Sum Rules for Virtual Compton Scattering of Pions*

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Sum rules for virtual Compton scattering of pions have been obtained by using the Weinberg sum rules. Saturation of the sum rules by π, ω, ϕ, A_1, and A_2 intermediate states is found to be still unsatisfactory, suggesting that the sum rules converge very slowly. Also, the connection between the calculation of Das *et al.* for the π^+-π^0 mass difference and those of the standard techniques is discussed and clarified.

IN the present paper, we shall study the virtual photon-pion forward-scattering sum rules, utilizing both the Weinberg sum rule[1] and dispersion relations. First, the Weinberg sum rule has been used to obtain low-energy theorems for the virtual photon-pion scattering amplitudes. Then by means of unsubtracted dispersion relations we get dispersion sum rules which we try to test by saturating them with low-lying intermediate states such as π, ω, ϕ, A_1, and A_2 mesons. In this way, one finds that our sum rules (we call them four-point sum rules hereafter) are inconsistent with three-point sum rules[2] for vertex functions which have been derived by several authors, unless intermediate states with spin 2 or higher are taken into account. Also, our method makes clear the connection between the calculation of π^+-π^0 mass difference by Das *et al.*[3] and those of the conventional treatments.

We have also investigated the sum rules involving matrix elements of two axial-vector currents between pion states. Combining the low-energy theorems obtained from the Weinberg sum rule with dispersion relations, one finds other sum rules which are again badly satisfied unless we take higher intermediate states with spin greater than, or equal to, 2 into account.

We start with the following linear combination of amplitudes:

$$F_{\mu\nu}(k,q)=(2k_0V)i \times\int d^4x\, e^{iqx}\{\langle\pi^+(k)|(V_\mu^{(3)}(x)V_\nu^{(3)}(0))_+|\pi^+(k)\rangle -\langle\pi^0(k)|(V_\mu^{(3)}(x)V_\nu^{(3)}(0))_+|\pi^0(k)\rangle\}, \quad (1)$$

where k and q are pion and photon four-momenta, respectively.

In the quark model, the vector current $V_\mu^{(\alpha)}(x)$ and the axial-vector current $A_\mu^{(\alpha)}(x)$ are defined by

$$V_\mu^{(\alpha)}(x)=\tfrac{1}{2}i\bar{q}(x)\gamma_\mu\lambda_\alpha q(x), \quad A_\mu^{(\alpha)}(x)=\tfrac{1}{2}i\bar{q}(x)\gamma_\mu\gamma_5\lambda_\alpha q(x). \quad (2)$$

In the usual viewpoint taken in the algebra of currents, the validity of the quark model is not essential for the final calculations.

If $q^2=0$, then Eq. (1) defines a difference of forward π^+ and π^0 Compton scattering amplitudes. The reason why we chose such a special combination will be made clear shortly.

From covariance alone, one can write

$$F_{\mu\nu}(k,q)=\delta_{\mu\nu}F_1+q_\mu q_\nu F_2+(q_\mu k_\nu+q_\nu k_\mu)F_3 +i(q_\mu k_\nu-q_\nu k_\mu)F_4+k_\mu k_\nu F_5, \quad (3)$$

where F_i $(i=1\cdots5)$ are functions of q^2 and ν;

$$F_i\equiv F_i(\nu,q^2), \quad \nu=-k\cdot q. \quad (4)$$

Now, the crossing symmetry implies that F_i $(i=1, 2, 4, 5)$ are even functions of ν, while F_3 is an odd function of ν. Further, the conservation law $\partial_\mu V_\mu^{(\alpha)}(x)=0$, together with the crossing relations, demands

$$F_1+q^2F_2-\nu F_3=0, \quad q^2F_3-\nu F_5=0, \quad F_4=0, \quad (5)$$

where we have assumed that the Schwinger terms are c numbers, or at least they do not contain $I=2$ components. Thus, only F_1 and F_2 are linearly independent.

Now, we use the soft-pion technique by letting $k\to0$ and utilize the partially conserved axial-vector current hypothesis (PCAC) in the following form:

$$\partial_\mu A_\mu^{(\alpha)}(x)=(1/\sqrt{2})f_\pi\mu^2\pi_\alpha(x), \quad (6)$$

where the charge-pion decay constant f_π is defined by

$$\langle0|A_\mu^{(\alpha)}(0)|\pi_\beta(k)\rangle=[i/(2k_0V)^{1/2}](f_\pi/\sqrt{2})k_\mu\delta_{\alpha\beta}; \quad (7)$$

α and β are isospin indices.

Standard calculation leads to

$$F_{\mu\nu}(0,q)=-i(4/f_\pi^2)[\Delta_{\mu\nu}^V(q)-\Delta_{\mu\nu}^A(q)], \quad (8)$$

where

$$\Delta_{\mu\nu}^V(q)=\int d^4x\, e^{iqx}\langle0|(V_\mu^{(3)}(x)V_\nu^{(3)}(0))_+|0\rangle,$$

and a similar expression for $\Delta_{\mu\nu}^A(q)$. Note that the so-called σ terms do not enter into our expressions, since we have taken a difference between π^+ and π^0 Compton amplitudes, provided that the σ term is an isoscalar, as is usually assumed.

Now, the familiar spectral representations of the

* Work supported in part by the U. S. Atomic Energy Commission.
[1] S. Weinberg, Phys. Rev. Letters 18, 507 (1967).
[2] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters 19, 1067 (1967); D. A. Geffen, *ibid.* 19, 770 (1967); S. G. Brown and G. B. West, *ibid.* 19, 812 (1967); H. J. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967).
[3] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Letters 18, 759 (1967).

propagator functions are given by

$$\Delta_{\mu\nu}{}^{V}(q)=\int_0^\infty d(m^2)\frac{(-i)}{q^2+m^2}\left[\delta_{\mu\nu}\rho_V(m)+\frac{\tilde{\rho}_V(m)}{m^2}q_\mu q_\nu\right]+i\delta_{\mu4}\delta_{\nu4}\int_0^\infty d(m^2)\frac{\tilde{\rho}_V(m)}{m^2},$$
$$\Delta_{\mu\nu}{}^{A}(q)=\int_0^\infty d(m^2)\frac{(-i)}{q^2+m^2}\left[\delta_{\mu\nu}\rho_A(m)+\frac{\tilde{\rho}_A(m)}{m^2}q_\mu q_\nu\right]+i\delta_{\mu4}\delta_{\nu4}\int_0^\infty d(m^2)\frac{\tilde{\rho}_A(m)}{m^2}. \quad (9)$$

Using the Weinberg sum rule,[1]

$$\int_0^\infty d(m^2)\frac{\tilde{\rho}_A(m)-\tilde{\rho}_V(m)}{m^2}=0. \quad (10)$$

Equations (7) and (3) now give us the following low-energy theorems:

$$F_1(0,q^2)=\frac{4}{f_\pi^2}\int_0^\infty d(m^2)\frac{\rho_A(m)-\rho_V(m)}{q^2+m^2},$$
$$F_2(0,q^2)=\frac{4}{f_\pi^2}\int_0^\infty d(m^2)\frac{\tilde{\rho}_A(m)-\tilde{\rho}_V(m)}{m^2(q^2+m^2)}, \quad (11)$$

provided that $F_i(\nu,q^2)$ remains nonsingular for $\nu\to 0$. On the other hand, when we set $\nu=0$, the first relation in Eq. (5) gives

$$F_1(0,q^2)+q^2F_2(0,q^2)=0. \quad (12)$$

We have to investigate the compatibility of Eq. (11) with Eq. (12). We find that this is assured only if we let μ^2 (μ=pion mass) go to zero on the right-hand side of (11). Indeed, because of the conservation law $\partial_\mu V_\mu{}^{(\alpha)}(x)=0$ and the hypothesis of partially conserved axial-vector currents (PCAC) $\partial_\mu A_\mu{}^{(\alpha)}(x)=0$ for $\mu^2\to 0$, we have

$$\tilde{\rho}_V(m)=\rho_V(m),$$
$$(1/m^2)[\tilde{\rho}_A(m)-\rho_A(m)]=\tfrac{1}{2}f_\pi{}^2\delta(m^2-\mu^2). \quad (13)$$

Hence, Eq. (11) is compatible with Eq. (12) if we take into account Eq. (10) together with $\mu^2\to 0$. In this connection, we further note that by setting $q^2=0$ in the first relation of Eq. (11), one finds

$$F_1(0,0)=-2, \quad (14)$$

where we used Eqs. (13) and (10) again. Note that the low-energy theorem Eq. (14) is exactly the same as the ordinary Thomson limit for Compton scattering.

