Existence Proof by a Fixed-Point Theorem for Solutions of the Low Equation*

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Schauder's fixed-point theorem may be used to show that certain crossing-symmetric S-matrix equations have solutions. The method is illustrated in the case of the one-meson Low equation. It is proved that a sufficient condition for the existence of a solution is that the coupling constant be less than a certain bound which depends on the cutoff and the crossing matrix. The proof works for an arbitrary $n\times n$ crossing matrix with weak conditions on the cutoff function [for instance, $v(k)=O(k^{-2-\epsilon})$, $k\to\infty$]. The allowed range of coupling constants is such as to rule out resonant scattering. A related circumstance is that for the solution in question the baryon is elementary in the sense that it corresponds to a Castillejo-Dalitz-Dyson pole of an appropriate D function. The technique of applying Schauder's theorem differs from that of Atkinson's similar work in that the dispersion relations are approached directly without the aid of the N/D method. Hence the problem of D-function ghosts is avoided, and complete crossing symmetry is ensured.

I. INTRODUCTION

PRECISE analysis of crossing-symmetric S-matrix equations proves to be difficult in general because the equations are essentially nonlinear. Such equations may be given the form

$$A\phi=\phi, \tag{1.1}$$

where A is a nonlinear operator in a linear vector space to which ϕ belongs. Finding a solution of the S-matrix equations is equivalent to finding a "fixed point" of A; that is, a vector ϕ invariant under the operation A. In the mathematical literature[1–3] one finds two standard approaches to the fixed-point problem. The first is analytical in character and is very elementary. It is merely the iterative method:

$$A\phi_0=\phi_1,\ A\phi_1=\phi_2,\ \cdots\ A\phi_n=\phi_{n+1},\ \cdots. \tag{1.2}$$

The Banach fixed-point theorem[1] states that under a simple condition on A, iteration leads to a unique solution of (1.1) in a subset of a Banach space. The second approach is by means of topological arguments which are more powerful in the sense that restrictions on the operator A are often effectively weaker. On the other hand, topological fixed-point theorems usually do not offer a method of computing the fixed point. They assert the existence of one fixed point and at best only estimates on the number of additional fixed points.

The purpose of this article is to test the effectiveness of the topological method in a nontrivial physical example. The well-known fixed-point theorem of Schauder is applied to the one-meson Low equation.[4] The Low equation is not a bad example since it has some of the essential difficulties common to all nontrivial S-matrix equations: It is strongly nonlinear (when the coupling constant has its physical value), it entails crossing symmetry, and it involves singular integrals of Cauchy type.

Our result is that the conditions for Schauder's theorem can be verified when the sum of the crossed and direct baryon-pole residues is sufficiently small. It follows that Low's equation has at least one solution when the coupling constant is small enough. The upper limit on the coupling constant is disappointingly small if the cutoff is like those usually considered.[4] In the symmetric pseudoscalar theory (Chew-Low theory[4]), the usual value $f^2/4\pi=0.08$ of the coupling constant far exceeds our upper limit. Whether the Chew-Low equation has an exact solution with $f^2/4\pi=0.08$ and a reasonable cutoff appears to be a completely open question despite proposals of approximate solutions.[5] In particular, there appears to be no evidence that the

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[1] J. Cronin, *Fixed Points and Topological Degree in Nonlinear Analysis* (American Mathematical Society, Providence, R. I., 1964), Chaps. 3, 4.

[2] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations* (Pergamon Press, Inc., London, 1964), Chap. 3.

[3] W. Pogorzelski, *Integral Equations and Their Applications* (Pergamon Press, Inc., London, 1966), Vol. I, Chap. 19.

[4] E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill Book Company, New York, 1962), Chap. 18.

[5] A reasonable cutoff is defined as one which puts the (33) resonance at about the correct energy when the Chew-Low equation is treated in one of the usual approximations. For the latter see, for instance, Sec. 18.7 of Ref. 4; G. Salzman and F. Salzman, Phys. Rev. **108**, 1619 (1957); M. Baker, Ann. Phys. (N. Y.) 4, 271 (1958); K. G. Wilson, thesis, California Institute of Technology, 1961 (unpublished); and Ref. 6. The exact sense in which these proposed approximations are supposed to be approximate is never discussed in the literature, as far as I know. Sometimes a comparison is made with the numerical work of Salzman and Salzman, but the amplitudes found by the latter authors through iteration of the inverse amplitude equations do not actually solve all three of the coupled Low equations. A difficulty of spurious zeros in the inverse amplitudes is encountered.

celebrated bootstrap solution[6] exists, even though the simple N/D approximations (for example) make plausible its existence. There is contrary evidence: Huang and Mueller[7] prove that the Chew-Low equation has no bootstrap solution if the cutoff function belongs to a certain class.

Despite the coupling-constant restriction we do gain an interesting piece of information that is not obvious from physical intuition. Namely, there is a solution of the Low equation for any crossing matrix whatever with any number of channels, provided only that the coupling constant is sufficiently small and the cutoff satisfies weak conditions of continuity and asymptotic behavior. It is seen that the requirements of analyticity, unitarity, and crossing are not sufficient to induce an internal symmetry (i.e., not sufficient to restrict the crossing matrix to one or a small number of values), at least within the context of Low's equation. Hopes for such an induction of symmetries have been expressed.[8] Analysis of the soluble two-channel model has already yielded a negative result in agreement with ours.[9] The imposition of the bootstrap requirement might very well narrow the allowed class of crossing matrices (hopefully not to the null set). For thinking along this line see Cunningham.[10]

The solution which is proved to exist by Schauder's theorem obeys the generalized Levinson relation

$$\delta_\alpha(\infty)-\delta_\alpha(1)=-\pi(n_b-n_c)_\alpha, \qquad (1.3)$$

where n_b is the number of stable particles in channel α, and n_c is the number of Castillejo-Dalitz-Dyson (CDD) poles of an appropriate D function for channel α. Equation (1.3) may be established under the same conditions assumed in our existence theorem provided the coupling constant is sufficiently small. In earlier proofs[11] of Eq. (1.3) a condition of no unit eigenvalue of the N/D kernel is assumed; that is avoided here. By analogy with soluble models n_{c_α} is *defined* as the number of elementary particles in channel α. In the Chew-Low model our solution has $n_b=n_c=1$ in the (11) state and $n_b=n_c=0$ in the other states. Hence the nucleon is elementary. Equation (1.3) makes sense as a definition of elementarity only in a purely elastic theory.[12]

