Current-Density–Current-Density Commutators and Vector-Meson Decays*

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The techniques developed in a previous paper for investigating equal-time current-density–current-density commutators by making use of the Dyson representation, and the Ward-identity techniques of Schnitzer and Weinberg are applied to a study of the three-point function relevant to the decays of the ω, φ, and π^o mesons. It is found that the algebra-of-fields commutation relations do not permit these mesons to decay, and that the pole model of Gell-Mann, Sharp, and Wagner is incorrect, in that the ω-ρ-π vertex is not a constant in the region appropriate to the decays but rather depends linearly on the squared momenta. The commutation relations of $U(12)$ are then assumed, and the Kawarabayashi-Suzuki relation $g_\rho=m_\rho f_\pi$ is obtained as a consistency condition. The unknown constants arising from the equal-time commutators of the π field with the vector currents, and from φ-ω mixing, are fixed by requiring, as a first approximation, the vanishing of the φ-ρ-π vertex, and by feeding the measured value of the width of the decay $\pi^o\to 2\gamma$. The results then obtained for the widths $\Gamma(\omega\to\pi^o+\gamma)$ and $\Gamma(\omega\to 3\pi)$ are in excellent agreement with experiment. This agreement is maintained when provision is made to allow for the decays $\varphi\to\rho+\pi$ and $\varphi\to\pi^o+\gamma$.

I. INTRODUCTION

IN a recent paper,[1] Schnitzer and Weinberg have developed systematic techniques for investigating the n-point functions of currents. These involve the use of vector-current conservation, the equal-time charge-density–charge-density and charge-density–current-density commutation relations, and the assumption that the currents are dominated by $j=0$ and $j=1$ mesons. This latter assumption is imposed in the form of a simplicity assumption applied to the momentum dependence of certain proper vertices, together with the requirement that the spectral functions of the propagators be dominated by the one-meson states.[2]

In the preceding paper,[3] we have shown how to incorporate information, obtained through the use of the above techniques, into the Dyson representations for the commutators and retarded commutators of the currents.[4] This enables one to obtain information about the equal-time current-density–current-density commutators. In the present paper we shall apply this technique to a study of the three-point function $\langle 0|T\{A_a{}^\mu(x)V_b{}^\nu(y)V_{(8,9)}{}^\lambda(0)\}|0\rangle$, where $V_a{}^\mu(x)$ and $A_a{}^\mu(x)$, $a=1, 2, 3$, are the currents of chiral $SU(2)\otimes SU(2)$, $V_8{}^\mu(x)$ is the eighth member of the vector-current octet, and $V_9{}^\mu(x)$ is an $SU(3)$ singlet, defined in terms of quark fields by[5]

$$V_9{}^\mu(x)=\bar{q}(x)\tfrac{1}{2}\lambda_9\gamma^\mu q(x)\,,\qquad \lambda_9=(\tfrac{2}{3})^{1/2}\mathbf{1}. \tag{1}$$

This three-point function determines the decays $\varphi\to\rho+\pi$, $\varphi\to\pi^o+\gamma$, $\omega\to\pi^o+\gamma$, $\omega\to 3\pi$,[6] $\rho\to\pi+\gamma$, and $\pi^o\to 2\gamma$. We shall see in Sec. II that in order for a theory to permit these decays to take place, it is necessary that the equal-time current-density–current-density commutators of the theory be nonvanishing. This excludes the algebra of fields as a possible model.[7] Also excluded is the pole model of Gell-Mann, Sharp, and Wagner,[8] in which the decay constants are independent of the commutation relations. In Sec. III we assume the commutation relations of $U(12)$.[5] We then obtain the Kawarabayashi-Suzuki relation,[9] $g_\rho=m_\rho f_\pi$, as a consistency condition, and are able to obtain excellent agreement between the calculated and measured widths $\Gamma(\omega\to\pi^o+\gamma)$ and $\Gamma(\omega\to 3\pi)$, by using as input the measured value of the width $\Gamma(\pi^o\to 2\gamma)$,[10] and requiring, as a first approximation, the vanishing of the φ-ρ-π vertex.[11] Some information is also obtained about certain tensor-current matrix elements. Concluding remarks are given in Sec. IV.

II. RELATIONS BETWEEN DECAY CONSTANTS AND EQUAL-TIME COMMUTATORS

We begin, in the manner of Ref. 1, by defining proper vertices $\Gamma^{(\beta)}(q^2,p^2,k^2)$ and $\Gamma_{\mu\nu\lambda}{}^{(\beta)}(p,k)$ by the following

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[1] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).

[2] In Ref. 1, the simplicity assumption is sufficient to ensure the one-particle dominance of the spectral functions. In general, this will not be the case.

[3] R. Perrin, preceding paper, Phys. Rev. **170**, 1365 (1968).

[4] F. J. Dyson, Phys. Rev. **110**, 1460 (1958).

[5] R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 142 (1965).

[6] The decay $\omega\to 3\pi$ is determined to the extent that it proceeds according to $\omega\to\rho+\pi\to 3\pi$. The width $\Gamma(\rho\to 2\pi)$ is assumed to be known.

[7] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).

[8] M. Gell-Mann, D. H. Sharp, and W. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[9] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966); F. J. Gilman and H. J. Schnitzer, *ibid.* **150**, 1362 (1966); J. J. Sakurai, Phys. Rev. Letters **17**, 552 (1966); M. Ademollo, Nuovo Cimento **46A**, 156 (1966).

[10] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).

[11] S. Okubo, Phys. Letters **5**, 165 (1963).

equations[12]:

$$\int d^4x \int d^4y\, e^{-iq\cdot x}e^{ip\cdot y}\langle 0|T\{\partial_\mu A_a{}^\mu(x)V_b{}^\nu(y)V_\alpha{}^\lambda(0)\}|0\rangle = \frac{\delta_{ab}(f_\pi/\sqrt{2})m_\pi{}^2 g_\rho{}^{-1}}{q^2-m_\pi{}^2}\Delta_\rho{}^{\nu\nu'}(p)\Delta_{\alpha\beta}{}^{\lambda\lambda'}(k)\epsilon_{\nu'\lambda'\sigma\tau}p^\sigma k^\tau \Gamma^{(\beta)}(q^2,p^2,k^2)\,, \quad (2)$$

$$\int d^4x \int d^4y\, e^{-iq\cdot x}e^{ip\cdot y}\langle 0|T\{A_a{}^\mu(x)V_b{}^\nu(y)V_\alpha{}^\lambda(0)\}|0\rangle = i\delta_{ab}g_{A_1}{}^{-1}g_\rho{}^{-1}\Delta_{A_1}{}^{\mu\mu'}(q)\Delta_\rho{}^{\nu\nu'}(p)\Delta_{\alpha\beta}{}^{\lambda\lambda'}(k)\Gamma_{\mu'\nu'\lambda'}{}^{(\beta)}(p,k) - \frac{i\delta_{ab}(f_\pi/\sqrt{2})g_\rho{}^{-1}}{q^2-m_\pi{}^2}q^\mu\Delta_\rho{}^{\nu\nu'}(p)\Delta_{\alpha\beta}{}^{\lambda\lambda'}(k)\epsilon_{\nu'\lambda'\sigma\tau}p^\sigma k^\tau\Gamma^{(\beta)}(q^2,p^2,k^2)\,. \quad (3)$$

Here $\alpha, \beta=8, 9$; $k=p-q$; and $\Delta_\rho{}^{\mu\nu}(p)$, $\Delta_{A_1}{}^{\mu\nu}(q)$, and $\Delta_{\alpha\beta}{}^{\mu\nu}(k)$ are the covariant spin-one parts of the vector and axial-vector propagators:

$$\Delta_\rho{}^{\mu\nu}(k)=\int_0^\infty d\mu^2\,\frac{\rho_V(\mu^2)}{k^2-\mu^2}(g^{\mu\nu}-k^\mu k^\nu/\mu^2)\,, \quad (4)$$

$$\langle 0|V_a{}^\mu(x)V_b{}^\nu(0)|0\rangle=\delta_{ab}(2\pi)^{-3}\int d^4p\,\theta(p^0)e^{ip\cdot x}\rho_V(p^2)(g^{\mu\nu}-p^\mu p^\nu/p^2)\,, \quad (5)$$

$$\Delta_{A_1}{}^{\mu\nu}(k)=\int_0^\infty d\mu^2\,\frac{\rho_A(\mu^2)}{k^2-\mu^2}(g^{\mu\nu}-k^\mu k^\nu/\mu^2)\,, \quad (6)$$

$$\langle 0|A_a{}^\mu(x)A_b{}^\nu(0)|0\rangle=\delta_{ab}(2\pi)^{-3}\int d^4p\,\theta(p^0)e^{ip\cdot x}[\rho_A(p^2)(g^{\mu\nu}-p^\mu p^\nu/p^2)-\tfrac{1}{2}f_\pi{}^2p^\mu p^\nu]\,, \quad (7)$$

$$\Delta_{\alpha\beta}{}^{\mu\nu}(k)=\int_0^\infty d\mu^2\,\frac{\rho_{\alpha\beta}(\mu^2)}{k^2-\mu^2}(g^{\mu\nu}-k^\mu k^\nu/\mu^2)\,, \quad (8)$$

$$\langle 0|V_\alpha{}^\mu(x)V_\beta{}^\nu(0)|0\rangle=(2\pi)^{-3}\int d^4p\,\theta(p^0)e^{ip\cdot x}\rho_{\alpha\beta}(p^2)(g^{\mu\nu}-p^\mu p^\nu/p^2)\,. \quad (9)$$