On the other hand, we expect to have dispersion relations[4] for $F_i(\nu,q^2)$ $(i=1,2)$. If the Regge theory is applicable to our virtual Compton scattering, then $F_i(\nu,q^2)$ should decrease faster than $\nu^{-\epsilon}$ with $\epsilon>0$, as has been noted by Harari,[5] since only $I=2$ particles can be exchanged in the t channel because of our special difference in Eq. (1). Of course, this argument neglects the possible Regge-cut contribution from the two-ρ exchange,[6] in the hope that somehow it will not contribute at all to the forward amplitude. Also, we hope that there is no fixed $J=0$, $I=2$ pole in the angular momentum plane. Indeed, a simple quark model will also lead to $F_i\to 0$ for $\nu\to\infty$. At any rate, one assumes unsubtracted dispersion relations

$$F_i(\nu,q^2)=\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu'\frac{1}{\nu'-\nu-i\epsilon}\,\mathrm{Im}F_i(\nu',q^2)\quad (i=1,2). \quad (15)$$

Then setting $\nu=0$ and noting Eq. (11), one has two sum rules

$$\frac{4}{f_\pi^2}\int_0^\infty d(m^2)\frac{\rho_A(m)-\rho_V(m)}{q^2+m^2}=\frac{1}{\pi}\int_{-\infty}^{\infty}d\nu'\frac{1}{\nu'}\,\mathrm{Im}F_1(\nu',q^2), \quad (16)$$

$$\frac{4}{f_\pi^2}\int_0^\infty d(m^2)\frac{\tilde{\rho}_A(m)-\tilde{\rho}_V(m)}{m^2(q^2+m^2)}=\frac{1}{\pi}\int_{-\infty}^{\infty}d\nu'\frac{1}{\nu'}\,\mathrm{Im}F_2(\nu',q^2). \quad (17)$$

The no-subtraction ansatz is crucial to our conclusions.

First, we note that Eq. (15) implies a superconvergent dispersion relation

$$\int_{-\infty}^{\infty}d\nu'\,\mathrm{Im}F_3(\nu',q^2)=0 \quad (18)$$

because of Eqs. (12) and (5). Then Eq. (17) is not independent, since it is derivable from Eqs. (16) and (18). Then, because of the second relation of Eq. (5), this in turn is equivalent to a superconvergent dispersion relation

$$\int_{-\infty}^{\infty}d\nu'\,\nu'\,\mathrm{Im}F_5(\nu',q^2)=0, \quad (19)$$

which has already been used by Gilman and Harari[7] for $q^2=m_\rho{}^2$ to derive sum rules. If we set $q^2=0$ in Eq. (16), it reduces to the sum rule given by Pagels[8] and Harari[7] because of Eq. (14). Hence, our sum rules Eqs. (16) and (17) are generalizations of the sum rules used by the above authors for $q^2\neq 0$, although we have to make one sacrifice by setting $\mu^2=0$.

[4] In the derivation of the dispersion relations, we first assume that q is a spacelike vector $q^2>0$ as in the ordinary proof. After that, we analytically continue q^2 to the general case, if we wish, provided of course that it is possible to do so.

[5] H. Harari, Phys. Rev. Letters **17**, 1303 (1966).
[6] I. J. Muzinich, Phys. Rev. Letters **18**, 381 (1967).
[7] H. Harari, Phys. Rev. Letters **18**, 319 (1967); F. J. Gilman and H. Harari, *ibid.* **18**, 1150 (1967); Phys. Rev. **165**, 1803 (1968).
[8] H. Pagels, Phys. Rev. Letters **18**, 316 (1967).

Now, we approximate the spectral functions by retaining only ρ, π, and A_1 poles:

$$\begin{aligned}\rho_V(m)&=\tilde{\rho}_V(m)=G_\rho^2\delta(m^2-m_\rho^2),\\ \rho_A(m)&=G_{A_1}^2\delta(m^2-m_{A_1}^2).\end{aligned}\tag{20}$$

Then Eqs. (16) and (17) reduce to

$$\begin{aligned}&\frac{4}{f_\pi^2}\left(\frac{G_{A_1}^2}{q^2+m_{A_1}^2}-\frac{G_\rho^2}{q^2+m_\rho^2}\right)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\nu'\frac{1}{\nu'}\,\mathrm{Im}F_1(\nu',q^2),\\ &\frac{2}{q^2}+\frac{4}{f_\pi^2}\left(\frac{G_{A_1}^2}{m_{A_1}^2}\frac{1}{q^2+m_{A_1}^2}-\frac{G_\rho^2}{m_\rho^2}\frac{1}{q^2+m_\rho^2}\right)\\ &\qquad=\frac{1}{\pi}\int_{-\infty}^{\infty}d\nu'\frac{1}{\nu'}\,\mathrm{Im}F_2(\nu',q^2),\end{aligned}\tag{21}$$

where m_ρ and m_{A_1} represent masses of ρ and A_1 mesons, respectively, and we have set $\mu^2=0$.

The right-hand side of Eq. (21) can be computed in the standard way, starting from

$$\begin{aligned}&\delta_{\mu\nu}\,\mathrm{Im}F_1+q_\mu q_\nu\,\mathrm{Im}F_2+(q_\mu k_\nu+q_\nu k_\mu)\,\mathrm{Im}F_3\\ &+i(q_\mu k_\nu-q_\nu k_\mu)\,\mathrm{Im}F_4+k_\mu k_\nu\,\mathrm{Im}F_5=-k_0V(2\pi)^4\\ &\times\sum_n\{\langle\pi^+(k)|V_\mu^{(3)}(0)|n\rangle\langle n|V_\nu^{(3)}(0)|\pi^+(k)\rangle\\ &\times\delta^{(4)}(q-k+p_n)-\langle\pi^+(k)|V_\nu^{(3)}(0)|n\rangle\\ &\times\langle n|V_\mu^{(3)}(0)|\pi^+(k)\rangle\delta^{(4)}(q-p_n+k)-(\pi^+\to\pi^0)\}.\end{aligned}\tag{22}$$

It is obvious that only intermediate states with $G=-1$ will give rise to nonzero contributions to the right-hand side summation of Eq. (22). Hence, we approximate the right-hand side by taking into account only π, ω, ϕ, A_1, and A_2 poles:

(i) Pion pole: We write

$$\begin{aligned}\langle\pi^+(k)|V_\nu^{(3)}(0)|\pi^+(p)\rangle&=(1/V)[1/(4p_0k_0)^{1/2}]\\ &\quad\times(p_\mu+k_\mu)F_\pi(q^2),\end{aligned}\tag{23}$$

where $F_\pi(q^2)$ is the pion electromagnetic form factor. In the ρ-dominance model, $F_\pi(q^2)=m_\rho^2/(q^2+m_\rho^2)$. A simple computation yields

$$\mathrm{Im}F_1^{(\pi)}(\nu,q^2)=0,\tag{24a}$$

$$\mathrm{Im}F_2^{(\pi)}(\nu,q^2)=\pi F_\pi^2(q^2)\delta(q^2-2\nu).\tag{24b}$$

The superscripts denote the contributing pole.

(ii) The ρ pole does not contribute because it has $G=+1$.