The usefulness of fixed-point theorems in physical problems might be very much greater than is indicated by the present modest study. The topological theorems, especially, are extremely general. They may be applied to the most diverse kinds of equations: differential, integral, integro-differential, algebraic, etc. There is no unique method of applying a given theorem, and the results obtained depend partly on how it is applied. The specific upper limit on the coupling constant in our proof, for instance, is related to a particular way of estimating integrals. A better way of doing the estimates might give a larger limit on the coupling strength. Also there is some freedom in choosing to analyze one of several equations describing the same physical situation. Instead of the Low equation one could examine the equation for the inverse Low amplitude, or the N/D equations. A fixed-point theorem might be made effectively more powerful by choosing the most advantageous equation. Atkinson[13] has already applied Schauder's theorem to the Chew-Mandelstam equations in the N/D formulation. His approach has an advantage over ours in that the question he asks is whether there is a solution with prescribed CDD poles. Our question is merely "Is there a solution?." The N/D method has the recognized disadvantage that the D function may develop spurious zeros (ghosts). If such zeros cannot be definitely ruled out, then the N/D method is not suitable for an existence proof.[14] Our method of working directly with the dispersion relations avoids the ghost problem, and it ensures full crossing symmetry. Atkinson's goal is to prove crossing symmetry only for the absorptive parts up to a finite energy. We expect to return to the Chew-Mandelstam equations and other applications of fixed-point theorems in later publications.

[*Note added in proof.* The difficulties of the N/D approach noted above can be overcome. In fact, existence theorems for the Low equation in the N/D and inverse amplitude formulations have been proved recently by McDaniel and the author.[14a] The theorems are stronger and more informative than that of the present paper. It should be mentioned that fixed-point theorems have been applied to equations of field theory by Taylor.[14b] Lovelace[14c] has employed other methods of nonlinear analysis in a study of S-matrix equations.]

[6] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962); F. E. Low, *ibid.* **9**, 277 (1962); K. Huang and F. E. Low, *ibid.* **13**, 596 (1964); J. Math. Phys. **6**, 795 (1965); see also Ref. 7.

[7] K. Huang and A. H. Mueller, Phys. Rev. Letters **14**, 396 (1965); Phys. Rev. **140**, B365 (1965). I follow Huang and Mueller in defining a bootstrap amplitude as one which satisfies the unsubtracted Low equation and obeys the Levinson relation without a CDD term.

[8] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963); R. H. Capps, Phys. Rev. Letters **10**, 312 (1963); Nuovo Cimento **30**, 340 (1963); Phys. Rev. **134**, B460 (1964); E. P. Wigner, Phys. Today **17**, 34 (1964).

[9] A. W. Martin and W. D. McGlinn, Phys. Rev. **136**, B1515 (1964).

[10] A. A. Cunningham, J. Math. Phys. **8**, 716 (1967).

[11] R. L. Warnock, Phys. Rev. **131**, 1320 (1963); Lectures at the American University of Beirut, 1967 (to be published by Academic Press Inc., New York).

[12] M. Bander, P. Coulter, and G. Shaw, Phys. Rev. Letters **14**, 207 (1965); D. Atkinson, K. Dietz, and D. Morgan, Ann. Phys. (N. Y.) **37**, 77 (1966).

[13] D. Atkinson, J. Math. Phys. **8**, 2281 (1967).

[14] In Sec. 5 of his paper, Atkinson states "there are several different ways in which one could define an N/D system which is free from ghosts." He then gives an admittedly incomplete sketch of how to rule out ghosts in the particular N/D system that he uses. I cannot see how to make these arguments into a proof.

[14a] H. McDaniel and R. L. Warnock, Bull. Am. Phys. Soc. **13**, 680 (1968).

[14b] J. G. Taylor, J. Math. Phys. **7**, 1720 (1966); **6**, 1148 (1965); *Lectures in Theoretical Physics*, edited by W. E. Brittin and A. O. Barut (Gordon and Breach, Science Publishers, Inc., New York, 1967), Vol. 9a, p. 353.

[14c] C. Lovelace, Commun. Math. Phys. **4**, 261 (1967).

In Sec. 2 the necessary theorems and definitions are stated, and the setup for applying Schauder's principle to the Low equation is described. Section 3 contains the necessary estimates of integrals. Section 4 includes a statement of our existence theorem, and some discussion of Levinson's relation and the physical meaning of the results. The work of Secs. 2 and 3 amounts to a straightforward adaptation of methods described by Pogorselski.[3]

II. SCHAUDER'S THEOREM APPLIED TO THE LOW EQUATION

With a complex variable ω identified as the meson energy, the Low equation reads[4]

$$f_\alpha(\omega)=\frac{\lambda_\alpha}{\omega}+\frac{1}{\pi}\int_1^\infty\frac{d\omega'\rho(\omega')f_{\alpha+}(\omega')f_{\alpha-}(\omega')}{\omega'-\omega}+\sum_{\beta=1}^{n}c_{\alpha\beta}\frac{1}{\pi}\int_1^\infty\frac{d\omega'\rho(\omega')f_{\beta+}(\omega')f_{\beta-}(\omega')}{\omega'+\omega},$$
$$\alpha=1,2,\cdots,n. \quad (2.1)$$

We allow any finite number n of channels, thereby generalizing slightly the original Chew-Low theory. The meson mass has the value 1, and all quantities appearing in (2.1) are dimensionless. The p-wave amplitude f_α is related to the phase shift for real ω above the threshold $\omega=1$ as follows:

$$f_{\alpha+}(\omega)=\sin\delta_\alpha(\omega)e^{i\delta_\alpha(\omega)}/\rho(\omega),\quad \omega\geqslant1. \quad (2.2)$$

Here $f_{\alpha\pm}(\omega)=f_\alpha(\omega\pm i0)$. In terms of the meson momentum k and the cutoff function $v(k)$, the ρ function is

$$\rho(\omega)=k^3v^2(k)/12\pi,\quad k=(\omega^2-1)^{1/2}. \quad (2.3)$$

The crossing matrix $c=[c_{\alpha\beta}]=c^*$ has the property $c^2=1$, but otherwise it is entirely arbitrary. The real constant λ_α is the sum of the direct and crossed-channel baryon pole residues

$$\lambda_\alpha=-g_\alpha^2+\sum_\beta c_{\alpha\beta}g_\beta^2. \quad (2.4)$$

Hence the vector $\lambda=[\lambda_\alpha]$ obeys $c\lambda=-\lambda$. Any solution of (2.1) obeys the crossing-symmetry equation

$$f_\alpha(\omega)=\sum_\beta c_{\alpha\beta}f_\beta(-\omega). \quad (2.5)$$

Since one is interested only in solutions of (2.1) which obey the reality requirements $f_\alpha(\omega)=f_\alpha^*(\omega^*)$, one may write $f_{\alpha+}f_{\alpha-}=|f_{\alpha+}|^2$.

