}(s_c)$, which, like the factor R of Eq. (31), is a negative number in this case.

Equations (44) are also interesting when there is repulsion in channel 1 ($a<0$). As the inelastic couplings are turned on, a zero of F comes out of the inelastic cut; in this case both R and $\mathfrak{D}(s_c)\Delta$ are positive. This conclusion holds whether there is a channel-2 bound state or not. If one exists, it must lie to the right of the zero of F.

[13] See, e.g., L. F. Cook and B. W. Lee, Phys. Rev. 127, 297 (1962).

In general, with $z_0 \neq z_1 \neq z_2$, a channel-2 bound state produces a D-function pole nearby if b^2 is weak; this is clear from expression (37) written as

$$F\Delta = [a/(s-z_0)]D^{(2)} + O(b^2). \quad (45)$$

This phenomenon has already been noted by many others.[1,5]

Another interesting set of circumstances can arise in which nothing is going on in the uncoupled second channel. This is the case in which $c=0$. For this illustration we rewrite (37) as

$$F\Delta = [a/(s-z_0)]D^{(12)} + b^2 Q_{11}{}^{(2)} \times [D^{(1)} + 2a(s-z_1)Q_{01}{}^{(1)}], \quad (46a)$$

or as

$$F\Delta = [a/(s-z_0)]D^{(12)} + b^2 Q_{11}{}^{(2)} \times [D^{(1)}(z_1) + a(s-z_1)Q_{01}{}^{(1)}]. \quad (46b)$$

We give two expressions because of the significance, for our purposes, of a particular choice for z_1. We have in mind an instance in which it is appropriate to put this off-diagonal driving pole at s_1, the elastic threshold. This procedure is quite reasonable for a large class of inelastic couplings between channels 1 and 2, typically, whenever an unstable particle appears in channel 2. In (46a) and (46b) this choice has the effect of making $D^{(12)}(s_1)=1$, of making the third term small near s_1, and of making $D^{(1)}(z_1=s_1)<0$ if there exists a bound state in the uncoupled first channel. We can now consider a variety of cases, each involving $a>0$. First suppose that a is not large enough to produce a channel-1 bound state; then $D^{(1)}(z_1)>0$. The combined second and third terms of (46b) will be positive below the second threshold[14] unless the integral

$$I_0{}^{(1)} = \int ds' \rho_1/(s'-s)(s'-z_0)$$

has begun to decrease to the point where, possibly, $Q_{01}{}^{(1)}<0$ for $s<s_I$ [note that $(s-z_1)Q_{01}{}^{(1)} = I_0{}^{(1)} - I_{01}{}^{(1)} = I_0{}^{(1)} - I_0{}^{(1)}(s_1)$]. If b^2 is large enough, however, $D^{(12)}$ can vanish and a zero of F can occur at a larger value of s. In this case the inelastic coupling is strong enough to serve as a resonance mechanism such as that used by Cook and Lee.[13] At s_c (the zero of F), the expression

$$\mathfrak{D}\Delta = -b^2(s_c-z_0)[D^{(1)}(s_1)]^2 Q_{11}{}^{(2)}(s_c)/a$$

is negative, like R. The situation is more interesting if a is large enough to produce an uncoupled channel-1 bound state. Now it is possible for a zero of F to come out of the inelastic cut with increasing b^2 without the necessity of $D^{(12)}$ vanishing anywhere. This case is shown in Fig. 3. If b^2 is further increased until $D^{(12)}$ can vanish, then the zero of F will have moved to the left and the zero of $D^{(12)}$ will occur to its right.

[14] Note that for $s<s_2$, the second threshold, $\mathrm{Im}F=0$, so that in the region between s_1 and s_2 it is valid to consider only the real parts of separate terms.

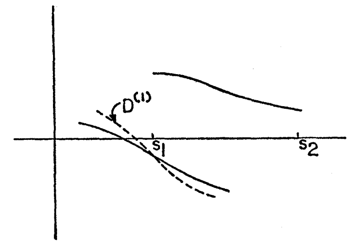

FIG. 3. Behavior of the terms in Eq. (46a). The upper and lower solid curves are, respectively, the first term and the combined second and third terms.

It is clear that the general model defined by Eq. (36), with a, b, and c nonzero, and with distinct poles z_i, is rich enough to yield a zero of F for a wide variety of physical circumstances. Some of these criteria for getting a D-function pole have already appeared in the literature. We wish to stress the fact that each circumstance is amenable to calculation in which the D-function pole parameters can be determined from specific models of multichannel scattering. In this way the origin of a D-function pole can be quantitatively assigned to a specific model involving a definite choice of inelastic states and couplings. A given problem will in general suggest the choice to make in much the same way in which inelastic mechanisms have been deduced to explain resonances.

We have indicated how s_c, R, and $\mathfrak{D}(s_c)$ can be calculated. The only remaining parameter in Eq. (31) is $N(s_c)$. Since the other factors in C have been determined from pole models in this section, we can pursue this approach to obtain $N(s_c)$. If we use $\mathrm{Im}B = -\pi a \times \delta(s-z_0)$ and manipulate Eqs. (22), (23), and (31), we get

$$N(s_c) = [a/(s_c-z_0)]\{1 + a[I_{00}{}^{(1)} + \mathfrak{D}(s_c)/R(s_c-s_0)^2]\}^{-1}. \quad (47)$$

Thus the function C is fully determinable from the pole model of $\mathbf{ND}^{-1}$. Of course, it depends on the problem at hand whether this determination of $N(s_c)$ is the most appropriate. Recall that we had an earlier formula for it, Eq. (35).