(iii) The ω pole contributes only to the π^0 term. The coupling used is

$$\begin{aligned}\langle\pi^0(k)|V_\nu^{(3)}(0)|\omega(p)\rangle&=(1/V)[i/(4p_0k_0)^{1/2}]\\ &\quad\times G_{\omega\pi\gamma}(q^2)\epsilon_{\nu\alpha\beta\gamma}k_\alpha p_\beta\epsilon_\gamma^{(\omega)}(p),\end{aligned}\tag{25}$$

where $\epsilon_{\nu\alpha\beta\gamma}$ is the completely antisymmetric symbol and $\epsilon_\gamma^{(\omega)}(p)$ is the ω-meson polarization vector. A simple calculation yields

$$\begin{aligned}\mathrm{Im}F_1^{(\omega)}(\nu,q^2)&=-\pi G_{\omega\pi\gamma}^2(q^2)\\ &\quad\times\delta(q^2+m_\omega^2-2\nu)\tfrac{1}{4}(q^2+m_\omega^2)^2,\end{aligned}\tag{26a}$$

$$\mathrm{Im}F_2^{(\omega)}(\nu,q^2)=0,\tag{26b}$$

where we set $\mu^2=0$ again. The ϕ contribution can be obtained from this by simply replacing ω by ϕ.

(iv) For the A_1-pole calculation, we use

$$\begin{aligned}\langle A_\beta(p)|V_\mu^{(\alpha)}(0)|\pi_\gamma(k)\rangle&=-\epsilon_{\alpha\gamma\beta}(1/V)[1/(4p_0k_0)^{1/2}]\\ &\times\{[\bar{\epsilon}_\mu(p)(p^2-k^2)+(p_\mu+k_\mu)(\bar{\epsilon}\cdot k)]C(q^2)\\ &+[\bar{\epsilon}_\mu(p)q^2+(p_\mu-k_\mu)(\bar{\epsilon}\cdot k)]D(q^2)\},\end{aligned}$$

where α, β, γ are isospin indices, $\epsilon(p)$ is the A_1 polarization, and $C(q^2)$ and $D(q^2)$ are $A_1\to\pi\gamma$ form factors.

After some calculation, one finds

$$\begin{aligned}\mathrm{Im}F_1^{(A_1)}(\nu,q^2)&=\pi\delta(q^2+m_{A_1}^2-2\nu)\\ &\quad\times[q^2D(q^2)-m_{A_1}^2C(q^2)]^2,\end{aligned}\tag{27a}$$

$$\begin{aligned}\mathrm{Im}F_2^{(A_1)}(\nu,q^2)&=\pi\delta(q^2+m_{A_1}^2-2\nu)\\ &\quad\times\{(1/m_{A_1}^2)[-\tfrac{1}{2}(q^2+3m_{A_1}^2)C(q^2)\\ &\quad+\tfrac{1}{2}(q^2-m_{A_1}^2)D(q^2)]^2\}.\end{aligned}\tag{27b}$$

We have again put $\mu^2=0$ in the above calculations.

For a while, we shall neglect the contribution from ϕ and A_2 intermediate states.

Combining these pole contributions, we obtain

$$\begin{aligned}F_1(0,q^2)&=-2G_{\omega\pi\gamma}^2(q^2)\tfrac{1}{2}(q^2+m_\omega^2)\\ &\quad+[2/(q^2+m_{A_1}^2)][q^2D(q^2)-m_{A_1}^2C(q^2)]^2.\end{aligned}\tag{28a}$$

Similarly,

$$\begin{aligned}F_2(0,q^2)&=(2/q^2)F_\pi^2(q^2)+[2/(q^2+m_{A_1}^2)](1/m_{A_1}^2)\\ &\quad\times[\tfrac{1}{2}(q^2-m_{A_1}^2)D(q^2)-\tfrac{1}{2}(q^2+3m_{A_1}^2)C(q^2)]^2.\end{aligned}\tag{28b}$$

By ρ dominance,

$$G_{\omega\pi\gamma}(q^2)=[m_\rho^2/(q^2+m_\rho^2)]g_{\omega\pi\gamma},$$

where $g_{\omega\pi\gamma}$ is the coupling constant to be determined from the $\omega\to\pi\gamma$ width.

We use the following parametrization on the basis of the ρ-dominance model

$$q^2D(q^2)-m_{A_1}^2C(q^2)=\theta_1+\lambda_1/(q^2+m_\rho^2)\tag{29}$$

and

$$\begin{aligned}&\tfrac{1}{2}(q^2-m_{A_1}^2)D(q^2)-\tfrac{1}{2}(q^2+3m_{A_1}^2)C(q^2)\\ &\qquad=\tfrac{1}{2}[\theta_2+\lambda_2/(q^2+m_\rho^2)].\end{aligned}\tag{30}$$

From (11) and (28)–(30), we have the following relations:

$$\begin{aligned}\frac{4}{f_\pi^2}\left[\frac{G_{A_1}^2}{q^2+m_{A_1}^2}-\frac{G_\rho^2}{q^2+m_\rho^2}\right]&=-\tfrac{1}{2}g_{\omega\pi\gamma}^2m_\rho^4\frac{q^2+m_\omega^2}{(q^2+m_\rho^2)^2}\\ &\quad+\frac{2}{q^2+m_{A_1}^2}\left[\theta_1+\frac{\lambda_1}{q^2+m_\rho^2}\right]^2\end{aligned}\tag{31}$$

and

$$\frac{2}{q^2}+\frac{4}{f_\pi^2}\left[\frac{G_{A_1}^2}{m_{A_1}^2}\frac{1}{q^2+m_{A_1}^2}-\frac{G_\rho^2}{m_\rho^2}\frac{1}{q^2+m_\rho^2}\right]$$
$$=\frac{2}{q^2}F_\pi^2(q^2)+\frac{1}{q^2+m_{A_1}^2}\frac{1}{2(m_{A_1}^2)}\left[\theta_2+\frac{\lambda_2}{q^2+m_\rho^2}\right]^2. \quad (32)$$

Note that the right-hand sides of Eqs. (31) and (32) contain double poles whereas the left-hand sides do not. Simplifying and breaking up into partial fractions, we obtain the following sum rules from (31) by comparing coefficients of $(m_\rho^2+q^2)^{-2}$, $(m_\rho^2+q^2)^{-1}$, and $(m_{A_1}^2+q^2)^{-1}$, respectively:

$$-\tfrac{1}{2}g_{\omega\pi\gamma}^2m_\rho^4(m_\omega^2-m_\rho^2)+2\lambda_1^2/(m_{A_1}^2-m_\rho^2)=0, \quad (33)$$

$$-\tfrac{1}{2}g_{\omega\pi\gamma}^2m_\rho^4+4\theta_1\lambda_1/(m_{A_1}^2-m_\rho^2)-2\lambda_1^2/(m_{A_1}^2-m_\rho^2)^2 =-4G_\rho^2/f_\pi^2, \quad (34)$$

$$2\theta_1^2-4\theta_1\lambda_1/(m_{A_1}^2-m_\rho^2)+2\lambda_1^2/(m_{A_1}^2-m_\rho^2)^2 =(4/f_\pi^2)G_{A_1}^2. \quad (35)$$

Similarly, Eq. (32) yields

$$-2m_\rho^2+(\lambda_2)^2/2m_{A_1}^2(m_{A_1}^2-m_\rho^2)=0, \quad (36)$$

$$-2+\frac{2\theta_2\lambda_2}{2m_{A_1}^2}\frac{1}{m_{A_1}^2-m_\rho^2}-\frac{(\lambda_2)^2}{2m_{A_1}^2}\frac{1}{(m_{A_1}^2-m_\rho^2)^2} =-\frac{4}{f_\pi^2}\frac{G_\rho^2}{m_\rho^2}, \quad (37)$$

$$\frac{\theta_2^2}{2m_{A_1}^2}-\frac{\theta_2\lambda_2}{m_{A_1}^2}\frac{1}{(m_{A_1}^2-m_\rho^2)}+\frac{(\lambda_2)^2}{2m_{A_1}^2}\frac{1}{(m_{A_1}^2-m_\rho^2)^2} =\frac{4}{f_\pi^2}\frac{G_{A_1}^2}{m_{A_1}^2}. \quad (38)$$

In the case of exact mass degeneracy of the ω and ρ mesons, Eq. (33) tells us that $\lambda_1=0$.

Now, spin and parity conservation allows $A_1\to\rho\pi$ decay to go via S and D waves, so there are two independent $A_1\rho\pi$ couplings

$$\langle A_\tau(p)|j_{\pi\alpha}(0)|\rho_\gamma(p')\rangle=-(\epsilon_{\alpha\gamma\tau}/V)(4p_0p_0')^{-1/2} \times\{G_S[\bar{\epsilon}^{A_1}(p)\cdot\epsilon^\rho(p')]+G_D[\bar{\epsilon}^{A}(p)\cdot p'][\epsilon^\rho(p')\cdot p]\}. \quad (39)$$

Here α, τ, γ are isospin indices, ϵ^{A_1} and ϵ^ρ are A_1 and ρ meson polarization vectors, and G_S and G_D are S- and D-wave coupling constants.