The mathematical problem is to find an analytic function having the representation (2.1). That problem is easily reduced to solving integral equations. Suppose that a function $f_\alpha(\omega)$ has the representation (2.1), and is such that boundary values of the integrals can be computed by the Plemelj rule $(x\pm i0)^{-1}=P(x^{-1})\mp i\pi\delta(x)$. Then $f_{\alpha+}$ obeys the integral equations

$$f_{\alpha+}(\omega)=\lambda_\alpha/\omega+i\rho f_{\alpha+}f_{\alpha+}^*(\omega)+\frac{P}{\pi}\int_1^\infty\frac{d\omega'\rho f_{\alpha+}f_{\alpha+}^*(\omega')}{\omega'-\omega}+\sum c_{\alpha\beta}\frac{1}{\pi}\int_1^\infty\frac{d\omega'\rho f_{\beta+}f_{\beta+}^*(\omega')}{\omega'+\omega},$$
$$\alpha=1,2,\cdots,n,\quad \omega\geqslant1. \quad (2.6)$$

Now suppose on the other hand that $f_{\alpha+}(\omega)$ is a solution of (2.6) such that $\rho f_{\alpha+}f_{\alpha+}^*$ is Hölder-continuous on any finite interval. [A function $y(\omega)$ is said to be Hölder-continuous or H-continuous in an interval if for all ω and ω' in that interval there exist positive numbers A and μ such that $|y(\omega)-y(\omega')|\leqslant A|\omega-\omega'|^\mu$.] From this solution of (2.6) construct the analytic function

$$F_\alpha(\omega)=\lambda_\alpha/\omega+\frac{1}{\pi}\int_1^\infty\frac{d\omega'\rho f_{\alpha+}f_{\alpha+}^*(\omega')}{\omega'-\omega}+\sum c_{\alpha\beta}\frac{1}{\pi}\int_1^\infty\frac{d\omega'\rho f_{\beta+}f_{\beta+}^*(\omega')}{\omega'+\omega}. \quad (2.7)$$

Because of the H continuity of $\rho f_{\alpha+}f_{\alpha+}^*$, the boundary value of (2.7) may be obtained from the Plemelj rule.[15] Hence by (2.6),

$$F_{\alpha+}(\omega)=f_{\alpha+}(\omega). \quad (2.8)$$

Substituting (2.8) in the right side of (2.7), and using the fact that $F_\alpha(\omega)=F_\alpha^*(\omega^*)$, we see that $F_\alpha(\omega)$ has the representation (2.1); i.e., it is a solution of Low's equation.

Our problem reduces to investigating Eq. (2.6), which we now transform to the interval [0,1] by the substitution $t=1/\omega$. We define $\phi_\alpha(t)=f_{\alpha+}(\omega)$ and write $\rho(t)$ for $\rho(\omega)$. The equations become

$$\phi_\alpha(t)=\lambda_\alpha t+i\rho\phi_\alpha\phi_\alpha^*(t)-\frac{t}{\pi}P\int_0^1\frac{d\tau}{\tau}\frac{\rho\phi_\alpha\phi_\alpha^*(\tau)}{\tau-t}+\sum c_{\alpha\beta}\frac{t}{\pi}\int_0^1\frac{d\tau}{\tau}\frac{\rho\phi_\beta\phi_\beta^*(\tau)}{\tau+t},$$
$$\alpha=1,\cdots,n,\quad 0\leqslant t\leqslant1. \quad (2.9)$$

Now we regard $\phi(t)=[\phi_1(t),\phi_2(t),\cdots,\phi_n(t)]$ as a member of an infinite-dimensional vector space, and write (2.9) as

$$\phi=A\phi, \quad (2.10)$$

where the operator A is defined by the right-hand side of (2.9). A will be called the Low operator.

Equation (2.10) is studied with the aid of Schauder's Fixed-Point Theorem[1]: *In a normed linear space let K be a convex closed set and A a compact operator such that $A(K)\subset K$. Then A has a fixed point $\phi\in K$*; i.e.,

[15] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

$A\phi=\phi$. The definitions involved in the theorem are as follows:

(i) A normed linear space N is a linear vector space in which every element ϕ is assigned a real number $\|\phi\|\geqslant 0$ called its *norm* in such a way that (a) if $\phi\neq 0$, then $\|\phi\|>0$; (b) $\|\lambda\phi\|=|\lambda|\,\|\phi\|$; and (c) $\|\phi+\psi\|\leqslant\|\phi\|+\|\psi\|$.

Limits in N will be defined with respect to the norm; i.e., a sequence $\{\phi_n\}$ tends to a limit ϕ if $\|\phi_n-\phi\|$ tends to zero. We write $\phi_n\rightarrow\phi$.

(ii) A set K in N is *bounded* if $\|\phi\|<c$ for all ϕ in K, c being fixed.

(iii) A set K in N is *convex* if $\lambda\phi+(1-\lambda)\psi\in K$ for all ϕ, $\psi\in K$ and all λ, $0\leqslant\lambda\leqslant 1$.

(iv) A set K in N is *closed* if it contains all of its limit points.

(v) A set K in N is *compact* if every infinite subset of K has a limit point belonging to N.

(vi) An operator A on N is *continuous* if $A\phi_n\rightarrow A\phi$ when $\phi_n\rightarrow\phi$.

(vii) Let K be a subset of N. An operator A from K into N is *compact* if (a) A is continuous; (b) if M is a bounded subset of K, then $A(M)$ is compact.