IV. CONCLUSIONS

It has been recognized for some time that inelastic effects are essential for the understanding of many scattering problems. A classic example already mentioned is P_{11} πN scattering. With Chew's mechanism[15] alone, the nucleon bound state is obtained but the phase shift does not become positive. Atkinson and Halpern[9] have given theoretical arguments for the need for a D-function pole, and, indeed, the data require that inelastic effects should be simulated in this way. They and others[10] go on to introduce such a pole phenomenologically. Such a procedure is not a fully dynamical

[15] G. F. Chew, Phys. Rev. Letters 9, 233 (1962).

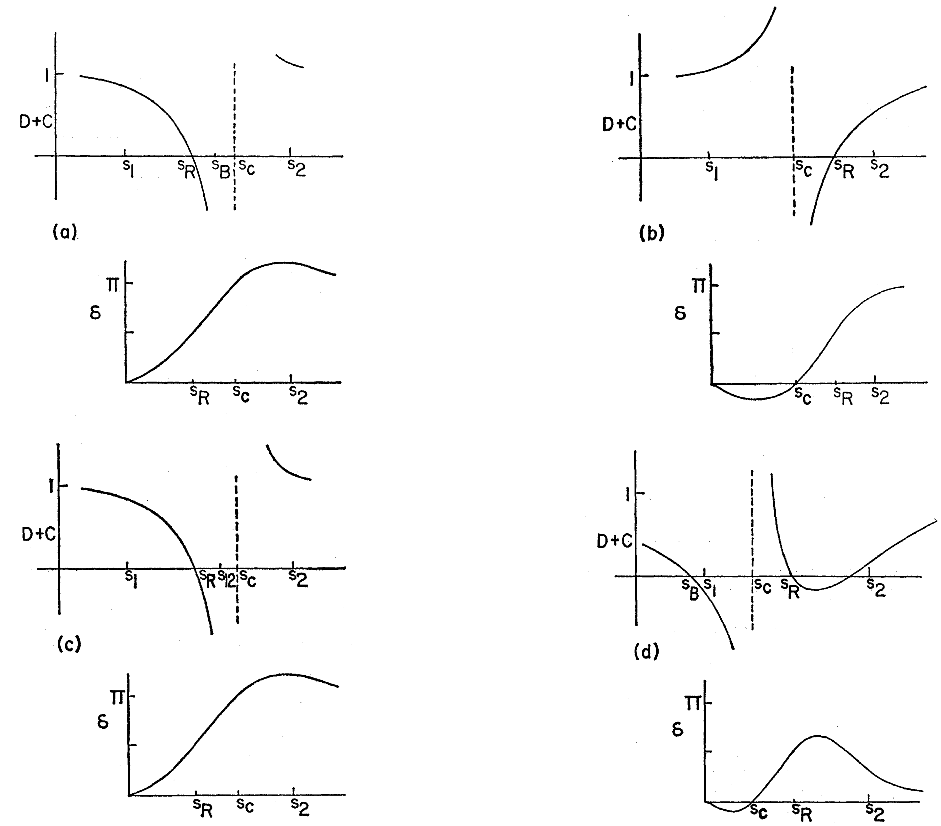

FIG. 4. The denominator function and phase shift for several cases: (a) when there exists a channel-2 bound state at s_B, (b) when there is elastic repulsion, (c) when there exists a zero of $D^{(12)}$ at the point s_{12}, (d) when there exists a channel-1 bound state between the points s_B and s_1 (this bound state is then shifted to position s_B by the inelasticity). The point s_R shown in the figures indicates where a resonance is possible in the results.

one, however, and the results of this paper would supply the missing link. Specifically, we have shown how D-function pole parameters can be calculated dynamically. These methods could be used, e.g., to study the validity of the conjecture[9] that the πN^* inelastic state is responsible for the P_{11} D-function pole.

The method presented here would seem to have an advantage over the usual approaches to solving the multichannel $\mathbf{ND}^{-1}$ problem. As we have indicated, gross methods can be used to derive the D-function pole parameters; the details of the elastic forces can be reserved for later treatment by more refined methods.

Clearly the method, as given here, satisfies only elastic unitarity. Thus we are entitled to simulate, by a D-function pole, only those inelastic effects which do not play a crucial role in the unitarity condition. Those inelastic channels which the data reveal to have large branching ratios should be incorporated in a multichannel way. We believe that this method can be readily extended to cover this case. If we again appeal to the P_{11} example, we would conclude that the πN and "σ"N channels should be treated with coupled-channel unitarity and that the πN^* channel, if it is relevant, may be simulated.

We have cited in Sec. III several circumstances leading to D-function poles. We will conclude by indicating what their effects are expected to be on the solution of the effective N/D method. In Fig. 4 we give the behavior of the denominator function $D+C$, and also the phase shift in each case. Evidently many resonance-generating possibilities exist and are worth pursuing by these methods. In particular, for πN P_{11}, dynamical circumstances should exist to yield the results of Fig. 4(d); for πN P_{33}, Fig. 4(a) or Fig. 4(c) would be the desired result. In the latter case note that δ crosses π and then asymptotically approaches π from above. Recent phase-shift analysis[16] shows that this trend is actually realized by the data.

[16] P. Bareyre, C. Brickman, A. V. Stirling, and G. Villet, Phys. Letters 18, 342 (1965).