Now, again assuming unsubtracted dispersion relations for $C(q^2)$ and $D(q^2)$ occurring in Eq. (27), one finds, with ρ dominance,

$$C(q^2)=-\tfrac{1}{2}G_\rho G_D/(q^2+m_\rho^2),$$
$$D(q^2)=-(G_\rho/m_\rho^2)[G_S-\tfrac{1}{2}m_{A_1}^2G_D]1/(q^2+m_\rho^2), \quad (40)$$

so that

$$\lambda_1=G_\rho G_S;\quad \theta_1=-(G_\rho/m_\rho^2)[G_S-\tfrac{1}{2}m_{A_1}^2G_D],$$
$$\lambda_2=(G_\rho/m_\rho^2)[(m_{A_1}^2+m_\rho^2)G_S-\tfrac{1}{2}(m_{A_1}^2-m_\rho^2)^2G_D], \quad (41)$$
$$\theta_2=-(G_\rho/m_\rho^2)[G_S-\tfrac{1}{2}(m_{A_1}^2+m_\rho^2)G_D].$$

Therefore, $\lambda_1=0$ implies $G_S=0$, which means that A_1 decay occurs only through D wave. If we take derivatives d/dq^2 of Eqs. (31) and (32) and set $q^2=0$, we obtain sum rules which are inconsistent with λ_1, θ_1, etc., given in Eq. (41).

Gilman and Harari[7] have studied forward π–ρ scattering superconvergent sum rules. They attempt to saturate the $I=1$ and $I=2$ sum rules with π, ω, and A_1 poles. In the limit of $m_\omega=m_\rho$, they also obtain that $G_S=0$ and that the A_1 decay is purely D wave. This is an unpleasant feature as will be clear from the following discussions. All recent calculations on 3-point functions show that $A_1\to\rho\pi$ must have both S- and D-wave parts. Many authors[2] have recently shown that

$$G_S-\tfrac{1}{2}(m_{A_1}^2-m_\rho^2)G_D=-\frac{\sqrt{2}}{f_\pi}\frac{G_{A_1}}{G_\rho}\frac{m_{A_1}^2-m_\rho^2}{m_{A_1}^2}m_\rho^2 \quad (42a)$$

and

$$G_\rho^2=m_\rho^2f_\pi^2/(1+\delta), \quad (42b)$$

where

$$\delta=\frac{m_\rho^2f_\pi}{\sqrt{2}m_{A_1}^2(m_{A_1}^2-m_\rho^2)}\frac{G_{A_1}}{G_\rho}[G_S+\tfrac{1}{2}(m_{A_1}^2-m_\rho^2)G_D] =(m_{A_1}^2-2m_\rho^2)/m_{A_1}^2. \quad (42c)$$

Therefore, $\delta=0$ and $G_S=0$ implies that $G_D=0$, i.e., A_1 cannot decay. From the point of view of the Schnitzer and Weinberg analysis,[2] pure D-wave A_1 decay corresponds to

$$\Gamma(A_1\to\rho\pi)=9\text{ MeV},$$
$$\Gamma(\rho\to\pi\pi)=265\text{ MeV},$$

in disagreement with experiments.

Substituting $(\lambda_2)^2$ from (36) and (37) and carrying out simple algebraic manipulations, one obtains

$$\frac{\theta_2\lambda_2}{m_{A_1}^2}\frac{1}{m_{A_1}^2-m_\rho^2}=-\frac{4}{f_\pi^2}\frac{G_\rho^2}{m_\rho^2}+\frac{2m_{A_1}^2}{m_{A_1}^2-m_\rho^2}. \quad (43)$$

Using the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KS-RF)[9] formula, $G_\rho^2=m_\rho^2f_\pi^2$ [see Eq. (42b) for $\delta=0$] and the Weinberg relations[1] $m_{A_1}^2=2m_\rho^2$ [$\delta=0$ for Eq. (42c)], we get $\theta_2=0$. But Eqs. (41) and (42) yield after some manipulation,

$$\theta_2=f_\pi m_{A_1}^2/\sqrt{2}G_{A_1}. \quad (44)$$

Thus $\theta_2=0$ implies $f_\pi=0$ which is definitely not good, although numerically it may not be very bad since

[9] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

$f_\pi \approx \mu$, and we have set $\mu^2=0$. Thus, we cannot stop with the A_1 intermediate state and must go to higher states such as 2^+ particles. The f and f' mesons do not contribute for reasons of G parity. However, the A_2 meson (1306 MeV) may contribute.

We choose the coupling

$$\langle \pi^+(k)| V_\nu^3(0)| A_2(p)\rangle = (i/V)(4p_0k_0)^{-1/2} \times f(q^2)\epsilon_{\nu\alpha\beta\lambda}k_\alpha q_\beta \epsilon_{\lambda\tau}(p)q_\tau, \quad (45)$$

where $\epsilon_{\lambda\tau}(p)$ is the A_2 polarization tensor. The propagators for spin-2 particles are very complicated. After carrying out an elaborate calculation, one finds for the contribution from the A_2 pole

$$F_1^{A_2}(0,q^2) = (1/16M_{A_2}^2)f^2(q^2)(q^2+M_{A_2}^2)^3,$$
$$F_2^{A_2}(0,q^2)=0.$$

ρ dominance gives $f(q^2)=G_\rho g_{A_2\rho\pi}/(q^2+m_\rho^2)$;

$$f(0)=G_\rho g_{A_2\rho\pi}/m_\rho^2 \equiv g_{A_2\pi\gamma},$$

$$F_1^{A_2}(0,q^2)=(1/16M_{A_2}^2)G_\rho^2 G_{A_2\rho\pi}^2 \times (q^2+M_{A_2}^2)^3/(q^2+m_\rho^2)^2, \quad (46)$$

which behaves as q^2 for $q\to\infty$ (M_{A_2} is the mass of the A_2 meson). Then the previous sum rule is inconsistent unless $g_{A_2\rho\pi}=0$, since the left-hand side decreases as $1/q^2$ while the right-hand side increases as q^2. The contribution from 2^- mesons (if they exist) can be calculated from the coupling

$$\langle \pi^+(k)| V_\nu^3(0)| 2^-(p)\rangle = (1/V)(4p_0k_0)^{-1/2}\epsilon_{\rho\lambda}(p) \times \{f_1(q^2)\delta_{\lambda\nu}k_\rho + [f_2(q^2)k_\nu + f_3(q^2)p_\nu]k_\lambda k_\rho\}. \quad (47)$$

Then their contributions to F_1 and F_2 are given by

$$F_1^{2^-}(0,q^2)=(1/4M_{2^-}^2)(q^2+M_{2^-})f_1^2(q^2), \quad (48a)$$

$$F_2^{2^-}(0,q^2)=(1/3M_{2^-}^4)(q^2+M_{2^-}^2) \times [f_1(q^2)-\tfrac{1}{2}(q^2+M_{2^-}^2)f_3(q^2)]^2. \quad (48b)$$