In our work the space N will consist of all n vectors, the components of which are complex, continuous functions on the closed interval $[0,1]$. The norm is

$$\|\phi\|=\sup_{\alpha,t}|\phi_\alpha(t)|\,, \tag{2.11}$$

the supremum being taken with respect to the index α as well as with respect to the variable t in the interval $[0,1]$. For the convex closed set K in the hypothesis of Schauder's theorem we take the subset of N satisfying the following conditions:

$$K:\quad |\phi_\alpha(t)|\leqslant a\,, \tag{2.12a}$$

$$|\phi_\alpha(t)-\phi_\alpha(t')|\leqslant b|t-t'|^\mu,\quad 0<\mu<1,\quad \alpha=1,\cdots,n. \tag{2.12b}$$

The constants a and b will eventually be determined so that the conditions of Schauder's theorem hold for the Low operator applied to K. The particular value of μ in the interval $[0,1]$ is not important. Convexity of the set K follows immediately from the definitions. To show that K is closed, let ϕ be a limit point in N of a sequence $\{\phi_n\}$ of points in K. We have

$$\|\phi\|=\|\phi-\phi_n+\phi_n\|\leqslant\|\phi-\phi_n\|+a. \tag{2.13}$$

Since $\|\phi-\phi_n\|$ tends to zero we have $\|\phi\|\leqslant a$, and ϕ meets condition (2.12a). Similarly,

$$|\phi_\alpha(t)-\phi_\alpha(t')|\leqslant|[\phi_\alpha(t)-\phi_{n\alpha}(t)]-[\phi_\alpha(t')-\phi_{n\alpha}(t')]+[\phi_{n\alpha}(t)-\phi_{n\alpha}(t')]|\leqslant 2\|\phi-\phi_n\|+b|t-t'|^\mu. \tag{2.14}$$

Thus (2.12b) also holds. ϕ belongs to K, which is to say that K is closed.

In order to test the operator A for compactness, a criterion for compactness of $A(M)$ is needed, where M is any subset of K. A criterion is provided by Ascoli's theorem[1]: *In an infinite set of uniformly bounded and equicontinuous real functions defined on a compact (i.e., closed and bounded) subset R of the real line there exists a sequence $\{f_n\}$ of functions which converges uniformly on R to a continuous function.* This clearly means that the set of functions described is a compact subset of the linear normed space consisting of all continuous functions f on R with norm $\|f\|=\sup|f|$. A set S of functions is uniformly bounded if $|f(x)|<c$ for all $f\in S$, c being a fixed constant. A set S of functions is equicontinuous if for all $f\in S$ we have $|f(x)-f(y)|<\epsilon$ for $|x-y|<\delta(\epsilon)$, where δ is independent of f.

Ascoli's theorem is easily extended to our case of complex vector-valued functions $\phi(t)=[\phi_\alpha(t)]$ if by "uniformly bounded" we mean $|\phi_\alpha(t)|<c(\alpha)$, and by "equicontinuous" we mean $|\phi_\alpha(t)-\phi_\alpha(t')|<\epsilon$ for $|t-t'|<\delta(\epsilon,\alpha)$, where $c(\alpha)$ and $\delta(\epsilon,\alpha)$ are independent of ϕ. We merely apply Ascoli's theorem separately to the real and imaginary parts of each component of ϕ. Since

$$\begin{gathered}|\mathrm{Re}\phi_\alpha|\leqslant|\phi_\alpha|,\quad |\mathrm{Im}\phi_\alpha|\leqslant|\phi_\alpha|\,,\\ |\mathrm{Re}\phi_\alpha(t)-\mathrm{Re}\phi_\alpha(t')|\leqslant|\phi_\alpha(t)-\phi_\alpha(t')|\,,\\ |\mathrm{Im}\phi_\alpha(t)-\mathrm{Im}\phi_\alpha(t')|\leqslant|\phi_\alpha(t)-\phi_\alpha(t')|\,,\end{gathered} \tag{2.15}$$

we see that if the functions ϕ are uniformly bounded and equicontinuous, then so are the functions $\mathrm{Re}\phi_\alpha$ and $\mathrm{Im}\phi_\alpha$. In that case there are sequences $\{\mathrm{Re}\phi_{n\alpha}\}$, $\{\mathrm{Im}\phi_{n\alpha}\}$ which converge uniformly to continuous functions $\mathrm{Re}\psi_\alpha$, $\mathrm{Im}\psi_\alpha$. That is to say,

$$|\psi_\alpha(t)-\phi_{n\alpha}(t)|<\epsilon,\quad n>N_\epsilon(\alpha)\,, \tag{2.16}$$

with $N_\epsilon(\alpha)$ independent of t. If $N_\epsilon=\sup_\alpha N_\epsilon(\alpha)$, then

$$\sup_{\alpha,t}|\psi_\alpha(t)-\phi_{n\alpha}(t)|<\epsilon,\quad n>N_\epsilon. \tag{2.17}$$

Thus an infinite set of uniformly bounded equicontinuous functions in our space N contains a sequence $\{\phi_n\}$ such that $\|\phi_n-\psi\|\rightarrow 0$, $\psi\in N$; i.e., such a set is compact.

In particular any infinite subset of the set K defined in (2.12) is compact, since conditions (2.12a) and (2.12b) express uniform boundedness and equicontinuity, respectively. If $A(K)\subset K$, then $A(M)$ is compact for any subset M of K. According to Schauder's theorem and the definition (vii) of a compact operator, there is a fixed point in K [i.e., a solution of Eq. (2.9) satisfying (2.12)] if the following conditions are met:

(a) $A(K)\subset K$,
(b) A is continuous in its action on K. (2.18)

The ρ function will be so restricted that conditions (2.12) ensure the Hölder continuity of $\rho f_{\alpha+}f_{\alpha+}^*$. Hence the solution of (2.9) that is proved to exist by Schauder's theorem will yield a solution of the Low equation via Eq. (2.7).

III. COMPACTNESS OF THE LOW OPERATOR

The first task is to meet the condition (2.18a): $A(K)\subset K$. Suppose $\phi\in K$, where K is defined in (2.12). We examine the function $\psi=A\phi$; i.e.,

$$\psi_\alpha(t)=\lambda_\alpha t+i\rho\phi_\alpha\phi_\alpha^*(t)-\frac{t}{\pi}P\int_0^1\frac{d\tau}{\tau}\frac{\rho\phi_\alpha\phi_\alpha^*(\tau)}{\tau-t}+\sum_\beta c_{\alpha\beta}\frac{t}{\pi}\int_0^1\frac{d\tau}{\tau}\frac{\rho\phi_\beta\phi_\beta^*(\tau)}{\tau+t},$$
$$\alpha=1,\cdots,n,\quad 0\leqslant t\leqslant 1. \tag{3.1}$$

We wish to determine the constants a and b in the definition of K in such a way that ψ belongs to K. The cutoff function is required to meet the following conditions:

$$|\rho(t)/t-\rho(t')/t'|\leqslant h|t-t'|^\nu,\quad 0\leqslant t,t'\leqslant 1 \tag{3.2}$$
$$\rho(t)/t|_{t=0}=\rho(t)/t|_{t=1}=0. \tag{3.3}$$

The exponent ν is to be greater than or equal to the exponent μ in the definition of K, but the latter may be taken arbitrarily small. The Hölder coefficient h is given the smallest possible value

$$h=\sup_{t,t'\in[0,1]}|\rho(t)/t-\rho(t')/t'|/|t-t'|^\nu. \tag{3.4}$$

Combining (3.3) and (3.2), we see that

$$\rho(t)=O(t^{1+\nu}),\quad t\to 0, \tag{3.5}$$
$$\rho(t)=O((1-t)^\nu),\quad t\to 1. \tag{3.6}$$