The constants appearing in (2) and (3) are defined by

$$(2\pi)^{3/2}(2k^0)^{1/2}\langle 0|A_a{}^\mu(0)|\pi_b(\mathbf{k})\rangle=(f_\pi/\sqrt{2})\delta_{ab}k^\mu\,,$$
$$(2\pi)^{3/2}(2k^0)^{1/2}\langle 0|A_a{}^\mu(0)|A_{1b}{}^\nu(\mathbf{k})\rangle = -g_{A_1}\delta_{ab}(g^{\mu\nu}-k^\mu k^\nu/m_{A_1}{}^2)\,, \quad (10)$$
$$(2\pi)^{3/2}(2k^0)^{1/2}\langle 0|V_a{}^\mu(0)|\rho_b{}^\nu(\mathbf{k})\rangle = -g_\rho\delta_{ab}(g^{\mu\nu}-k^\mu k^\nu/m_\rho{}^2)\,.$$

In addition, we define the constants g_ω, g_φ, f_ω, and f_φ by

$$(2\pi)^{3/2}(2k^0)^{1/2}\langle 0|V_{(8,9)}{}^\mu(0)|\omega^\nu(\mathbf{k})\rangle = -(g_\omega,f_\omega)(g^{\mu\nu}-k^\mu k^\nu/m_\omega{}^2)\,,$$
$$(2\pi)^{3/2}(2k^0)^{1/2}\langle 0|V_{(8,9)}{}^\mu(0)|\varphi^\nu(\mathbf{k})\rangle = -(g_\varphi,f_\varphi)(g^{\mu\nu}-k^\mu k^\nu/m_\varphi{}^2)\,. \quad (11)$$

Lorentz invariance restricts $\Gamma_{\mu\nu\lambda}{}^{(\beta)}(p,k)$ to have the form

$$\Gamma_{\mu\nu\lambda}{}^{(\beta)}(p,k)=\epsilon_{\nu\lambda\mu\tau}p^\tau\gamma_1{}^{(\beta)}+\epsilon_{\nu\lambda\mu\tau}k^\tau\gamma_2{}^{(\beta)} +p_\mu\epsilon_{\nu\lambda\sigma\tau}p^\sigma k^\tau\gamma_3{}^{(\beta)}+k_\mu\epsilon_{\nu\lambda\sigma\tau}p^\sigma k^\tau\gamma_4{}^{(\beta)} +p_\nu\epsilon_{\sigma\lambda\mu\tau}p^\sigma k^\tau\gamma_5{}^{(\beta)}+k_\nu\epsilon_{\sigma\lambda\mu\tau}p^\sigma k^\tau\gamma_6{}^{(\beta)} +p_\lambda\epsilon_{\nu\sigma\mu\tau}p^\sigma k^\tau\gamma_7{}^{(\beta)}+k_\lambda\epsilon_{\nu\sigma\mu\tau}p^\sigma k^\tau\gamma_8{}^{(\beta)}\,, \quad (12)$$

where

$$\gamma_j{}^{(\beta)}=\gamma_j{}^{(\beta)}(q^2,p^2,k^2)\,. \quad (13)$$

From (2)–(9), (12), vector-current conservation, and the chiral commutation relations,[13] we obtain the following Ward identities:

$$k^\lambda\Gamma_{\mu\nu\lambda}{}^{(\beta)}(p,k)=0\,,$$
$$p^\nu\Gamma_{\mu\nu\lambda}{}^{(\beta)}(p,k)=0\,, \quad (14)$$
$$g_{A_1}{}^{-1}C_{A_1}q^\mu\Gamma_{\mu\nu\lambda}{}^{(\beta)}(p,k)+(f_\pi/\sqrt{2})\times\epsilon_{\nu\lambda\sigma\tau}p^\sigma k^\tau\Gamma^{(\beta)}(q^2,p^2,k^2)=0\,,$$

where

$$C_{A_1}=\int_0^\infty d\mu^2\,\rho_A(\mu^2)/\mu^2\,. \quad (15)$$

[12] Our metric has $g^{00}=1$, $g^{11}=g^{22}=g^{33}=-1$.

[13] M. Gell-Mann, Physics (N. Y.) 1, 63 (1964).

From (12) and (14) there follows

$$\gamma_1^{(\beta)}=-\tfrac{1}{2}\gamma_7^{(\beta)}q^2+\tfrac{1}{2}\gamma_7^{(\beta)}p^2+(\tfrac{1}{2}\gamma_7^{(\beta)}+\gamma_8^{(\beta)})k^2,$$

$$\gamma_2^{(\beta)}=\tfrac{1}{2}\gamma_6^{(\beta)}q^2-(\gamma_5^{(\beta)}+\tfrac{1}{2}\gamma_6^{(\beta)})p^2-\tfrac{1}{2}\gamma_6^{(\beta)}k^2,$$

$$\Gamma^{(\beta)}(q^2,p^2,k^2)=\frac{g_{A_1}^{-1}C_{A_1}}{(f_\pi/\sqrt{2})}[(-\tfrac{1}{2}\gamma_3^{(\beta)}+\tfrac{1}{2}\gamma_4^{(\beta)}-\tfrac{1}{2}\gamma_6^{(\beta)}+\tfrac{1}{2}\gamma_7^{(\beta)})q^2 \tag{16}$$

$$+(-\tfrac{1}{2}\gamma_3^{(\beta)}-\tfrac{1}{2}\gamma_4^{(\beta)}+\gamma_5^{(\beta)}+\tfrac{1}{2}\gamma_6^{(\beta)}-\tfrac{1}{2}\gamma_7^{(\beta)})p^2+(\tfrac{1}{2}\gamma_3^{(\beta)}+\tfrac{1}{2}\gamma_4^{(\beta)}+\tfrac{1}{2}\gamma_6^{(\beta)}-\tfrac{1}{2}\gamma_7^{(\beta)}-\gamma_8^{(\beta)})k^2]$$

$$\equiv\Gamma_1^{(\beta)}q^2+\Gamma_2^{(\beta)}p^2+\Gamma_3^{(\beta)}k^2.$$

Since $\gamma_j^{(\beta)}(q^2,p^2,k^2)$ cannot be singular as its arguments go to zero,[14] we see from (16) that the leading terms in $\Gamma^{(\beta)}(q^2,p^2,k^2)$ are at least linear in the squared momenta. This result contradicts the assumption of the Gell-Mann, Sharp, and Wagner pole model,[8] that $\Gamma^{(\beta)}(q^2,p^2,k^2)$ is a constant.

Up to this point we have made no approximations. We now impose meson dominance by assuming one-particle dominance of the propagator spectral functions, and by requiring that the momentum dependence of the proper vertices be as smooth as possible.[1] This latter requirement we impose by assuming that $\gamma_j^{(\beta)}=(\text{const})^{(\beta)}$, $j=3,\cdots,8$. From (16) it then follows that $\Gamma_j^{(\beta)}=(\text{const})^{(\beta)}$, $j=1, 2, 3$.

Equations (2) and (3) now become

$$\int d^4x\int d^4y\; e^{-iq\cdot x}e^{ip\cdot y}\langle 0|T\{\partial_\mu A_a{}^\mu(x)V_b{}^\nu(y)V_{(8,9)}{}^\lambda(0)\}|0\rangle$$

$$=\frac{\delta_{ab}(f_\pi/\sqrt{2})m_\pi{}^2g_\rho(g_\omega,f_\omega)}{(q^2-m_\pi{}^2)(p^2-m_\rho{}^2)(k^2-m_\omega{}^2)}\epsilon^{\nu\lambda\sigma\tau}p_\sigma k_\tau[g_\omega\Gamma^{(8)}(q^2,p^2,k^2)+f_\omega\Gamma^{(9)}(q^2,p^2,k^2)]+(\omega\rightarrow\varphi), \tag{17}$$

$$\int d^4x\int d^4y\; e^{-iq\cdot x}e^{ip\cdot y}\langle 0|T\{A_a{}^\mu(x)V_b{}^\nu(y)V_{(8,9)}{}^\lambda(0)\}|0\rangle$$

$$=\frac{i\delta_{ab}g_{A_1}g_\rho(g_\omega,f_\omega)}{(q^2-m_{A_1}{}^2)(p^2-m_\rho{}^2)(k^2-m_\omega{}^2)}\{g_\omega[\Gamma^{(8)\mu\nu\lambda}(p,k)+g_{A_1}{}^{-1}(f_\pi/\sqrt{2})q^\mu\epsilon^{\nu\lambda\sigma\tau}p_\sigma k_\tau\Gamma^{(8)}(q^2,p^2,k^2)]$$

$$+f_\omega[\Gamma^{(9)\mu\nu\lambda}(p,k)+g_{A_1}{}^{-1}(f_\pi/\sqrt{2})q^\mu\epsilon^{\nu\lambda\sigma\tau}p_\sigma k_\tau\Gamma^{(9)}(q^2,p^2,k^2)]\}$$

$$-\frac{i\delta_{ab}(f_\pi/\sqrt{2})g_\rho(g_\omega,f_\omega)}{(q^2-m_\pi{}^2)(p^2-m_\rho{}^2)(k^2-m_\omega{}^2)}q^\mu\epsilon^{\nu\lambda\sigma\tau}p_\sigma k_\tau[g_\omega\Gamma^{(8)}(q^2,p^2,k^2)+f_\omega\Gamma^{(9)}(q^2,p^2,k^2)]+(\omega\rightarrow\varphi), \tag{18}$$

where $(\omega\rightarrow\varphi)$ means that $m_\omega\rightarrow m_\varphi$, $g_\omega\rightarrow g_\varphi$, and $f_\omega\rightarrow f_\varphi$, wherever these quantities explicitly appear.