Since the ρ dominance model suggests that $f_3(q^2) \sim 1/(q^2+m_\rho^2)$ for $q^2\to\infty$, again $F_2(0,q^2)\sim q^2$ as $q\to\infty$ and our sum rule becomes meaningless unless $f_3(q^2)\equiv 0$. Note that both bad terms coming from 2^+ and 2^- mesons cannot cancel each other since the former contributes only to F_1 whereas the latter contributes to F_2. The problem we are encountering may be due to kinematical complexities of spin $\geqslant 2$ particles. Although the contributions from these high-spin particles are expected to be small, as we see from the argument given by Gilman and Harari[7] for the $q^2=-m_\rho^2$ case, we shall try to remedy the situation by assuming that these high-spin particles cannot be regarded as elementary particles but as resonances built out of a two-particle $\rho\pi$ intermediate state. Then the calculation is rather involved, but we give a reasonable argument in the Appendix that the A_2 contribution as a $\rho\pi$ resonance is now given by the following instead of Eq. (46):

$$F_1^{2^+}(0,q^2) = -\tfrac{1}{16}m_\rho^4 M_{A_2}^2 g_{A_2\pi\gamma}^2 \times \left[\frac{1}{q^2+m_\rho^2}+\frac{M_{A_2}^2-m_\rho^2}{(q^2+m_\rho^2)^2}\right], \quad (49)$$

where $g_{A_2\pi\gamma}$ is the coupling constant for $A_2\to\pi\gamma$ and in the ρ-dominance model is given by

$$g_{A_2\pi\gamma}=G_\rho g_{A_2\rho\pi}/m_\rho^2. \quad (50)$$

Then we modify the left-hand sides of Eqs. (33) and (34) by adding

$-\tfrac{1}{16}m_\rho^4 M_{A_2}^2 g_{A_2\pi\gamma}^2$ and $-\tfrac{1}{16}m_\rho^4 M_{A_2}^2 g_{A_2\pi\gamma}^2(M_{A_2}^2-m_\rho^2)$,

respectively. Similarly, the contribution from the ϕ meson can be easily found by replacing ω by ϕ everywhere. Then Eqs. (33) and (34) are modified as follows, while Eq. (35) remains unaltered:

$$-\tfrac{1}{2}g_{\omega\pi\gamma}^2 m_\rho^4(m_\omega^2-m_\rho^2)-\tfrac{1}{2}g_{\phi\pi\gamma}^2 m_\rho^4(m_\phi^2-m_\rho^2) -\tfrac{1}{16}g_{A_2\pi\gamma}^2 m_\rho^4 M_{A_2}^2(M_{A_2}^2-m_\rho^2)+2\lambda_1^2/m_\rho^2=0, \quad (51)$$

$$-\tfrac{1}{2}g_{\omega\pi\gamma}^2 m_\rho^4-\tfrac{1}{2}g_{\phi\pi\gamma}^2 m_\rho^4+4\theta_1\lambda_1/m_\rho^4-2(\lambda_1)^2/m_\rho^4 -\tfrac{1}{16}g_{A_2\pi\gamma}^2 m_\rho^4 m_{A_2}^2=-4m_\rho^2, \quad (52)$$

$$2\theta_1^2-4\theta_1\lambda_1/m_\rho^2+2(\lambda_1)^2/m_\rho^4=4m_\rho^2. \quad (53)$$

We have used the Weinberg relations $m_{A_1}^2=2m_\rho^2$; $G_{A_1}=G_\rho$ and the KS-RF[9] relation $G_\rho^2=m_\rho^2 f_\pi^2$ [$\delta=0$ in Eq. (42)] and $g_{A_2\pi\gamma}=g_{A_2\rho\pi}G_\rho/m_\rho^2$ can be computed from the experimental $A_2\to\rho\pi$ width. Similarly, $g_{\phi\pi\gamma}$ can be computed by assuming $\Gamma(\phi\to\rho\pi)/\Gamma(\phi\to\text{all}) \simeq 10\%$ as an upper limit by using ρ dominance for $\phi\to\pi\gamma$ modes. From Eq. (51) we obtain

$$|\lambda_1|=6.203\times10^8. \quad (54)$$

Using this value of λ_1, Eq. (52) yields $\theta_1=149.7$ with $m_\rho\simeq 770$ MeV; the left-hand side of (53) is 1.43×10^6, but the right-hand side is 2.37×10^6. This indicates that the sum rules still are not well saturated, and contributions from still higher states should be considered. The $\pi\gamma$-scattering sum rules are thus difficult to saturate.

For the sum rules (36), (37), and (38), we have to add at least a contribution from the 2^- meson in order to get reasonable agreement. But, first, the existence of a 2^- particle is experimentally uncertain[10] and second, we have to introduce many unknown parameters. Therefore, we do not have much to say for this case, except for the fact that the saturation of the sum rule is slow and we again have to include higher resonant intermediate states with $J\geqslant 2$. However, we have to bear in mind that, for the above conclusion, we used the result based upon three-point sum rules Eqs. (42a)–(42c). If we do not use these 3-point vertex sum rules, based upon the Weinberg sum rule for $\langle\pi_\alpha(k)|A_\mu^{(\beta)}(0)|\rho_\gamma(p)\rangle$ and $\langle\pi_\alpha(k)|V_\mu^{(\beta)}(0)|A_{1\gamma}(p)\rangle$, then our four-point sum rules Eqs. (33)–(38) may be still consistent. However, it may

[10] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).

be worthwhile to emphasize that for the three-point vertex sum rules, the only intermediate states to be taken into account are those with $J=0$ or 1, and no higher states with $J=2$ are needed, in contrast with the 4-point sum rules. Hence, probably, the three-point sum rules are more trustworthy. Further, we encountered some ambiguities from $J=2^{\pm}$ intermediate states. But their contributions are not so large, in general, as to offset the conclusion.

Now let us consider the connection between the calculation of Das *et al.*[3] for the π^+-π^0 mass difference and those of the standard methods.[11]

The electromagnetic mass difference of pions may be expressed as

$$\Delta(\mu^2) \propto \int K_{\mu\nu}(q^2) M_{\mu\nu}(k,q) d^4q\,, \tag{55}$$

where $K_{\mu\nu}(q^2)$ is the photon propagator and $M_{\mu\nu}$ is the forward Compton scattering amplitude for virtual photons.

In the old pion mass-difference calculations,[11] one considers only the pion pole and obtains

$$\Delta(\mu^2) \propto e^2 \int \frac{d^4q}{q^2-i\epsilon}\left[-4+\frac{q^2-4k\cdot q-4\mu^2}{q^2-2k\cdot q-i\epsilon}\right] F_\pi^2(q^2)\,. \tag{56}$$

From ρ-pole dominance, the pion electromagnetic form factor is

$$F_\pi(q^2) = m_\rho^2/(q^2+m_\rho^2)\,. \tag{57}$$

Thus, the integrand in the Eq. (56) has a double pole at the ρ mass: $1/(m_\rho^2+q^2)^2$.

The modern calculation of the pion mass difference by Das *et al.*[3] gives

$$\Delta(\mu^2) \propto e^2 \int \frac{d^4q}{q^2-i\epsilon} \int_0^\infty d(m^2) \frac{\rho_V(m)-\rho_A(m)}{q^2+m^2}\,, \tag{58}$$

which does not have a double pole.

Our calculations indicate that the π-γ scattering sum rules are very slowly converging, and that, in addition to the pion pole, one must consider ω, ϕ, A_1, A_2, and even higher states. Significant contributions from these states indicate that the older calculation with only a pion pole gave good results only accidentally. Since the current-algebra calculation did not give rise to a double pole, as seen from Eqs. (28)–(32), the sum of the residues at the double pole obtained from dispersion theory must vanish as displayed in (51).

We now consider sum rules resulting from the following combination of amplitudes:

$$G_{\mu\nu}(k,q) = (2k_0V)i \int d^4x\, e^{iqx} \times\{\langle\pi^+(k)|(A_\mu^{(3)}(x)A_\nu^{(3)}(0))_+|\pi^+(k)\rangle - \langle\pi^0(k)|(A_\mu^{(3)}(x)A_\nu^{(3)}(0))_+|\pi^0(k)\rangle\}\,. \tag{59}$$

[11] Riazuddin, Phys. Rev. 114, 1184 (1959); V. Barger and E. Kazes, Nuovo Cimento 28, 385 (1963).