Equation (3.6) is consistent with the p-wave threshold behavior $\rho(t)=O((1-t)^{3/2})$. By the definition (2.3) of the cutoff function $v(k)$, (3.5) implies

$$v(k)=O(k^{-2-\nu/2}),\quad k\to\infty. \tag{3.7}$$

By (3.2) and the fact that a product of H-continuous functions is itself H-continuous, the integrals in (3.1) are seen to exist. Equation (3.2) implies that $\rho(t)$ is H-continuous with exponent ν;

$$|\rho(t)-\rho(t')|=|(t-t')(\rho(t)/t)+(\rho(t)/t-\rho(t')/t')t'|$$
$$\leqslant\sup|\rho(t)/t|\;|t-t'|+h|t-t'|^\nu\leqslant k|t-t'|^\nu. \tag{3.8}$$

We write $\|\rho/t^\alpha\|=\sup_t|\rho(t)/t^\alpha|$, for $\alpha\leqslant 1+\nu$, $0\leqslant t\leqslant 1$.

From (3.1) and (2.12) we have

$$|\psi_\alpha(t)|\leqslant|\lambda_\alpha|+a^2\|\rho\|+\left|\frac{P}{\pi}\int_0^1\frac{d\tau}{\tau}\frac{\rho\phi_\alpha\phi_\alpha^*(\tau)}{\tau-t}\right|+\sum|c_{\alpha\beta}|\left|\frac{1}{\pi}\int_0^1\frac{d\tau}{\tau}\frac{\rho\phi_\beta\phi_\beta^*(\tau)}{\tau+t}\right|$$
$$=|\lambda_\alpha|+a^2\|\rho\|+|I_1|+\sum|c_{\alpha\beta}|\;|I_{2\beta}|. \tag{3.9}$$

To investigate the integral I_1 it is convenient to extend the integration formally to a larger interval $[-\epsilon, 1+\epsilon]$, $\epsilon>0$. In $[-\epsilon, 0]$ we put $\rho(\tau)/\tau=0$ and $\phi_\alpha(\tau)=\phi_\alpha(0)$, while in $[1, 1+\epsilon]$ we have $\rho(\tau)/\tau=0$ and $\phi_\alpha(\tau)=\phi_\alpha(1)$. After subtraction and addition of the pole contribution to the integral, I_1 becomes

$$I_1=\frac{P}{\pi}\int_{-\epsilon}^{1+\epsilon}\frac{d\tau}{\tau}\frac{\rho\phi\phi^*(\tau)}{\tau-t}=\frac{P}{\pi}\int_{-\epsilon}^{1+\epsilon}d\tau\left[\frac{\rho\phi\phi^*(\tau)/\tau-\rho\phi\phi^*(t)/t}{\tau-t}\right]+\frac{\rho\phi\phi^*(t)}{t}\frac{P}{\pi}\int_{-\epsilon}^{1+\epsilon}\frac{d\tau}{\tau-t}. \tag{3.10}$$

The second integral of (3.10) is bounded by a constant A_1:

$$\left|\frac{P}{\pi}\int_{-\epsilon}^{1+\epsilon}\frac{d\tau}{\tau-t}\right|=\frac{1}{\pi}\left|\ln\left(\frac{1+\epsilon-t}{\epsilon+t}\right)\right|\leqslant A_1,\quad 0\leqslant t\leqslant 1. \tag{3.11}$$

(Here and in the following, various constant bounds are denoted by A_i.) To handle the first integral of (3.10) note the identity

$$(\rho/\tau)\phi\phi^*(\tau)-(\rho/t)\phi\phi^*(t)=\phi\phi^*(\tau)[\rho(\tau)/\tau-\rho(t)/t]+[\rho(t)/t][\phi(\tau)(\phi^*(\tau)-\phi^*(t))+\phi^*(t)(\phi(\tau)-\phi(t))]. \tag{3.12}$$

By (3.2) and (3.3) it is seen that $\rho(t)/t$ satisfies the Hölder condition (3.2) on the entire interval $[-\epsilon, 1+\epsilon]$. Suppose for example that $-\epsilon\leqslant t'\leqslant 0$, and $0\leqslant t\leqslant 1$. Then

$$|\rho(t)/t-\rho(t')/t'|=|\rho(t)/t|\leqslant ht^\nu\leqslant h|t-t'|^\nu. \tag{3.13}$$

Hence by (3.12) and (2.12) we have

$$|(\rho/\tau)\phi\phi^*(\tau)-(\rho/t)\phi\phi^*(t)|\leqslant a^2h|\tau-t|^\nu+2ab\|\rho/t\|\;|\tau-t|^\mu, \tag{3.14}$$
$$-\epsilon\leqslant\tau\leqslant 1+\epsilon,\quad 0\leqslant t\leqslant 1.$$

The first integral in (3.10) is bounded in magnitude by

$$\frac{1}{\pi}\int_{-\epsilon}^{1+\epsilon}d\tau[a^2h|\tau-t|^{\nu-1}+2ab\|\rho/t\|\;|\tau-t|^{\mu-1}]\leqslant a^2hA_2+2ab\|\rho/t\|A_3,\quad 0\leqslant t\leqslant 1. \tag{3.15}$$

The integral I_2 of (3.9) is easily bounded;

$$|I_2|\leqslant\frac{1}{\pi}\int_0^1\frac{d\tau\,\tau^\nu}{\tau+t}a^2\|\rho/t^{1+\nu}\|\leqslant a^2\|\rho/t^{1+\nu}\|A_4. \tag{3.16}$$

Now $A(K)\subset K$ requires $\|\psi_\alpha\|\leqslant a$. Collecting (3.9)–(3.11), (3.15), and (3.16), we see that this inequality holds if

$$\sup_\alpha|\lambda_\alpha|+a^2[\|\rho\|+\|\rho/t\|A_1+hA_2+\sum_\beta|c_{\alpha\beta}|\;\|\rho/t^{1+\nu}\|A_4]+ab[2\|\rho/t\|A_3]\leqslant a. \tag{3.17}$$