We now use the results of Ref. 3 to obtain from (17) and (18) the following representations for the current commutators[15]:

$$(2\pi)^{3/2}(2q^0)^{1/2}\langle 0|[V_b{}^\nu(\tfrac{1}{2}x),V_{(8,9)}{}^\lambda(-\tfrac{1}{2}x)]|A_{1a}{}^\mu(\mathbf{q})\rangle=\delta_{ab}(2\pi)^{-3}\int d^4l\; e^{-il\cdot x}\int d^4u\int_0^\infty d\lambda^2\;\epsilon(l^0-u^0)\delta[(l-u)^2-\lambda^2]$$

$$\times[\{g_\omega[\Gamma^{(8)\mu\nu\lambda}(l+\tfrac{1}{2}q,\,l-\tfrac{1}{2}q)+q_{A_1}{}^{-1}(f_\pi/\sqrt{2})q^\mu\epsilon^{\nu\lambda\sigma\tau}q_\sigma l_\tau\Gamma^{(8)}(m_{A_1}{}^2,(l+\tfrac{1}{2}q)^2(l-\tfrac{1}{2}q)^2)]$$

$$+f_\omega[\Gamma^{(9)\mu\nu\lambda}(l+\tfrac{1}{2}q,\,l-\tfrac{1}{2}q)+g_{A_1}{}^{-1}(f_\pi/\sqrt{2})q^\mu\epsilon^{\nu\lambda\sigma\tau}q_\sigma l_\tau\Gamma^{(9)}(m_{A_1}{}^2,(l+\tfrac{1}{2}q)^2(l-\tfrac{1}{2}q)^2)]\}\Phi_1(u,\lambda^2)+(\omega\rightarrow\varphi)], \tag{19}$$

$$\Phi_1(u,\lambda^2)=-g_\rho(g_\omega,f_\omega)\int_0^1 d\alpha\;\delta^{(4)}[u-\tfrac{1}{2}(1-2\alpha)q](d/d\lambda^2)\delta[\lambda^2+m_{A_1}{}^2\alpha(1-\alpha)-m_\omega{}^2(1-\alpha)-m_\rho{}^2\alpha], \tag{20}$$

[14] The opposite would imply the existence of zero-mass particles in the theory, satisfying at least one of the three conditions $\langle 0|A_a{}^\mu(0)|p^2=0\rangle\neq 0$, $\langle 0|V_b{}^\nu(0)|p^2=0\rangle\neq 0$, or $\langle 0|V_{(8,9)}{}^\lambda(0)|p^2=0\rangle\neq 0$.

[15] Implicit in (19)–(28) is the assumption that, within the context of the meson-dominance approximation, the retarded commutators and the time-ordered products have the same pole structure. That this is indeed true is shown in Appendix A.

$$(2\pi)^{3/2}(2q^0)^{1/2}\langle 0|[V_b{}^\nu(\tfrac{1}{2}x),V_{(8,9)}{}^\lambda(-\tfrac{1}{2}x)]|\pi_a(\mathbf{q})\rangle$$
$$=\delta_{ab}(2\pi)^{-3}\int d^4l\, e^{-il\cdot x}\epsilon^{\nu\lambda\sigma\tau}q_\sigma l_\tau\int d^4u\int_0^\infty d\lambda^2\,\epsilon(l^0-u^0)\delta[(l-u)^2-\lambda^2]$$
$$\times\{g_\rho(g_\omega,f_\omega)[g_\omega\Gamma^{(8)}(m_\pi{}^2,(l+\tfrac{1}{2}q)^2,(l-\tfrac{1}{2}q)^2)+f_\omega\Gamma^{(9)}(m_\pi{}^2,(l+\tfrac{1}{2}q)^2,(l-\tfrac{1}{2}q)^2)]\Phi_2(u,\lambda^2)+(\omega\to\varphi)\},\quad(21)$$

$$\Phi_2(u,\lambda^2)=-\int_0^1 d\alpha\,\delta^{(4)}[u-\tfrac{1}{2}(1-2\alpha)q](d/d\lambda^2)\delta[\lambda^2+m_\pi{}^2\alpha(1-\alpha)-m_\omega{}^2(1-\alpha)-m_\rho{}^2\alpha],\quad(22)$$

$$(2\pi)^{3/2}(2p^0)^{1/2}\langle\rho_b{}^\nu(\mathbf{p})|[\partial_\mu A_a{}^\mu(\tfrac{1}{2}x),V_{(8,9)}{}^\lambda(0)]|0\rangle$$
$$=-i\delta_{ab}(2\pi)^{-3}\int d^4l\,e^{-il\cdot x}\epsilon^{\nu\lambda\sigma\tau}p_\sigma l_\tau\int d^4u\int_0^\infty d\lambda^2\,\epsilon(l^0-u^0)\delta[(l-u)^2-\lambda^2]\{(f_\pi/\sqrt{2})m_\pi{}^2(g_\omega,f_\omega)$$
$$\times[g_\omega\Gamma^{(8)}((l-\tfrac{1}{2}p)^2,m_\rho{}^2,(l+\tfrac{1}{2}p)^2)+f_\omega\Gamma^{(9)}((l-\tfrac{1}{2}p)^2,m_\rho{}^2,(l+\tfrac{1}{2}p)^2)]\Phi_3(u,\lambda^2)+(\omega\to\varphi)\},\quad(23)$$

$$\Phi_3(u,\lambda^2)=-\int_0^1 d\alpha\,\delta^{(4)}[u-\tfrac{1}{2}(1-2\alpha)p](d/d\lambda^2)\delta[\lambda^2+m_\rho{}^2\alpha(1-\alpha)-m_\pi{}^2(1-\alpha)-m_\omega{}^2\alpha],\quad(24)$$

$$(2\pi)^{3/2}(2k^o)^{1/2}\langle 0|[\partial_\mu A_a{}^\nu(\tfrac{1}{2}x),V_b{}^\nu(-\tfrac{1}{2}x)]|\omega^\lambda(\mathbf{k})\rangle$$
$$=-i\delta_{ab}(2\pi)^{-3}\int d^4l\,e^{-il\cdot x}\epsilon^{\nu\lambda\sigma\tau}k_\sigma l_\tau\int d^4u\int_0^\infty d\lambda^2\,\epsilon(l^0-u^0)\delta[(l-u)^2-\lambda^2]$$
$$\times(f_\pi/\sqrt{2})m_\pi{}^2g_\rho[g_\omega\Gamma^{(8)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\omega{}^2)+f_\omega\Gamma^{(9)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\omega{}^2)]\Phi_4(u,\lambda^2),\quad(25)$$

$$\Phi_4(u,\lambda^2)=-\left[\int_{-\infty}^0 d\alpha+\int_1^\infty d\alpha\right]\delta^{(4)}[u-\tfrac{1}{2}(1-2\alpha)k](d/d\lambda^2)\delta[\lambda^2+m_\omega{}^2\alpha(1-\alpha)-m_\rho{}^2(1-\alpha)-m_\pi{}^2\alpha],\quad(26)$$

$$(2\pi)^{3/2}(2k^o)^{1/2}\langle 0|[\partial_\mu A_a{}^\mu(\tfrac{1}{2}x),V_b{}^\nu(-\tfrac{1}{2}x)]|\varphi^\lambda(\mathbf{k})\rangle$$
$$=-i\delta_{ab}(2\pi)^{-3}\int d^4l\,e^{-il\cdot x}\epsilon^{\nu\lambda\sigma\tau}k_\sigma l_\tau\int d^4u\int_0^\infty d\lambda^2\epsilon(l^0-u^0)\delta[(l-u)^2-\lambda^2](f_\pi/\sqrt{2})m_\pi{}^2g_\rho$$
$$\times[g_\varphi\Gamma^{(8)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\varphi{}^2)+f_\varphi\Gamma^{(9)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\varphi{}^2)]\Phi_5(u,\lambda^2),\quad(27)$$

$$\Phi_5(u,\lambda^2)=-\left[\int_{-\infty}^0 d\alpha+\int_1^\infty d\alpha\right]\delta^{(4)}[u-\tfrac{1}{2}(1-2\alpha)k](d/d\lambda^2)\delta[\lambda^2+m_\omega{}^2\alpha(1-\alpha)-m_\rho{}^2(1-\alpha)-m_\pi{}^2\alpha].\quad(28)$$