We have replaced the vector currents in Eq. (1) by axial-vector currents. Let us use the decomposition

$$G_{\mu\nu}(k,q) = \delta_{\mu\nu}G_1 + q_\mu q_\nu G_2 + \cdots\,. \tag{60}$$

The calculations proceed exactly as before, and low-energy theorems similar to Eq. (11) may be written down by simply making the replacement $V_\mu^{(3)} \leftrightarrow A_\mu^{(3)}$ in the previous results. Also, we assume unsubtracted dispersion relations for $G_i(\nu,q^2)$ ($i=1$, 2). Then the relations analogous to Eq. (21) are

$$G_1(0,q^2) = \frac{4}{f_\pi^2}\left[\frac{G_\rho^2}{q^2+m_\rho^2} - \frac{G_{A_1}^2}{q^2+m_{A_1}^2}\right] = \frac{1}{\pi}\int_{-\infty}^{\infty} d\nu' \frac{1}{\nu'} \operatorname{Im} G_1(\nu',q^2)\,,$$

$$G_2(0,q^2) = -\frac{2}{q^2} + \frac{4}{f_\pi^2}\left[\frac{G_\rho^2}{m_\rho^2}\frac{1}{q^2+m_\rho^2} - \frac{G_{A_1}^2}{m_A^2}\frac{1}{q^2+m_{A_1}^2}\right] = \frac{1}{\pi}\int_{-\infty}^{\infty} d\nu' \frac{1}{\nu'} \operatorname{Im} G_2(\nu',q^2)\,. \tag{61}$$

The right-hand sides of Eq. (61) are computed from

$$\begin{aligned}\delta_{\mu\nu} \operatorname{Im} G_1 + q_\mu q_\nu \operatorname{Im} G_2 + \cdots &= -k_0 V(2\pi)^4 \sum_n \{\langle\pi^+(k)|A_\mu^{(3)}(0)|n\rangle \\ &\times\langle n|A_\nu^{(3)}(0)|\pi^+(k)\rangle\delta^{(4)}(q-k+p_n) \\ &-\langle\pi^+(k)|A_\nu^{(3)}(0)|n\rangle\langle n|A_\mu^{(3)}(0)|\pi^+(k)\rangle \\ &\times\delta^{(4)}(q-p_n+k) - (\pi^+\to\pi^0)\}\,.\end{aligned} \tag{62}$$

Only $G=+1$ intermediate states may have a nonvanishing contribution in this case; hence π, ω, A_1, ϕ, and A_2 poles do not contribute. The η-pole contribution vanishes because of parity. Let us calculate the ρ-pole contribution. From covariance alone, we can write

$$\begin{aligned}\langle\pi_\alpha(k)|A_\mu^{(\beta)}(0)|\rho_\gamma(p)\rangle &= \epsilon_{\alpha\beta\gamma}(1/V)(4p_0k_0)^{-1/2} \\ &\times\{\epsilon_\mu(p)K_1(q^2) + [\epsilon(p)\cdot k](p_\mu-k_\mu)K_2(q^2) \\ &+[\epsilon(p)\cdot k](p_\mu+k_\mu)K_3(q^2)\}\,,\end{aligned} \tag{63}$$

where α, β, γ are isospin indices, ϵ_μ is the ρ-meson polarization vector, $q=p-k$, and K_1, K_2, K_3 are appropriate form factors.

After some calculation one finds

$$\begin{aligned}G_1^{(\rho)}(0,q^2) &= [2/(q^2+m_\rho^2)]K_1^2(q^2)\,, \\ G_2^{(\rho)}(0,q^2) &= (1/4m_\rho^2)[8K_1^2(q^2)/(q^2+m_\rho^2) \\ &-2K_1(q^2)K_2(q^2)+2K_2(q^2)K_3(q^2)(q^2+m_\rho^2) \\ &-2K_1(q^2)K_3(q^2)+K_3^2(q^2)(q^2+m_\rho^2)]\,.\end{aligned} \tag{64}$$

We shall now assume that the form factors $K_2(q^2)$ and $K_3(q^2)$ satisfy unsubtracted dispersion relations and $K_1(q^2)$ satisfies a once-subtracted dispersion relation.[2]

We dominate them by π and A_1 poles and obtain

$$K_1(q^2)=\sqrt{2}G_\rho/f_\pi+(q^2+m_\rho^2)/(q^2+m_{A_1}^2)\times G_{A_1}G_S/(m_{A_1}^2-m_\rho^2),$$
$$K_2(q^2)=\sqrt{2}f_\pi g_{\rho\pi\pi}/q^2+G_{A_1}(G_S-\tfrac{1}{2}m_\rho^2G_D)/m_{A_1}^2(q^2+m_{A_1}^2), \quad (65)$$
$$K_3(q^2)=G_{A_1}G_D/2(q^2+m_{A_1}^2),$$

where we have set $\mu^2=0$. G_S and G_D are defined by Eq. (39). Now, we substitute (65) into (64). The first equation in (61) leads to the following sum rules:

$$2G_{A_1}^2G_S^2/(m_{A_1}^2-m_\rho^2)=0, \quad (66)$$

$$2G_{A_1}^2G_S^2/(m_{A_1}^2-m_\rho^2)^2+4\sqrt{2}G_\rho G_{A_1}G_S/f_\pi(m_{A_1}^2-m_\rho^2)=-4G_{A_1}^2/f_\pi^2. \quad (67)$$

Equation (66) tells us that $G_S=0$ and (67) gives $G_{A_1}=0$.

To avoid this, we are again forced to take the higher state contributions into account. We consider the f meson (1250) as a resonance constructed out of a two-particle π-π intermediate state; it contributes only to the π^0 term. Calculations are very similar to the one sketched in the Appendix. The result is

$$G_1^{(f)}(0,q^2)=\frac{1}{4m_f}f_1^2(0)\frac{m_{A_1}^4}{m_f}\times\left[\frac{1}{q^2+m_{A_1}^2}+\frac{m_f^2-m_{A_1}^2}{(q^2+m_{A_1}^2)^2}\right]. \quad (68)$$

m_f is the f-meson mass and f_1 is one of the f-meson form factors occurring in

$$\langle\pi^+(k)|A_\nu^{(3)}(0)|f(p)\rangle=(1/V)(4p_0k_0)^{-1/2}\epsilon_{\lambda\rho}(p)\times\{f_1(q^2)\delta_{\lambda\rho}k_\rho+[f_2(q^2)k_\nu+f_3(q^2)p_\nu]k_\lambda k_\rho\}, \quad (69)$$

where $\epsilon_{\lambda\rho}(p)$ is the f polarization tensor.

Equations (66) and (67) are then modified and one obtains

$$2G_{A_1}^2G_S^2/(m_\rho^2-m_{A_1}^2)+\tfrac{1}{4}f_1^2(0)(m_{A_1}^4/m_f^2)\times(m_f^2-m_{A_1}^2)=0, \quad (70)$$

$$2G_{A_1}^2G_S^2/(m_{A_1}^2-m_\rho^2)^2+4\sqrt{2}G_\rho G_{A_1}G_S/f_\pi(m_{A_1}^2-m_\rho^2)+(m_{A_1}^4/4m_f^2)f_1^2(0)=-4G_{A_1}^2/f_\pi^2. \quad (71)$$

Substituting for the masses in Eq. (70), we obtain

$$|G_S|=782.1\times f_1(0).$$

The second equation of (61), upon using (64) and (65), yields the following sum rules:

$$\tfrac{1}{16}G_D^2(m_\rho^2-m_{A_1}^2)+\tfrac{1}{4}G_SG_D+(G_D^2/4m_{A_1}^2)[G_S-\tfrac{1}{2}m_\rho^2G_D](m_\rho^2-m_{A_1}^2)+G_S[G_S-\tfrac{1}{2}m_\rho^2G_D]/2m_{A_1}^2+2G_S^2/(m_{A_1}^2-m_\rho^2)=0, \quad (72)$$

$$\tfrac{1}{16}G_D^2-G_SG_D/4(m_{A_1}^2-m_\rho^2)-\sqrt{2}G_D/4f_\pi+(G_D/4m_{A_1}^2)\times[G_S-\tfrac{1}{2}m_\rho^2G_D]-\sqrt{2}f_\pi g_{\rho\pi\pi}G_D(m_\rho^2-m_{A_1}^2)-G_S[G_S-\tfrac{1}{2}m_\rho^2G_D]/2m_{A_1}^2(m_{A_1}^2-m_\rho^2)-f_\pi g_{\rho\pi\pi}G_S/\sqrt{2}m_{A_1}^2G_{A_1}-[G_S-\tfrac{1}{2}m_\rho^2G_D]/\sqrt{2}f_\pi m_{A_1}^2+4\sqrt{2}G_S/f_\pi(m_{A_1}^2-m_\rho^2)+2G_S^2/(m_{A_1}^2-m_\rho^2)^2=-4m_\rho^2/f_\pi m_{A_1}^2, \quad (73)$$

$$G_D/2\sqrt{2}-G_S/\sqrt{2}(m_{A_1}^2-m_\rho^2)=-2m_{A_1}^2/f_\pi g_{\rho\pi\pi}G_{A_1}. \quad (74)$$