Next we must estimate $|\psi_\alpha(t)-\psi_\alpha(t')|$. Define

$$\chi(t)=\rho\phi\phi^*(t)/t. \qquad (3.18)$$

According to (3.14) the variation of χ has the bound

$$|\chi(t)-\chi(t')|\leqslant(a^2h+2ab\|\rho/t\|)|t-t'|^\mu =\alpha|t-t'|^\mu,\quad 0\leqslant t,t'\leqslant 1, \qquad (3.19)$$

since $|t-t'|^\mu\geqslant|t-t'|^\nu$ when t and t' are both confined to the interval $[0,1]$. In a similar way

$$|\rho\phi\phi^*(t)-\rho\phi\phi^*(t')|\leqslant(a^2k+2ab\|\rho\|)|t-t'|^\mu =\beta|t-t'|^\mu. \qquad (3.20)$$

If we treat I_1 as in (3.10) we have the following (subscripts are dropped when no confusion arises):

$$|\psi(t)-\psi(t')|\leqslant\sup_\alpha|\lambda_\alpha|\ |t-t'|+\beta|t-t'|^\mu +\left|\frac{\rho\phi\phi^*(t)}{\pi}\ln\frac{1+\epsilon-t}{\epsilon+t}-\frac{\rho\phi\phi^*(t')}{\pi}\ln\frac{1+\epsilon-t'}{\epsilon+t'}\right| +\left|\frac{t}{\pi}\int_{-\epsilon}^{1+\epsilon}d\tau\left[\frac{\chi(\tau)-\chi(t)}{\tau-t}\right]-\frac{t'}{\pi}\int_{-\epsilon}^{1+\epsilon}d\tau\left[\frac{\chi(\tau)-\chi(t')}{\tau-t'}\right]\right| +\sum_\beta|c_{\alpha\beta}|\left|\frac{1}{\pi}\int_0^1 d\tau\,\chi(\tau)\left[\frac{t}{\tau+t}-\frac{t'}{\tau+t'}\right]\right|. \qquad (3.21)$$

With

$$l(t)=\pi^{-1}\ln[(1+\epsilon-t)/(\epsilon+t)],\ \|l\|=A_1,\ \|dl/dt\|=A_5,$$

the third term of (3.21) is

$$|l(t)[\rho\phi\phi^*(t)-\rho\phi\phi^*(t')]+[l(t)-l(t')]\rho\phi\phi^*(t')| \leqslant(\beta A_1+a^2\|\rho\|A_5)|t-t'|^\mu. \qquad (3.22)$$

The integral in the last term in (3.21) is bounded as follows:

$$|t-t'|\left|\frac{1}{\pi}\int_0^1\frac{d\tau\chi(\tau)\tau}{(\tau+t)(\tau+t')}\right| \leqslant|t-t'|a^2\|\rho/t^{1+\nu}\|\frac{1}{\pi}\int_0^1\frac{d\tau\,\tau^{1+\nu}}{(\tau+t)(\tau+t')} \leqslant a^2\|\rho/t^{1+\nu}\|A_6|t-t'|. \qquad (3.23)$$

The remaining (fourth) term of (3.21) has the form

$$|tI(t)-t'I(t')|\leqslant|t-t'|\ |I(t)|+|I(t)-I(t')| \leqslant(a^2hA_2+2ab\|\rho/t\|A_3)|t-t'|+|I(t)-I(t')|, \qquad (3.24)$$

where the bound of $I(t)$ was obtained from (3.15). We now treat the variation of $I(t)$ by a method copied from a known proof of the Privalov-Plemelj theorem on Hölder continuity of Cauchy integrals.[15] Let $t'=t+\theta$; without losing generality we may take $\theta\geqslant 0$. Denote the interval $[-\epsilon,1+\epsilon]$ as γ, and define a subinterval γ_1 as follows:

$$\gamma_1=\{\tau|t-2\theta\leqslant\tau\leqslant t+2\theta\}. \qquad (3.25)$$

To allow all possible values of θ and at the same time ensure $\gamma_1\subset\gamma$ we suppose $\epsilon\geqslant 2$. Now $I(t+\theta)-I(t)=\int_{\gamma_1}+\int_{\gamma-\gamma_1}$, and the integral over γ_1 is seen to be of order θ^μ;

$$\left|\frac{1}{\pi}\int_{\gamma_1}d\tau\left[\frac{\chi(\tau)-\chi(t+\theta)}{\tau-t-\theta}-\frac{\chi(\tau)-\chi(t)}{\tau-t}\right]\right| \leqslant\frac{\alpha}{\pi}\int_{\gamma_1}|\tau-t-\theta|^{\mu-1}d\tau+\frac{\alpha}{\pi}\int_{\gamma_1}|\tau-t|^{\mu-1}d\tau =(\alpha/\pi\mu)(1+3^\mu+2^{1+\mu})\theta^\mu. \qquad (3.26)$$

The remaining integral over $\gamma-\gamma_1$ is divided into two parts;

$$I_1+I_2=(1/\pi)[\chi(t)-\chi(t+\theta)]\int_{\gamma-\gamma_1}\frac{d\tau}{\tau-t} +\frac{1}{\pi}\int_{\gamma-\gamma_1}[\chi(\tau)-\chi(t+\theta)]\left(\frac{1}{\tau-t-\theta}-\frac{1}{\tau-t}\right)d\tau. \qquad (3.27)$$

By direct evaluation

$$|I_1|=|\chi(t)-\chi(t+\theta)|\ \left|\frac{1}{\pi}\ln\left(\frac{1+\epsilon-t}{\epsilon+t}\right)\right|\leqslant\alpha A_1\theta^\mu. \qquad (3.28)$$

For I_2 we have

$$|I_2|\leqslant\frac{\alpha\theta}{\pi}\int_{\gamma-\gamma_1}\frac{d\tau}{|\tau-t-\theta|^{1-\mu}|\tau-t|} =\frac{\alpha\theta}{\pi}\int_{\gamma-\gamma_1}\frac{d\tau}{|\tau-t|^{2-\mu}|1-\theta/(\tau-t)|^{1-\mu}}. \qquad (3.29)$$

Since $|\tau-t|\geqslant 2\theta$, one has the bound $|1-\theta/(\tau-t)|^{-1+\mu}\leqslant 2^{1-\mu}$. Thus,

$$|I_2|\leqslant\frac{2^{1-\mu}\alpha\theta}{\pi}\int_{\gamma-\gamma_1}\frac{d\tau}{|\tau-t|^{2-\mu}} =\frac{\alpha\theta^\mu}{(1-\mu)\pi}\{2-(2\theta)^{1-\mu}[(1+\epsilon-t)^{\mu-1}+(\epsilon+t)^{\mu-1}]\} \leqslant\alpha\theta^\mu A_7. \qquad (3.30)$$