From (15), (16), and (19)–(28), we obtain for the matrix elements of the equal-time commutators

$$(2\pi)^{3/2}(2q^0)^{1/2}\langle 0|[V_b{}^\nu(\tfrac{1}{2}x),V_{(8,9)}{}^\lambda(-\tfrac{1}{2}x)]_{x^o=0}|A_{1a}{}^\nu(\mathbf{q})\rangle=-\delta_{ab}(g_\sigma{}^\mu-q^\mu q_\sigma/m_{A_1}{}^2)\epsilon^{\nu\lambda\sigma 0}\delta(\mathbf{x})$$
$$\times g_\rho g_{A_1}{}^{-1}(f_\pi/\sqrt{2})m_{A_1}{}^2\{(g_\omega,f_\omega)[g_\omega(\Gamma_2{}^{(8)}+\Gamma_3{}^{(8)})+f_\omega(\Gamma_2{}^{(9)}+\Gamma_3{}^{(9)})]$$
$$+(g_\varphi,f_\varphi)[g_\varphi(\Gamma_2{}^{(8)}+\Gamma_3{}^{(8)})+f_\varphi(\Gamma_2{}^{(9)}+\Gamma_3{}^{(9)})]\},\quad(29)$$

$$(2\pi)^{3/2}(2q^0)^{1/2}\langle 0|[V_b{}^\nu(\tfrac{1}{2}x),V_{(8,9)}{}^\lambda(-\tfrac{1}{2}x)]_{x^o=0}|\pi_a(\mathbf{q})\rangle=\delta_{ab}\epsilon^{\nu\lambda\sigma 0}q_\sigma\delta(\mathbf{x})$$
$$\times g_\rho\{(g_\omega,f_\omega)[g_\omega(\Gamma_2{}^{(8)}+\Gamma_3{}^{(8)})+f_\omega(\Gamma_2{}^{(9)}+\Gamma_3{}^{(9)})]+(g_\varphi,f_\varphi)[g_\varphi(\Gamma_2{}^{(8)}+\Gamma_3{}^{(8)})+f_\varphi(\Gamma_2{}^{(9)}+\Gamma_3{}^{(9)})]\},\quad(30)$$

$$(2\pi)^{3/2}(2p^0)^{1/2}\langle\rho_b{}^\nu(\mathbf{p})|[\partial_\mu A_a{}^\mu(\tfrac{1}{2}x),V_{(8,9)}{}^\lambda(0)]_{x^o=0}|0\rangle=i\delta_{ab}\epsilon^{\nu\lambda\sigma 0}p_\sigma\delta(\mathbf{x})$$
$$\times(f_\pi/\sqrt{2})m_\pi{}^2\{(g_\omega,f_\omega)[g_\omega(\Gamma_1{}^{(8)}+\Gamma_3{}^{(8)})+f_\omega(\Gamma_1{}^{(9)}+\Gamma_3{}^{(9)})]+(g_\varphi,f_\varphi)[g_\varphi(\Gamma_1{}^{(8)}+\Gamma_3{}^{(8)})+f_\varphi(\Gamma_1{}^{(9)}+\Gamma_3{}^{(9)})]\},\quad(31)$$

$$(2\pi)^{3/2}(2k^o)^{1/2}\langle 0|[\partial_\mu A_a{}^\mu(\tfrac{1}{2}x),V_b{}^\nu(-\tfrac{1}{2}x)]_{x^o=0}|\omega^\lambda(\mathbf{k})\rangle=-i\delta_{ab}\epsilon^{\nu\lambda\sigma 0}k_\sigma\delta(\mathbf{x})$$
$$\times(f_\pi/\sqrt{2})m_\pi{}^2g_\rho[g_\omega(\Gamma_1{}^{(8)}+\Gamma_2{}^{(8)})+f_\omega(\Gamma_1{}^{(9)}+\Gamma_2{}^{(9)})],\quad(32)$$

$$(2\pi)^{3/2}(2k^o)^{1/2}\langle 0|[\partial_\mu A_a{}^\mu(\tfrac{1}{2}x),V_b{}^\nu(-\tfrac{1}{2}x)]_{x^o=0}|\varphi^\lambda(\mathbf{k})\rangle=-i\delta_{ab}\epsilon^{\nu\lambda\sigma 0}k_\sigma\delta(\mathbf{x})$$
$$\times(f_\pi/\sqrt{2})m_\pi{}^2g_\rho[g_\varphi(\Gamma_1{}^{(8)}+\Gamma_2{}^{(8)})+f_\varphi(\Gamma_1{}^{(9)}+\Gamma_2{}^{(9)})].\quad(33)$$

The constants $\Gamma_j^{(\beta)}$ determine the widths of the decays $\varphi\to\rho+\pi$, $\varphi\to\pi^0+\gamma$, $\omega\to\pi^0+\gamma$, $\omega\to 3\pi$,[6] $\rho\to\pi+\gamma$, and $\pi^0\to 2\gamma$, and are expressible, through Eqs. (29)–(33), in terms of the equal-time commutation relations of the theory. This latter fact is to be contrasted with the situation obtaining in the Gell-Mann, Sharp, and Wagner pole model,[8] in which the decay constants are independent of the commutation relations. We note that if one assumes the commutation relations of the algebra of fields,[7] in which all current-density–current-density commutators vanish, then $\Gamma_j^{(\beta)}=0$, $j=1, 2, 3$, $\beta=8, 9$ and all of the above decays are forbidden.

III. RESULTS FROM CHIRAL $U(12)$ COMMUTATION RELATIONS

In this section we assume the commutation relations of $U(12)$,[5] and use the resulting expressions for the constants $\Gamma_j^{(\beta)}$ to calculate the decay widths. The currents we are concerned with are taken to be defined in terms of quark fields by the following equations[5]:

$$V_\alpha^\mu(x)=\bar{q}(x)\tfrac{1}{2}\lambda_\alpha\gamma^\mu q(x),\qquad \alpha=1,\cdots,9 \tag{34}$$

$$A_a^\mu(x)=\bar{q}(x)\tfrac{1}{2}\lambda_a\gamma^\mu\gamma_5 q(x),\quad a=1,2,3 \tag{35}$$

$$\partial_\mu A_a^\mu(x)=\eta\bar{q}(x)\tfrac{1}{2}\lambda_a\gamma_5 q(x),\qquad a=1,2,3 \tag{36}$$

where λ_9 is defined in (1), and η is some unknown constant. We also define tensor currents $V_\alpha^{\mu\nu}(x)$ by

$$V_\alpha^{\mu\nu}(x)=\tfrac{1}{2}\eta\bar{q}(x)\tfrac{1}{2}\lambda_\alpha\sigma^{\mu\nu}q(x),\quad \alpha=1,\cdots,9 \tag{37}$$

where $\sigma^{\mu\nu}=\frac{1}{2}[\gamma^\mu,\gamma^\nu]$, and η is the same constant that appears in (36).

The equal-time commutators that follow from (34)–(37) are

$$[V_a^i(\tfrac{1}{2}x),V_8^j(-\tfrac{1}{2}x)]_{x^0=0}=(\tfrac{1}{3})^{1/2}i\epsilon_{ijk}A_a^k(0)\delta(\mathbf{x}), \tag{38}$$

$$[V_a^i(\tfrac{1}{2}x),V_9^j(-\tfrac{1}{2}x)]_{x^0=0}=(\tfrac{2}{3})^{1/2}i\epsilon_{ijk}A_a^k(0)\delta(\mathbf{x}), \tag{39}$$

$$[\partial_\mu A_a^\mu(\tfrac{1}{2}x),V_b^j(-\tfrac{1}{2}x)]_{x^0=0}=(\tfrac{1}{3})^{1/2}i\delta_{ab}\epsilon^{0j}{}_{\sigma\tau}\times[V_8^{\sigma\tau}(0)+\sqrt{2}V_9^{\sigma\tau}(0)]\delta(\mathbf{x}), \tag{40}$$

$$[\partial_\mu A_a^\mu(\tfrac{1}{2}x),V_8^j(-\tfrac{1}{2}x)]_{x^0=0}=(\tfrac{1}{3})^{1/2}i\delta_{ab}\times\epsilon^{0j}{}_{\sigma\tau}V_a^{\sigma\tau}(0)\delta(\mathbf{x}), \tag{41}$$

$$[\partial_\mu A_a^\mu(\tfrac{1}{2}x),V_9^j(-\tfrac{1}{2}x)]_{x^0=0}=(\tfrac{2}{3})^{1/2}i\delta_{ab}\times\epsilon^{0j}{}_{\sigma\tau}V_a^{\sigma\tau}(0)\delta(\mathbf{x}). \tag{42}$$

We need not worry about the existence of Schwinger terms in the above equal-time commutators, since, within the context of our meson-dominance approximation, they do not contribute to (29)–(33).[16]

From (10), (29), (30), and (38) there follows

$$g_{A_1}{}^2=(\tfrac{1}{2}f_\pi{}^2)m_{A_1}{}^2. \tag{43}$$

[16] The fact that gradients of δ functions do not appear in (29)–(33) is a consequence of the fact that we have assumed, as part of our meson-dominance approximation, that $\Gamma^{(\beta)}(q^2,p^2,k^2)$ is linear in the squared momenta [see (19)–(28)].