Now, freely using the KS-RF[9] and Weinberg[1] formulas, we simplify Eqs. (72)–(74) to

$$-\frac{9}{4}\frac{G_s^2}{m_\rho^2}=0, \quad (72')$$

$$G_D(f_\pi g_{\rho\pi\pi}/4\sqrt{2}G_\rho-1/2\sqrt{2}f_\pi)+(G_s/m_\rho^2)(4\sqrt{2}/f_\pi-f_\pi g_{\rho\pi\pi}/2\sqrt{2}G_{A_1}-1/2\sqrt{2}f_\pi)+(7/4)G_s^2/m_\rho^4=-2/f_\pi^2, \quad (73')$$

$$G_D/2\sqrt{2}-G_s/\sqrt{2}m_\rho^2=-4m_\rho^2/f_\pi g_{\rho\pi\pi}G_\rho. \quad (74')$$

Equation (72) tells us that $G_S=0$, again suggesting that the $J\geqslant 2$ state must be included. We refer to the discussions following Eq. (54).

Setting $q^2=0$ in the left-hand side of Eq. (61), we obtain $G_1(0,0)=2$. A knowledge of the matrix elements $\langle\pi|(A_\mu^{(3)}(x)A_\nu^{(3)}(0))_+|\pi\rangle$ and $\langle\pi|(V_\mu^{(3)}(x)V_\nu^{(3)}(0))_+|\pi\rangle$ enables us to calculate the total ν-π scattering cross section at zero energy. The results will be published elsewhere.

Note added in proof. If we assume that the electromagnetic form factor of pion obeys a once-subtracted dispersion relation, then, using the parametrization of Schnitzer and Weinberg,[2]

$$F_\pi(q^2)=\tfrac{1}{4}(1+\delta)+\frac{3-\delta}{4}\frac{m_\rho^2}{q^2+m_\rho^2}.$$

Equations (36)–(38) would now contain the free parameter δ. θ_2 is seen to be quadratic in δ which if equated to the ρ-dominated expression [Eq. (44)] yields $\delta=9.3$ or -2.1. The second solution corresponds to the Gilman-Harari result that we have already discussed. Note that $\delta=-1$ leads to the unsubtracted form of $F_\pi(q^2)$ and gives $\theta_2=0$ or $f_\pi=0$ as shown earlier.

APPENDIX

Here we shall sketch the derivation of Eq. (49). The two-particle intermediate-state calculation proceeds as follows (see Fig. 1):

$$I_{\mu\nu}=\delta_{\mu\nu}\,\mathrm{Im}F_1(\nu,q^2)+q_\mu q_\nu\,\mathrm{Im}F_2(\nu,q^2)+\cdots$$
$$=+k_0V(2\pi)^4\frac{V^2}{(2\pi)^6}\sum_\lambda\left\{\int d^3q_1d^3q_2\times\langle\pi^+(k)|V_\nu^{(3)}(0)|\mathbf{q}_1,\mathbf{q}_2,\lambda\rangle\langle\mathbf{q}_1,\mathbf{q}_2,\lambda|V_\mu^{(3)}(0)|\pi^+(k)\rangle\times\delta^4(q+k-q_1-q_2)-\text{the cross term}\right\}, \quad (75)$$

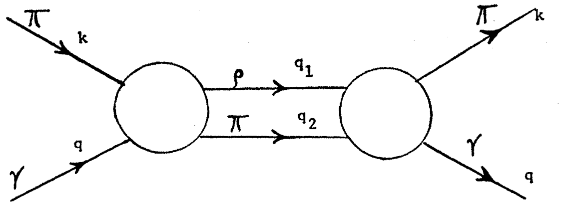

FIG. 1. Two-particle intermediate state in forward π-γ scattering.

where λ is the ρ helicity. The A^0 meson cannot contribute to the π^0 term, because of C invariance.

Let us now introduce the variables $\mathbf{Q}=\mathbf{q}_1+\mathbf{q}_2$ and $\mathbf{p}=\frac{1}{2}(\mathbf{q}_1-\mathbf{q}_2)$. The Jacobian of this transformation is unity, so $d^3q_1d^3q_2=d^3Qd^3p$. We carry out the d^3Q integration and work in the center-of-mass (c.m.) system

$$I_{\mu\nu}=\frac{k_0V^3}{(2\pi)^2}\sum_\lambda\int p^2dpd\Omega\ \delta^{(1)}(q_0+k_0-q_{10}-q_{20})$$
$$\times\langle\pi^+(k)|V_\nu^{(3)}(0)|W,\mathbf{p},\lambda\rangle\langle W,\mathbf{p},\lambda|V_\mu^{(3)}(0)|\pi^+(k)\rangle. \quad (76)$$

W is the c.m. energy of the system and $d\Omega$ is the element of solid angle. We now construct states of definite angular momentum out of states containing particles of definite helicity.[12]

$$|W,\mathbf{p},\lambda\rangle=\sum_{J,M}\left(\frac{2J+1}{4\pi}\right)^{1/2}D_{M,\lambda}{}^J(\phi,\theta,0)|W,J,M,\lambda\rangle. \quad (77)$$

ϕ and θ are Euler angles. The direction of motion of the center of mass is taken as the z direction. Substituting back in Eq. (76), the angular integration can be easily carried out using orthogonality properties of the D matrices.[12]

$$I_{\mu\nu}=\frac{k_0V^3}{(2\pi)^2}\sum_{J,\lambda}\int p^2dp$$
$$\times\delta^{(1)}(q_0+k_0-q_{10}-q_{20})\langle\pi^+(k)|V_\nu^{(3)}(0)|W,J,M,\lambda\rangle$$
$$\times[\langle\pi^+(k)|V_\mu^{(3)}(0)|W,J,M,\lambda\rangle]^*. \quad (78)$$

Symmetry properties and multipole decompositions of the electromagnetic vertex functions are discussed in detail by Durand *et al.*,[13] and for details we refer to that paper and the references quoted there.

Let us use the following notation (see Fig. 2):

$$p_a=k;\quad m_a=m_\pi,\quad p_c=q+k,$$

and $m_c=$ the invariant mass of the $\rho\pi$ system $=m_{A_2}$. We pick the A_2 state as a $\rho\pi$ resonance.

$$\Gamma_{s,\lambda;0,0}{}^{(0)}=\langle\pi^+(k)|V_0{}^{(3)}(0)|W,J,M,\lambda\rangle,$$
$$\Gamma_{s,\lambda;0,0}{}^{\pm1}=\langle\pi^+(k)|V_{\pm1}{}^{(3)}(0)|W,J,M,\lambda\rangle,$$

[12] See, for example, S. Gasiorowicz, *Elementary Particle Physics* (John Wiley & Sons, Inc., New York, 1966), p. 78.

[13] L. Durand, III, P. C. De Celles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).

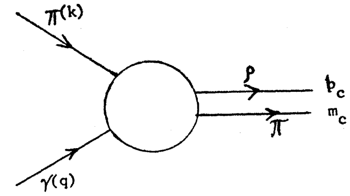

FIG. 2. p_c and m_c are the 4-momentum and invariant mass of the two-particle intermediate state.

where

$$V_{\pm1}=\mp\tfrac{1}{2}\sqrt{2}(V_1\pm iV_2),$$

the subscripts s,λ refer to the spin and helicity of the final state, and the subscripts 0,0 refer to those of the pion state.