After collecting results from (3.21)–(3.24), (3.26), (3.28), and (3.30) we see that $A(K)\subset K$ leads to the following inequality:

$$|\psi_\alpha(t)-\psi_\alpha(t')|\leqslant[\sup_\alpha|\lambda_\alpha|+\beta+\beta A_1+a^2\|\rho\|A_5 +\sum_\beta|c_{\alpha\beta}|a^2\|\rho/t^{1+\nu}\|A_6+a^2hA_2+2ab\|\rho/t\|A_3 +\frac{\alpha}{\pi\mu}(1+3^\mu+2^{1+\mu})+\alpha A_1+\alpha A_7]|t-t'|^\mu \leqslant b|t-t'|^\mu. \qquad (3.31)$$

From (3.17) and (3.31), and the definitions of α and β in (3.19) and (3.20), one sees that $A(K)\subset K$ if

$$\begin{aligned}\sup|\lambda_\alpha|+M_1a^2+M_2ab\leqslant a\,,\\ \sup|\lambda_\alpha|+M_3a^2+M_4ab\leqslant b.\end{aligned}\tag{3.32}$$

The positive constants M_i depend on the cutoff function, the crossing matrix, the exponent μ in the definition (2.12) of K, and on geometrical factors that arose in estimating the integrals. The M_i are independent of the coupling constant. The conditions may always be fulfilled if $\sup|\lambda_\alpha|$ is sufficiently small. For example, if we choose $a=b=2\sup|\lambda_\alpha|$, then conditions (3.32) are met if

$$\begin{aligned}4(M_1+M_2)\sup|\lambda_\alpha|\leqslant 1\,,\\ 4(M_3+M_4)\sup|\lambda_\alpha|\leqslant 1\,.\end{aligned}\tag{3.33}$$

It remains to show that the operator A is continuous in its action on the set K. Suppose that a sequence $\{\phi_n\}$ of elements of K converges to a limit ϕ in K. We must show that $\|\psi_n-\psi\|=\|A\phi_n-A\phi\|$ tends to zero. We have

$$\begin{aligned}\psi_{n\alpha}(t)-\psi_\alpha(t)&=i\rho(t)[\phi_{n\alpha}\phi_{n\alpha}^*(t)-\phi_\alpha\phi_\alpha^*(t)]\\ &\quad-\frac{t}{\pi}P\int_0^1\frac{d\tau\rho(\tau)}{\tau}\frac{[\phi_{n\alpha}\phi_{n\alpha}^*(\tau)-\phi_\alpha\phi_\alpha^*(\tau)]}{\tau-t}\\ &\quad+\sum_\beta c_{\alpha\beta}\frac{t}{\pi}\int_0^1\frac{d\tau\rho(\tau)}{\tau}\frac{[\phi_{n\beta}\phi_{n\beta}^*(\tau)-\phi_\beta\phi_\beta^*(\tau)]}{\tau+t}.\end{aligned}\tag{3.34}$$

The first term on the right side of (3.34) clearly tends uniformly to zero. To handle the second term we extend the principal-value integral to the interval $[-\epsilon, 1+\epsilon]$, and subtract and add a pole term as was done in Eq. (3.10). The pole term tends uniformly to zero. The other term is divided into an integral over a small neighborhood Δ of $\tau=t$, and a remainder; $\int_\Gamma=\int_\Delta+\int_{\Gamma-\Delta}$. By choosing Δ sufficiently small, the integral over Δ may be made less than $\frac{1}{3}\epsilon$:

$$\begin{aligned}&\bigg|\frac{t}{\pi}\int_\Delta\frac{d\tau}{\tau-t}[\rho\phi_n\phi_n^*(\tau)/\tau-\rho\phi_n\phi_n^*(t)/t\\ &\qquad-\rho\phi\phi^*(\tau)/\tau+\rho\phi\phi^*(t)/t]\bigg|\\ &\qquad\qquad\leqslant\frac{2\alpha}{\pi}\int_\Delta d\tau|\tau-t|^{\mu-1}<\tfrac{1}{3}\epsilon.\end{aligned}\tag{3.35}$$

With Δ fixed to satisfy (3.35), one may choose $N(\epsilon)$ so that $\int_{\Gamma-\Delta}$ is less in magnitude than $\frac{1}{3}\epsilon$, because in the latter integral the factor $1/(\tau-t)$ is bounded. In a similar way the last term of (3.34) is less than $\frac{1}{3}\epsilon$ for sufficiently large n. The continuity of A follows.

IV. EXISTENCE THEOREM AND CHARACTERISTICS OF SOLUTION

After returning to the variable ω, we may state the result of the preceding section as follows. *Theorem:* Suppose that $\rho(\omega)$ satisfies the following conditions:

(i) $|\omega\rho(\omega)-\omega'\rho(\omega')|\leqslant h\left|\dfrac{\omega-\omega'}{\omega\omega'}\right|^\nu,\quad 1\leqslant\omega,\omega'<\infty,$

$\nu>0,\quad h>0;$

(ii) $\lim_{\omega\to 1}\omega\rho(\omega)=\lim_{\omega\to\infty}\omega\rho(\omega)=0.$

[(i) and (ii) together imply $\rho(\omega)=O(\omega^{-1-\nu})$, $\omega\to\infty$.] Let the set K be defined by (2.12), with any $\mu\leqslant\nu$, and with the positive numbers a and b satisfying conditions (3.32). Then the Low equation has at least one solution with boundary values $f_{\alpha+}(\omega)=\phi_\alpha(t)$ belonging to K.

We have seen that when the coupling constant is sufficiently small (or more generally when $\sup|\lambda_\alpha|$ is sufficiently small), such numbers a and b certainly exist. The allowed range of coupling strength depends on the crossing matrix, the cutoff, the exponent $\mu\leqslant\nu$, and on geometrical terms obtained in estimating the integrals. We may get some idea of the limitation on coupling by looking at the usual symmetric pseudoscalar theory (Chew-Low theory) with a conventional type of cutoff. Salzman and Salzman[5] treated the Chew-Low equation numerically with the following choice of cutoff:

$$\begin{aligned}\rho(\omega)=(k^3/12\pi)e^{-k^2/m^2},\\ m=7m_\pi.\end{aligned}\tag{4.1}$$

This ρ function, which clearly satisfies the conditions of our theorem, has its maximum value at $k^2=\frac{3}{2}(49)$, hence $\|\rho\|\approx 3.7$. Also, for the Chew-Low theory one has

$$\sup|\lambda_\alpha|=8f^2\approx 8.75\,,\tag{4.2}$$

with $f^2/4\pi=0.087$. According to (3.17), $a\geqslant\sup|\lambda_\alpha|$. Noting the term $a^2\|\rho\|$ in (3.17), we see that (3.17) is far from being satisfied if ρ and f^2 are chosen as in (4.1) and (4.2). Other choices of the cutoff which are made in the literature are roughly similar to (4.1), and lead to similar restrictions on the coupling strength.