Using Weinberg's sum rule[17]

$$g_{A_1}{}^2m_{A_1}{}^{-2}+\tfrac{1}{2}f_\pi{}^2=g_\rho{}^2m_\rho{}^{-2}, \tag{44}$$

there then follows

$$g_\rho{}^2=m_\rho{}^2f_\pi{}^2, \tag{45}$$

which is the Kawarabayashi-Suzuki relation.[9] It is to be noted that our derivation of this relation does not suffer from the problems associated with soft-pion extrapolations, as do the derivations in Ref. 9.[18]

From Lorentz invariance, and the fact that the currents transform as an octet and a singlet, there follows

$$\begin{aligned}(2k^0)^{1/2}\langle 0|\partial_\mu V_\alpha^{\mu\nu}(0)|v_\beta^\lambda(\mathbf{k})\rangle&=C^{(8)}(2k^0)^{1/2}\langle 0|V_\alpha^\nu(0)|v_\beta^\lambda(\mathbf{k})\rangle,\\ \alpha=1,\cdots,8,&\quad \beta=1,\cdots,9\\ (2k^0)^{1/2}\langle 0|\partial_\mu V_9^{\mu\nu}(0)|v_\beta^\lambda(\mathbf{k})\rangle&=C^{(1)}(2k^0)^{1/2}\langle 0|V_9^\nu(0)|v_\beta^\lambda(\mathbf{k})\rangle,\\ \beta&=1,\cdots,9\end{aligned} \tag{46}$$

where $v_\beta^\lambda(\mathbf{k})$ represent the nine physical vector mesons and $C^{(8)}$ and $C^{(1)}$ are constants, independent of α and β. The corresponding matrix elements of the tensor currents are

$$\begin{aligned}(2\pi)^{3/2}(2k^0)^{1/2}\langle 0|V_\alpha^{\mu\nu}(0)|v_\beta^\lambda(\mathbf{k})\rangle&=i(\delta_{\alpha\beta}+\delta_{\alpha8}\delta_{\beta9})g_\beta m_\beta{}^{-2}C^{(8)}(g^{\mu\lambda}k^\nu-g^{\nu\lambda}k^\mu),\\ (2\pi)^{3/2}(2k^0)^{1/2}\langle 0|V_9^{\mu\nu}(0)|v_\beta^\lambda(\mathbf{k})\rangle&=i(\delta_{\beta8}+\delta_{\beta9})f_\beta m_\beta{}^{-2}C^{(1)}(g^{\mu\lambda}k^\nu-g^{\nu\lambda}k^\mu),\end{aligned} \tag{47}$$

where $\alpha=1,\cdots,8$, $\beta=1,\cdots,9$, and the constants g_β and f_β are defined in (10) and (11).

Using (10), (11), (30)–(33), (38)–(42), and (47), we obtain the following equations for the constants $\Gamma_j^{(\beta)}$:

$$g_\omega(\Gamma_1{}^{(8)}+\Gamma_2{}^{(8)})+f_\omega(\Gamma_1{}^{(9)}+\Gamma_2{}^{(9)})=i2\sqrt{2}(g_\omega C^{(8)}+\sqrt{2}f_\omega C^{(1)})/\sqrt{3}f_\pi m_\pi{}^2g_\rho m_\omega{}^2, \tag{48}$$

$$g_\varphi(\Gamma_1{}^{(8)}+\Gamma_2{}^{(8)})+f_\varphi(\Gamma_1{}^{(9)}+\Gamma_2{}^{(9)})=i2\sqrt{2}(g_\varphi C^{(8)}+\sqrt{2}f_\varphi C^{(1)})/\sqrt{3}f_\pi m_\pi{}^2g_\rho m_\varphi{}^2, \tag{49}$$

$$(g_\omega{}^2+g_\varphi{}^2)(\Gamma_1{}^{(8)}+\Gamma_3{}^{(8)})+(g_\omega f_\omega+g_\varphi f_\varphi)(\Gamma_1{}^{(9)}+\Gamma_3{}^{(9)})=i2\sqrt{2}g_\rho C^{(8)}/\sqrt{3}f_\pi m_\pi{}^2m_\rho{}^2, \tag{50}$$

$$(g_\omega f_\omega+g_\varphi f_\varphi)(\Gamma_1{}^{(8)}+\Gamma_3{}^{(8)})+(f_\omega{}^2+f_\varphi{}^2)\times(\Gamma_1{}^{(9)}+\Gamma_3{}^{(9)})=i4g_\rho C^{(8)}/\sqrt{3}f_\pi m_\pi{}^2m_\rho{}^2, \tag{51}$$

$$(g_\omega{}^2+g_\varphi{}^2)(\Gamma_2{}^{(8)}+\Gamma_3{}^{(8)})+(g_\omega f_\omega+g_\varphi f_\varphi)\times(\Gamma_2{}^{(9)}+\Gamma_3{}^{(9)})=-if_\pi/(\sqrt{6})g_\rho, \tag{52}$$

$$(g_\omega f_\omega+g_\varphi f_\varphi)(\Gamma_2{}^{(8)}+\Gamma_3{}^{(8)})+(f_\omega{}^2+f_\varphi{}^2)\times(\Gamma_2{}^{(9)}+\Gamma_3{}^{(9)})=-if_\pi/\sqrt{3}g_\rho. \tag{53}$$

[17] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967). The assumption, made in this reference, that no $I=1$ Schwinger terms exist in the local chiral $SU(2)\otimes SU(2)$ current algebra, can be weakened. The presence of $I=1$ vector and axial-vector Schwinger terms does not affect the proof of the sum rule (44). However, if an $I=1$ pseudoscalar Schwinger term exists, then one must assume that its coupling to the pion is much weaker than that of the axial-vector current. See R. Perrin, Phys. Rev. Letters **20**, 306 (1968).

[18] D. A. Geffen, Phys. Rev. Letters **19**, 770 (1967); R. Arnowitt, M. H. Friedman, and P. Nath, *ibid.* **19**, 1085 (1967).

The solutions to these equations are listed in Appendix B. Since we do not know the constants $C^{(8)}$, $C^{(1)}$, g_ω, and f_ω,[19] we shall have to use information about the decays to fix them. This we do by first noting that the widths of the decays $\varphi\to\rho+\pi$ and $\varphi\to\pi^o+\gamma$ are several orders of magnitude smaller than one would expect on the basis of a simple pole-model calculation with $f_{\varphi\rho\pi}=f_{\omega\rho\pi}$.[20] We take this into account by assuming, as a first approximation, that the φ-ρ-π vertex vanishes in the region appropriate to the decays. Using (16) and (17), this gives

$$g_\varphi\Gamma_j^{(8)}+f_\varphi\Gamma_j^{(9)}=0,\quad j=1,2,3. \tag{54}$$

From (B4)–(B6) there then follow

$$f_\omega=\sqrt{2}g_\omega \tag{55}$$

and

$$g_\varphi C^{(8)}+\sqrt{2}f_\varphi C^{(1)}=0. \tag{56}$$

Using the sum rules[21]

$$g_\omega^2/m_\omega^2+g_\varphi^2/m_\varphi^2=g_\rho^2/m_\rho^2 \tag{57}$$

and

$$g_\omega f_\omega/m_\omega^2+g_\varphi f_\varphi/m_\varphi^2=0, \tag{58}$$

we obtain from (56)

$$(2g_\omega^2/m_\omega^2)C^{(1)}=(g_\rho^2/m_\rho^2-g_\omega^2/m_\omega^2)C^{(8)}. \tag{59}$$

From (45), (55), (59), and (B1)–(B3) we now have

$$\begin{aligned}g_\omega\Gamma_1^{(8)}+f_\omega\Gamma_1^{(9)}&=(i/2(\surd 6)m_\rho g_\omega)(1+8C^{(8)}/m_\pi^2),\\ g_\omega\Gamma_2^{(8)}+f_\omega\Gamma_2^{(9)}&=g_\omega\Gamma_3^{(8)}+f_\omega\Gamma_3^{(9)}\\ &=-i/2(\surd 6)m_\rho g_\omega.\end{aligned} \tag{60}$$

Using (10), (11), (17), and (60), and assuming that the electromagnetic current is given by

$$J^{(\mathrm{em})\mu}(x)=e[V_3^\mu(x)+\tfrac{1}{3}\sqrt{3}V_8^\mu(x)], \tag{61}$$

we obtain for the decay constants

$$f_{\pi\gamma\gamma}=f_\pi(m_\pi^2+8C^{(8)})/3\sqrt{2}m_\rho^2m_\omega^2, \tag{62}$$

$$f_{\rho\pi\gamma}=[(m_\pi^2+8C^{(8)})-m_\rho^2]/6\sqrt{2}m_\rho m_\omega^2, \tag{63}$$

$$f_{\omega\pi\gamma}=[(m_\pi^2+8C^{(8)})-m_\omega^2]g_\rho/2(\surd 6)m_\rho^3g_\omega, \tag{64}$$

$$f_{\omega\rho\pi}=[(m_\pi^2+8C^{(8)})-m_\rho^2-m_\omega^2]/2(\surd 6)m_\rho g_\omega, \tag{65}$$

where the equations relating the decay widths to the decay constants are given in Appendix C.