Let us introduce two orthogonal 4-vectors:

$$P_\mu=(p_a+p_c)_\mu=2k_\mu+q_\mu,$$
$$K_\mu=(p_a-p_c)_\mu-P_\mu(m_c{}^2-m_a{}^2)/P^2, \quad (79)$$
$$K\cdot P=0,\quad P^2=q^2-4\nu.$$

In an arbitrary Lorentz frame, we have[13]

$$\sum_\lambda\langle\pi^+(k)|V_\nu^{(3)}(0)|W,J,M,\lambda\rangle\langle W,J,M,\lambda|V_\mu^{(3)}(0)|\pi^+(k)\rangle$$
$$=\sum_\lambda\{(\Gamma_{s\lambda,00}{}^{(0)})^2(P^2)^{-1}[-P_\mu P_\nu+(P_\mu K_\nu+K_\mu P_\nu)$$
$$\times(m_c{}^2-m_a{}^2)/K^2-K_\mu K_\nu(m_c{}^2-m_a{}^2)^2/K^4]$$
$$+(\Gamma_{s\lambda,00}{}^{(+)})^2(\delta_{\mu\nu}-K_\mu K_\nu/K^2-P_\mu P_\nu/P^2)\}. \quad (80)$$

To carry out the multipole decomposition, it is convenient to work in the brick-wall frame of the initial and final states. For any two states s,λ and s',λ' having direction of motion along the z direction

$$\Gamma_{s',\lambda';s,\lambda}{}^{(0)}=\langle p,0,s',\lambda'|e^{i\pi J_2}V_0{}^{(3)}(0)|p0s\lambda\rangle. \quad (81)$$

By applying suitable boosting, this can be expressed in terms of the rest states $|s,\lambda\rangle$:

$$\Gamma_{s'\lambda';s,\lambda}{}^{(0)}=\langle s'\lambda'|e^{i\xi'K_3}e^{i\pi J_2}V_0{}^{(3)}(0)e^{-i\xi K_3}|s,\lambda\rangle, \quad (82)$$

where K_3 is the boosting operator, $\xi=\sinh^{-1}(p_c/m_c)$, and $\xi'=\sinh^{-1}(p_a/m_a)$. Let

$$\alpha=\xi+\xi'=\sinh^{-1}\left[\left(\frac{p_a\cdot p_c}{m_am_c}\right)^2-1\right]^{1/2}. \quad (83)$$

A typical term in Eq. (82) will involve terms like $V_0(K_3)^n$, which can be decomposed into sets of spherical tensors. Using the Wigner-Eckart theorem, we finally get

$$\Gamma_{s',\lambda';s,\lambda}{}^{(0)}=(-)^{2s'}\sum_{J=0}^\infty\begin{pmatrix}s' & J & s\\ \lambda' & 0 & \lambda\end{pmatrix}Q_J(s',s), \quad (84)$$

where the charge-transition form factor $Q_J(s',s)$

$=\langle s'\|T_J^{(0)}\|s\rangle$ satisfies

$$Q_J \underset{\alpha\to 0}{\longrightarrow} \alpha^J. \tag{85}$$

Proceeding in a similar manner, the transverse components are given by

$$\Gamma_{s',\lambda';s,\lambda}{}^{\pm 1}=(-1)^{2s'}\sum_{J=1}^{\infty}\begin{pmatrix} s' & J & s\\ \lambda' & 1 & \lambda\end{pmatrix}\times\{\tfrac{1}{2}[1+(-1)^{J+\pi}]E_J(s',s)+\tfrac{1}{2}[1-(-1)^{J+\pi}]M_J(s',s)\}. \tag{86}$$

The form factors M_J and E_J correspond to the magnetic and electric transition multipole moments, and $\pi=0$ (1) for even (odd) relative parity.

It is possible to show that

$$E_J \underset{\alpha\to 0}{\longrightarrow} \alpha^{J-1},\quad M_J \underset{\alpha\to 0}{\longrightarrow} \alpha^J. \tag{87}$$

When we pick up the 2^+ (A_2) resonance, we find $s=2$ and $s'=0$, so that $J=2$ is the only possibility. This means that as $\alpha\to 0$, $\Gamma^{(\pm 1)}$ has to vanish at least as fast as α, and $\Gamma^{(0)}\to\alpha^2$ as $\alpha\to 0$. From now on, our analysis is valid for small α.

The relativistic generalization of the c.m. energy is

$$W=(-q^2+2\nu)^{1/2}=[-(q+k)^2]^{1/2}.$$

In the resonance approximation, we write

$$(\Gamma^{(+1)})^2=-\frac{C(q^2)\alpha^2}{2k_0V^3}\delta((-q^2+2\nu)^{1/2}-m_{A_2}), \tag{88}$$

where $C(q^2)$ is a real A_2 parameter.

For small α,

$$\alpha\approx\left[\left(\frac{p_a\cdot p_c}{m_am_c}\right)^2-1\right]^{1/2},\quad \alpha^2\approx(1/m_\pi^2m_{A_2}^2)[(-m_\pi^2-\nu)^2-m_\pi^2m_{A_2}^2]\approx\nu^2/m_\pi^2m_{A_2}^2. \tag{89}$$

The m_π^2 in the denominator is incorporated in the definition of physical coupling constants, as will be seen shortly.

Now, we pick the coefficient of $\delta_{\mu\nu}$ from Eq. (80) and

$$\mathrm{Im}F_1(0,q^2)=-\frac{1}{(2\pi)^2}\int_0^\infty p^2dp\,\delta^{(1)}(q_0+k_0-q_{10}-q_{20})\times\frac{C(q^2)}{m_\pi^2m_{A_2}^2}\alpha^2\delta((-q^2+2\nu)^{1/2}-m_{A_2}). \tag{90}$$

In the c.m. frame $\mathbf{p}^2=\mathbf{q}_1^2=q_{10}^2-m_\rho^2$.

The phase-space integration can be carried out in a covariant manner by using the following trick:

$$\int_0^\infty \mathbf{p}^2d\mathbf{p}\delta(q_0+k_0-q_{10}-q_{20})=\int(q_{10}^2-m_\rho^2)^{1/2}q_{10}dq_{10}\times\delta^{(1)}(q_0+k_0-q_{10}-q_{20})=\int\left[-\frac{(Q\cdot q_1)^2}{Q^2}-m_\rho^2\right]^{1/2}\times(Q\cdot q_1)\frac{d(Q\cdot q_1)}{(-Q^2)^{1/2}}\delta[-Q_0(q_0+k_0-q_{10}-q_{20})].$$

where

$$Q=(0,iQ_0),\quad (-Q^2)^{1/2}=(-q^2+2\nu)^{1/2}.$$

The phase-space factor turns out to be

$$A=\frac{m_\rho^2+m_{A_2}^2-m_\pi^2}{2m_{A_2}^3}\left[\frac{(m_\rho^2+m_{A_2}^2-m_\pi^2)^2}{4m_{A_2}^2}-m_\rho^2\right]^{1/2}\times\left(\frac{3m_{A_2}^2+m_\rho^2-m_\pi^2}{2}\right), \tag{91}$$

$$\mathrm{Im}F_1(\nu,q^2)=-\frac{1}{(2\pi)^2}\frac{AC(q^2)}{m_{A_2}^2m_\pi^2}\times\nu^2\delta((-q^2+2\nu)^{1/2}-m_{A_2}). \tag{92}$$

From (17), the contribution of A_2 as a $\rho\pi$ resonance is

$$F_1^{2+}(0,q^2)=-AC(q^2)(q^2+m_{A_2}^2)/8\pi^3m_{A_2}m_\pi^2. \tag{93}$$

ρ dominance gives

$$C(q^2)=C'm_\rho^4/(q^2+m_\rho^2)^2. \tag{94}$$

Substituting (94) into (93) and using partial fractions, we have

$$F_1^{2+}(0,q^2)=-\frac{AC'}{8\pi^3m_\pi^2}\frac{m_\rho^4}{m_{A_2}}\left[\frac{1}{q^2+m_\rho^2}+\frac{m_{A_2}^2-m_\rho^2}{(q^2+m_\rho^2)^2}\right]. \tag{95}$$

The parameter C' is related to the physical coupling constant $g_{A_2\pi\gamma}$ by equating the A_2-pole calculation (46) to the magnitude of (93) at the physical photon limit $q^2=0$. This yields

$$AC'/\pi^3m_\pi^2=\tfrac{1}{2}g_{A_2\pi\gamma}{}^2m_{A_2}{}^3. \tag{96}$$

Substituting (96) into (95), we obtain Eq. (49).

It may be mentioned that by considering the A_1 meson as a $J=1$ $\rho\pi$ resonance, we obtain exactly the same q dependence as in the A_1-pole calculation.