Using (62), (C1), the experimental values[10,22]

$$\Gamma(\pi^o\to 2\gamma)=(7.3\pm1.5)\text{ eV}, \tag{66}$$

$$f_\pi^2=0.94m_\pi^2, \tag{67}$$

and the following mass values[10]:

$$\begin{aligned}m_\rho&=770\text{ MeV},\\ m_\omega&=783\text{ MeV},\\ m_\varphi&=1019\text{ MeV},\end{aligned} \tag{68}$$

we obtain

$$C^{(8)}=-(3.80\pm0.39)\times10^5\text{ (MeV)}^2, \tag{69}$$

where the negative sign has been chosen so as to obtain agreement with the remaining decay widths.

From (63), (68), (69), and (C2), we obtain

$$\Gamma(\rho^0\to\pi^o+\gamma)=(0.10\pm0.02)\text{ MeV}, \tag{70}$$

which is consistent with the experimental upper bound[10]

$$\Gamma_{\text{expt}}(\rho^0\to\pi^o+\gamma)<0.56\text{ MeV}. \tag{71}$$

To calculate the widths $\Gamma(\omega\to\pi^o+\gamma)$ and $\Gamma(\omega\to3\pi)$, we must determine the constant g_ω. We do this by assuming that the currents transform as a nonet, rather than as an octet and a singlet.[11,23] This gives $C^{(8)}=C^{(1)}$ and, from (55)–(59),

$$\begin{aligned}g_\omega^2&=\tfrac{1}{3}(m_\omega^2/m_\rho^2)g_\rho^2=0.35g_\rho^2,\\ g_\varphi^2&=\tfrac{2}{3}(m_\varphi^2/m_\rho^2)g_\rho^2=1.17g_\rho^2,\\ f_\omega&=\sqrt{2}g_\omega,\\ f_\varphi&=-(\tfrac{1}{2}\sqrt{2})g_\varphi.\end{aligned} \tag{72}$$

We shall shortly break this nonet symmetry to account for the decays $\varphi\to\rho+\pi$ and $\varphi\to\pi^o+\gamma$.

Using (64), (65), (68), (69), (72), (C3)–(C5), and the measured value for the ρ width,[24]

$$\Gamma(\rho\to2\pi)=(129\pm15)\text{ MeV}, \tag{73}$$

we obtain

$$\begin{aligned}\Gamma(\omega\to\pi^o+\gamma)&=(1.02\pm0.17)\text{ MeV},\\ \Gamma(\omega\to3\pi)&=(11.3\pm2.0)\text{ MeV},\end{aligned} \tag{74}$$

in excellent agreement with the experimental results[10]

$$\begin{aligned}\Gamma_{\text{expt}}(\omega\to\pi^o+\gamma)&=(1.15\pm0.17)\text{ MeV},\\ \Gamma_{\text{expt}}(\omega\to3\pi)&=(10.7\pm1.4)\text{ MeV}.\end{aligned} \tag{75}$$

We must now take into account the fact that the decay $\varphi\to\rho+\pi$ does take place.[10] We do this by assuming that the relations $f_\omega=\sqrt{2}g_\omega$ and $f_\varphi=-(\frac{1}{2}\sqrt{2})g_\varphi$ are maintained, but that $C^{(8)}\neq C^{(1)}$. The new decay constants which result from this assumption are

$$f_{\varphi\rho\pi}=\sqrt{2}g_\varphi(m_\pi^2+m_\rho^2-m_\varphi^2)(C^{(8)}-C^{(1)})/\sqrt{3}f_\pi g_\rho m_\pi^2m_\varphi^2, \tag{76}$$

$$f_{\varphi\pi\gamma}=\sqrt{2}g_\varphi(m_\pi^2-m_\rho^2)(C^{(8)}-C^{(1)})/\sqrt{3}f_\pi m_\pi^2m_\rho^2m_\varphi^2, \tag{77}$$

$$f_{\pi\gamma\gamma}=(f_\pi/3\sqrt{2}m_\rho^2m_\omega^2)[(m_\pi^2+8C^{(8)})+4g_\varphi^2m_\omega^2(C^{(8)}-C^{(1)})/f_\pi^2m_\varphi^4], \tag{78}$$

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[19] The constants g_φ and f_φ are determined in terms of g_ρ, g_ω, and f_ω by the sum rules (57) and (58).

[20] R. F. Dashen and D. H. Sharp, Phys. Rev. **133**, B1585 (1964).

[21] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 470 (1967); R. J. Oakes and J. J. Sakurai, *ibid.* **19**, 1266 (1967).

[22] In obtaining (67) we have assumed that $\pi^\pm$ β decay is determined by $f_\pi\cos\theta$ with $\theta\approx0.25$. See N. Cabbibo, Phys. Rev. Letters **10**, 531 (1963).

[23] In Sec. IV, we consider an alternative possibility for fixing the constant g_ω.

[24] M. Roos, CERN Report No. TH. 798, 1967 (unpublished).

$$f_{\rho\pi\gamma}=(1/6\sqrt{2}m_\rho m_\omega^2)\times\{[(m_\pi^2+8C^{(8)})-m_\rho^2]+4g_\varphi^2m_\omega^2(m_\pi^2+m_\rho^2)\times(C^{(8)}-C^{(1)})/f_\pi^2m_\pi^2m_\varphi^4\},\quad(79)$$

$$f_{\omega\rho\pi}=[(m_\pi^2+8C^{(8)})-m_\rho^2-m_\omega^2]/2(\sqrt{6})m_\rho g_\omega,\quad(80)$$

$$f_{\omega\pi\gamma}=[(m_\pi^2+8C^{(8)})-m_\omega^2]g_\rho/2(\sqrt{6})m_\rho^3g_\omega.\quad(81)$$

Using the experimental value[10]

$$\Gamma(\varphi\to\rho+\pi)=(0.48\pm0.20)\text{ MeV},\quad(82)$$

we obtain from (76) and (C6)

$$|C^{(8)}-C^{(1)}|=(0.75\pm0.16)\times10^4\text{ (MeV)}^2.\quad(83)$$

This gives

$$\Gamma(\varphi\to\pi^o+\gamma)=(0.056\pm0.023)\text{ MeV},\quad(84)$$

which is consistent with the present experimental upper limit of 0.08 MeV.[25]

From (66), (78), (83), and (C1), we obtain, as before [see Eq. (69)],

$$C^{(8)}=-(3.80\pm0.39)\times10^5\text{ (MeV)}^2.\quad(85)$$

The widths $\Gamma(\omega\to\pi^o+\gamma)$ and $\Gamma(\omega\to3\pi)$ are still as given in (74), while the width $\Gamma(\rho^0\to\pi^o+\gamma)$ is now given by

$$\Gamma(\rho^0\to\pi^o+\gamma)=(0.12\pm0.02)\text{ MeV}$$

for

$$C^{(8)}-C^{(1)}=(0.75\pm0.16)\times10^4\text{ (MeV)}^2,$$

or by

$$\Gamma(\rho^0\to\pi^o+\gamma)=(0.08\pm0.02)\text{ MeV}$$

for

$$C^{(8)}-C^{(1)}=-(0.75\pm0.16)\times10^4\text{ (MeV)}^2.\quad(86)$$

Lacking a more precise experimental determination of this width, we have no way of choosing between the two alternatives in (86).

IV. CONCLUDING REMARKS

The constants g_ρ, g_ω, and g_φ determine the lepton-pair decays of ρ^0, ω, and φ. Assuming that the electromagnetic current is as given in (61), there follows

$$\frac{\Gamma(\omega\to l^+l^-)}{\Gamma(\rho^0\to l^+l^-)}=\frac{1}{3}\left(\frac{m_\rho}{m_\omega}\right)^3\frac{g_\omega^2}{g_\rho^2},\quad(87)$$

$$\frac{\Gamma(\varphi\to l^+l^-)}{\Gamma(\rho^0\to l^+l^-)}=\frac{1}{3}\left(\frac{m_\rho}{m_\varphi}\right)^3\frac{g_\varphi^2}{g_\rho^2}.\quad(88)$$

From (72), (87), and (88), we have

$$\frac{\Gamma(\omega\to l^+l^-)}{\Gamma(\rho^0\to l^+l^-)}=\frac{1}{9}\left(\frac{m_\rho}{m_\omega}\right)=0.11,\quad(89)$$

$$\frac{\Gamma(\varphi\to l^+l^-)}{\Gamma(\rho^0\to l^+l^-)}=\frac{2}{9}\left(\frac{m_\rho}{m_\varphi}\right)=0.17.\quad(90)$$

Using the experimental result[26]

$$\frac{\Gamma(\rho^0\to\mu^+\mu^-)}{\Gamma(\rho^0\to\text{all})}=(5.1\pm1.2)\times10^{-5}\quad(91)$$

and the measured widths of ρ^0, ω, and φ,[10,24] there follows

$$\frac{\Gamma(\omega\to l^+l^-)}{\Gamma(\omega\to\text{all})}=(6.0\pm2.0)\times10^{-5},\quad(92)$$

$$\frac{\Gamma(\varphi\to l^+l^-)}{\Gamma(\varphi\to\text{all})}=(2.8\pm1.2)\times10^{-4}.\quad(93)$$

The data presently available are not sufficiently accurate to check the results (92) and (93). If it turns out that these results are in conflict with experiment, then we shall have to drop the assumption $C^{(8)}\approx C^{(1)}$ which we used to fix g_ω. The experimental value of g_ω could then be used to calculate the widths $\Gamma(\omega\to\pi^o+\gamma)$ and $\Gamma(\omega\to3\pi)$. This could be done by simply multiplying the results (74) by $(g_\omega^{(\text{theor})}/g_\omega^{(\text{expt})})^2$, where $g_\omega^{(\text{theor})}$ is given in (72). The constant $C^{(1)}$ could be determined from (59). There would, however, be no simple way to account for the decays $\varphi\to\rho+\pi$ and $\varphi\to\pi^o+\gamma$.

To illustrate the effect of a change in the decay constants, we consider an alternate possibility for fixing the constant g_ω, this one proposed by Das, Mathur, and Okubo.[21] In addition to the sum rule (57) they propose the condition

$$g_\omega^2+g_\varphi^2=\tfrac{1}{3}(4g_{K^*}^2-g_\rho^2)=\tfrac{1}{3}(4m_{K^*}^2/m_\rho^2-1)g_\rho^2,\quad(94)$$

from which there follow $g_\omega^2=0.43g_\rho^2$, $g_\varphi^2=1.03g_\rho^2$, and

$$\frac{\Gamma(\omega\to l^+l^-)}{\Gamma(\rho^0\to l^+l^-)}=0.14,\quad\frac{\Gamma(\varphi\to l^+l^-)}{\Gamma(\rho^0\to l^+l^-)}=0.15.\quad(95)$$

The decay widths $\Gamma(\omega\to\pi^o+\gamma)$ and $\Gamma(\omega\to3\pi)$ which follow from this new value of g_ω are

$$\Gamma(\omega\to\pi^o+\gamma)=(0.83\pm0.14)\text{ MeV},\quad\Gamma(\omega\to3\pi)=(9.2\pm1.6)\text{ MeV}.\quad(96)$$

Results (96) are in poorer agreement with experiment than those given in (74) [see (75)]. We note, in fact, that since results (74) are in such close agreement with experiment, any change in g_ω is going to worsen the results for the above widths.

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[25] J. S. Lindsey and G. A. Smith, Phys. Letters **20**, 93 (1966).

[26] A. Wehmann *et al.*, Phys. Rev. Letters **18**, 929 (1967).

We remark, finally, that the methods used in this paper are applicable to a study of the decays $\eta(\chi^0)\rightarrow 2\gamma$ and $\eta(\chi^0)\rightarrow \pi^+ + \pi^- + \gamma$. Such an application is currently in progress.

{*Note added in proof*. Implicit in the foregoing work is the assumption that a pole-dominance approximation for the time-ordered products is a good approximation at high energies as well as at low energies. This assumption is sufficient to ensure consistency between our work and that of Bjorken [J. B. Bjorken, Phys. Rev. **148**, 1467 (1966); B. L. Young, *ibid.* **161**, 1615 (1967); C. S. Lai and P. D. De Souza (to be published)].}

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APPENDIX A

We show here that, within the context of our meson-dominance approximation, the retarded commutators and the time-ordered products, corresponding to expressions (19)–(28), have the same pole structure. Specifically, we consider the matrix elements $\langle 0|R\{\partial_\mu A_a{}^\mu(\frac{1}{2}x)V_b{}^\nu(-\frac{1}{2}x)\}|\omega^\lambda(\mathbf{k})\rangle$ and $\langle 0|T\{\partial_\mu A_a{}^\mu(\frac{1}{2}x)V_b{}^\nu(-\frac{1}{2}x)\}|\omega^\lambda(\mathbf{k})\rangle$.

Assuming that an intermediate-state sum inserted into $\langle 0|\partial_\mu A_a{}^\mu(\frac{1}{2}x)V_b{}^\nu(-\frac{1}{2}x)|\omega^\lambda(\mathbf{k})\rangle$ is dominated by the π-meson state, and using (10), we have

$$(2k^o)^{1/2}\langle 0|\partial_\mu A_a{}^\mu(\tfrac{1}{2}x)V_b{}^\nu(-\tfrac{1}{2}x)|\omega^\lambda(\mathbf{k})\rangle = -i(f_\pi/\sqrt{2})m_\pi{}^2(2\pi)^{-3/2} \times\int d^4l\,\theta(l^0)\delta(l^2-m_\pi{}^2)(4k^ol^0)^{1/2}\langle\pi_a(\mathbf{l})|V_b{}^\nu(0)|\omega^\lambda(\mathbf{k})\rangle e^{-i(l-\frac{1}{2}k)\cdot x}. \quad \text{(A1)}$$

From (10), (11), (17), and (A1), there follows

$$(2\pi)^{3/2}(2k^o)^{1/2}\langle 0|\partial_\mu A_a{}^\mu(\tfrac{1}{2}x)V_b{}^\nu(-\tfrac{1}{2}x)|\omega^\lambda(\mathbf{k})\rangle = -i\delta_{ab}(2\pi)^{-3}(f_\pi/\sqrt{2})m_\pi{}^2g_\rho\epsilon^{\nu\lambda\sigma\tau}\partial_\sigma k_\tau\int d^4l\,e^{-il\cdot x}\theta(l^0+\tfrac{1}{2}k^o)\delta[(l+\tfrac{1}{2}k)^2-m_\pi{}^2][(l-\tfrac{1}{2}k)^2-m_\rho{}^2+i\epsilon]^{-1} \times[g_\omega\Gamma^{(8)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\omega{}^2)+f_\omega\Gamma^{(9)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\omega{}^2)]. \quad \text{(A2)}$$

In a similar manner, we obtain

$$(2\pi)^{3/2}(2k^o)^{1/2}\langle 0|V_b{}^\nu(-\tfrac{1}{2}x)\partial_\mu A_a{}^\mu(\tfrac{1}{2}x)|\omega^\lambda(\mathbf{k})\rangle = -i\delta_{ab}(2\pi)^{-3}(f_\pi/\sqrt{2})m_\pi{}^2g_\rho\epsilon^{\nu\lambda\sigma\tau}\partial_\sigma k_\tau\int d^4l\,e^{-il\cdot x}\theta[-(l^0-\tfrac{1}{2}k^o)]\delta[(l-\tfrac{1}{2}k)^2-m_\rho{}^2][(l+\tfrac{1}{2}k)^2-m_\pi{}^2\pm i\epsilon]^{-1} \times[g_\omega\Gamma^{(8)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\omega{}^2)+f_\omega\Gamma^{(9)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\omega{}^2)], \quad \text{(A3)}$$

where the $+i\epsilon$ $(-i\epsilon)$ must be used to obtain the time-ordered product (retarded commutator).

Using

$$\delta(x)=(1/2\pi i)[1/(x-i\epsilon)-1/(x+i\epsilon)], \quad \text{(A4)}$$

there then follow

$$(2\pi)^{3/2}(2k^o)^{1/2}\langle 0|T\{\partial_\mu A_a{}^\mu(\tfrac{1}{2}x)V_b{}^\nu(-\tfrac{1}{2}x)\}|\omega^\lambda(\mathbf{k})\rangle = i\delta_{ab}(2\pi)^{-4}(f_\pi/\sqrt{2})m_\pi{}^2g_\rho \times\int d^4l\,e^{-il\cdot x}\epsilon^{\nu\lambda\sigma\tau}k_\sigma l_\tau[(l+\tfrac{1}{2}k)^2-m_\pi{}^2+i\epsilon]^{-1}[(l-\tfrac{1}{2}k)^2-m_\rho{}^2+i\epsilon]^{-1} \times[g_\omega\Gamma^{(8)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\omega{}^2)+f_\omega\Gamma^{(9)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\omega{}^2)] \quad \text{(A5)}$$

and

$$(2\pi)^{3/2}(2k^o)^{1/2}\langle 0|R\{\partial_\mu A_a{}^\mu(\tfrac{1}{2}x)V_b{}^\nu(-\tfrac{1}{2}x)\}|\omega^\lambda(\mathbf{k})\rangle = \delta_{ab}(2\pi)^{-4}(f_\pi/\sqrt{2})m_\pi{}^2g_\rho \times\int d^4l\,e^{-il\cdot x}\epsilon^{\nu\lambda\sigma\tau}k_\sigma l_\tau[(l^0+(\tfrac{1}{2}k^o)+i\epsilon)^2-(\mathbf{l}+\tfrac{1}{2}\mathbf{k})^2-m_\pi{}^2]^{-1}[(l^0-(\tfrac{1}{2}k^o)+i\epsilon)^2-(\mathbf{l}-\tfrac{1}{2}\mathbf{k})^2-m_\rho{}^2]^{-1} \times[g_\omega\Gamma^{(8)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\omega{}^2)+f_\omega\Gamma^{(9)}((l+\tfrac{1}{2}k)^2,(l-\tfrac{1}{2}k)^2,m_\omega{}^2)]. \quad \text{(A6)}$$

APPENDIX B

The following expressions for the constants $\Gamma_j^{(\beta)}$, $j=1, 2, 3$, $\beta=8, 9$ are the solutions to Eqs. (48)–(53):

$$\Gamma_1^{(8)}=(i/2\sqrt{3})(g_\omega f_\varphi-g_\varphi f_\omega)^{-1}\{[2\sqrt{2}g_\rho C^{(8)}/f_\pi m_\pi^2 m_\rho^2+f_\pi/\sqrt{2}g_\rho](g_\omega f_\varphi-g_\varphi f_\omega)^{-1}[(f_\omega^2+f_\varphi^2)-\sqrt{2}(g_\omega f_\omega+g_\varphi f_\varphi)] \\ +(2\sqrt{2}/f_\pi m_\pi^2 g_\rho)[(g_\omega f_\varphi/m_\omega^2-g_\varphi f_\omega/m_\varphi^2)C^{(8)}+\sqrt{2}f_\omega f_\varphi(1/m_\omega^2-1/m_\varphi^2)C^{(1)}]\}, \quad \text{(B1)}$$

$$\Gamma_2^{(8)}=(i/2\sqrt{3})(g_\omega f_\varphi-g_\varphi f_\omega)^{-1}\{-[2\sqrt{2}g_\rho C^{(8)}/f_\pi m_\pi^2 m_\rho^2+f_\pi/\sqrt{2}g_\rho](g_\omega f_\varphi-g_\varphi f_\omega)^{-1}[(f_\omega^2+f_\varphi^2)-\sqrt{2}(g_\omega f_\omega+g_\varphi f_\varphi)] \\ +(2\sqrt{2}/f_\pi m_\pi^2 g_\rho)[(g_\omega f_\varphi/m_\omega^2-g_\varphi f_\omega/m_\varphi^2)C^{(8)}+\sqrt{2}f_\omega f_\varphi(1/m_\omega^2-1/m_\varphi^2)C^{(1)}]\}, \quad \text{(B2)}$$

$$\Gamma_3^{(8)}=(i/2\sqrt{3})(g_\omega f_\varphi-g_\varphi f_\omega)^{-1}\{[2\sqrt{2}g_\rho C^{(8)}/f_\pi m_\pi^2 m_\rho^2-f_\pi/\sqrt{2}g_\rho](g_\omega f_\varphi-g_\varphi f_\omega)^{-1}[(f_\omega^2+f_\varphi^2)-\sqrt{2}(g_\omega f_\omega+g_\varphi f_\varphi)] \\ -(2\sqrt{2}/f_\pi m_\pi^2 g_\rho)[(g_\omega f_\varphi/m_\omega^2-g_\varphi f_\omega/m_\varphi^2)C^{(8)}+\sqrt{2}f_\omega f_\varphi(1/m_\omega^2-1/m_\varphi^2)C^{(1)}]\}, \quad \text{(B3)}$$

$$\Gamma_1^{(9)}=(i/2\sqrt{3})(g_\omega f_\varphi-g_\varphi f_\omega)^{-1}\{[2\sqrt{2}g_\rho C^{(8)}/f_\pi m_\pi^2 m_\rho^2+f_\pi/\sqrt{2}g_\rho](g_\omega f_\varphi-g_\varphi f_\omega)^{-1}[\sqrt{2}(g_\omega^2+g_\varphi^2)-(g_\omega f_\omega+g_\varphi f_\varphi)] \\ +(2\sqrt{2}/f_\pi m_\pi^2 g_\rho)[g_\omega f_\varphi(1/m_\varphi^2-1/m_\omega^2)C^{(8)}+\sqrt{2}(g_\omega f_\varphi/m_\varphi^2-g_\varphi f_\omega/m_\omega^2)C^{(1)}]\}, \quad \text{(B4)}$$

$$\Gamma_2^{(9)}=(i/2\sqrt{3})(g_\omega f_\varphi-g_\varphi f_\omega)^{-1}\{-[2\sqrt{2}g_\rho C^{(8)}/f_\pi m_\pi^2 m_\rho^2+f_\pi/\sqrt{2}g_\rho](g_\omega f_\varphi-g_\varphi f_\omega)^{-1}[\sqrt{2}(g_\omega^2+g_\varphi^2)-(g_\omega f_\omega+g_\varphi f_\varphi)] \\ +(2\sqrt{2}/f_\pi m_\pi^2 g_\rho)[g_\omega g_\varphi(1/m_\varphi^2-1/m_\omega^2)C^{(8)}+\sqrt{2}(g_\omega f_\varphi/m_\varphi^2-g_\varphi f_\omega/m_\omega^2)C^{(1)}]\}, \quad \text{(B5)}$$

$$\Gamma_3^{(9)}=(i/2\sqrt{3})(g_\omega f_\varphi-g_\varphi f_\omega)^{-1}\{[2\sqrt{2}g_\rho C^{(8)}/f_\pi m_\pi^2 m_\rho^2-f_\pi/\sqrt{2}g_\rho](g_\omega f_\varphi-g_\varphi f_\omega)^{-1}[\sqrt{2}(g_\omega^2+g_\varphi^2)-(g_\omega f_\omega+g_\varphi f_\varphi)] \\ -(2\sqrt{2}/f_\pi m_\pi^2 g_\rho)[g_\omega g_\varphi(1/m_\varphi^2-1/m_\omega^2)C^{(8)}+\sqrt{2}(g_\omega f_\varphi/m_\varphi^2-g_\varphi f_\omega/m_\omega^2)C^{(1)}]\}. \quad \text{(B6)}$$

APPENDIX C

The following equations express the decay widths in terms of the decay constants (62)–(65) and (76)–(81)[8]:

$$\Gamma(\pi^o\to 2\gamma)=(\tfrac{1}{4}\pi)\alpha^2 m_\pi^3 f_{\pi\gamma\gamma}^2, \quad \text{(C1)}$$

$$\Gamma(\rho^0\to\pi^o+\gamma)=(\alpha/24)(m_\rho^2-m_\pi^2)^3 m_\rho^{-3} f_{\rho\pi\gamma}^2, \quad \text{(C2)}$$

$$\Gamma(\omega\to\pi^o+\gamma)=(\alpha/24)(m_\omega^2-m_\pi^2)^3 m_\omega^{-3} f_{\omega\pi\gamma}^2, \quad \text{(C3)}$$

$$\Gamma(\omega\to 3\pi)=(m_\omega-3m_\pi)^4(m_\rho^2-4m_\pi^2)^{-2}m_\omega m_\pi^2 3^{-3/2}W(m_\omega)(f_{\rho\pi\pi}^2/4\pi)(f_{\omega\rho\pi}^2/4\pi), \quad \text{(C4)}$$

$$W(3m_\pi)=1,\quad W(783\text{ MeV})=3.53,$$

$$\Gamma(\rho\to 2\pi)=\tfrac{1}{3}(m_\rho^2-4m_\pi^2)^{3/2}m_\rho^{-2}(f_{\rho\pi\pi}^2/4\pi), \quad \text{(C5)}$$

$$\Gamma(\varphi\to\rho+\pi)=[\tfrac{1}{4}(m_\varphi^2+m_\rho^2-m_\pi^2)^2 m_\varphi^{-2}-m_\rho^2]^{3/2}(f_{\varphi\rho\pi}^2/4\pi),$$

$$\Gamma(\varphi\to\pi^o+\gamma)=(\alpha/24)(m_\varphi^2-m_\pi^2)^3 m_\varphi^{-3} f_{\varphi\pi\gamma}^2. \quad \text{(C6)}$$