Even if $\sup|\lambda_\alpha|$ is small, satisfaction of the condition (3.32) requires a^2 and ab to be small with respect to 1, since the M_i are large compared to 1. In view of (2.12a) and (2.2) it follows that for the solution in question

$$|\sin\delta(\omega)|\leqslant a\rho(\omega)\ll 1\,.\tag{4.3}$$

(These statements are made assuming again a conventional cutoff.) Thus our solution is nonresonant. We are not able to study by the present method the physically interesting (and strongly nonlinear) case in which resonances may appear.

With the solution in question the nucleon is elementary in the technical sense that it corresponds to a Castillejo-Dalitz-Dyson (CDD) pole in the D function of an appropriate N/D representation. This is seen from the generalized Levinson relation

$$\delta_\alpha(\infty)=-\pi(n_b-n_c)_\alpha,\quad [\delta_\alpha(1)=0],\tag{4.4}$$

where n_b is the number of stable particles and n_c the number of CDD poles. In soluble models each CDD pole corresponds to a discrete eigenstate of the free Hamiltonian H_0, the latter being what one means by an elementary particle. Hence it is reasonable, even if the theory is not explicitly soluble, to say that a CDD pole in a particle channel corresponds to an elementary particle with the quantum numbers of that channel. Now note that for small a we have a solution in which $\delta(\infty)$ is zero, because by (4.3) the phase δ tends to an integral multiple of π at infinity, while $|\delta|$ never exceeds $\frac{1}{2}\pi$ (say) at any energy. At the same time we have $n_b=1$ in the (11) channel of the Chew-Low theory, and therefore $n_c=1$ in that channel; the nucleon is elementary. All other channels have $n_b=n_c=0$.

To show that Eq. (4.4) actually holds in the present case, we need only explain that (4.4) follows from our definition of n_c provided $\delta(\infty)\geqslant-\pi n_b$, and that the latter inequality is easily seen to hold for a solution corresponding to small a and small coupling constant. The most general D function bounded by a polynomial at infinity and meromorphic in the cut plane is[11]

$$D(z)=R(z)\mathfrak{D}(z),$$
$$\mathfrak{D}(z)=\exp\left[-\frac{z}{\pi}\int_1^\infty\frac{\delta(\omega)d\omega}{\omega(\omega-z)}\right],\tag{4.5}$$

where $R(z)=R^*(z^*)$ is a rational function. The integer n_c is defined as the number of poles that R must have if D is to tend to 1 at infinity, supposing that R has zeros only at the positions of the stable particle poles of f_α. When $\delta(\omega)$ tends to its limit $\delta(\infty)$ as rapidly as a power (as it does in our example), one has the asymptotic behavior[11]

$$\mathfrak{D}(z)\sim z^{\delta(\infty)/\pi},\quad |z|\to\infty.\tag{4.6}$$

Hence (4.4) holds if $\delta(\infty)\geqslant-\pi n_b$. To see that $\delta(\infty)<-\pi n_b$ is impossible when a is sufficiently small, we note that a certain integral equation is a necessary condition on the imaginary part of the following D function:

$$D(z)=\prod_{i=1}^{n_b}(1-z/\omega_i)\mathfrak{D}(z).\tag{4.7}$$

Here the ω_i are the energies of the stable particles. If $\delta(\infty)<-\pi n_b$, then from (4.6) and the fact $\delta(\infty)$ is an integral multiple of π we have

$$D(z)=O(z^{-1}).\tag{4.8}$$

Hence D certainly may be represented as follows:

$$D(z)=1-\frac{z}{\pi}\int_1^\infty\frac{\rho(\omega)N(\omega)d\omega}{\omega(\omega-z)},$$
$$-\rho(\omega)N(\omega)=\text{Im}D_+(\omega).\tag{4.9}$$

It follows by a standard argument[11] that $N(\omega)$ obeys the integral equation

$$\rho^{1/2}(\omega)N(\omega)=\rho^{1/2}(\omega)B(\omega)+\frac{1}{\pi}\int_1^\infty\left[\rho^{1/2}(\omega)\frac{B(\omega)-B(\omega')}{\omega-\omega'}\rho^{1/2}(\omega')\right]\times\rho^{1/2}(\omega')N(\omega'),\tag{4.10}$$

where $B(\omega)$ is the contribution to $f_\alpha(\omega)$ of all singularities to the left of $\omega=1$, except for the direct-channel particle poles at $\omega=\omega_i$. With the Low equation one has

$$B_\alpha(\omega)=\sum_\beta c_{\alpha\beta}\left[g_\beta^2/\omega+\frac{1}{\pi}\int_1^\infty\frac{d\omega'\rho(\omega')f_{\beta+}(\omega')f_{\beta+}^*(\omega')}{\omega'+\omega}\right].\tag{4.11}$$

With this definition of B, Eq. (4.10) is a Fredholm equation in the space of square integrable functions. According to (4.8) the function $\bar{D}(z)=(1+z)D(z)$ also has a representation like (4.9), and consequently the corresponding $\bar{N}(\omega)=-\text{Im}\bar{D}_+(\omega)/\rho(\omega)$ also satisfies (4.10). Hence $N(\omega)-\bar{N}(\omega)$ satisfies the homogeneous equation corresponding to (4.10), which is impossible if the kernel does not have a characteristic value equal to one. An upper bound on $1/\lambda^2$, where λ is any characteristic value, is provided by the squared norm of the kernel[16]:

$$\frac{1}{\lambda^2}\leqslant\frac{1}{\pi^2}\int_1^\infty d\omega\int_1^\infty d\omega'\left[\frac{B(\omega)-B(\omega')}{\omega-\omega'}\right]^2\rho(\omega)\rho(\omega').\tag{4.12}$$

If a and g_β^2 are sufficiently small, it is clear that the right side of (4.12) is less than 1, hence $\lambda^2>1$ for every λ. It follows that Levinson's relation (4.4) holds for our solution of the Low equation, provided a and g_β^2 are sufficiently small.

The dodge of *defining* n_c as we did following Eq. (4.5) while still identifying n_c with the number of elementary particles is justified only by analogy with soluble field theories. In any case such a definition of elementarity is not realistic, because it makes sense only in a theory lacking inelastic effects.[12]

16 R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. I.

It should be obvious that the methods we have described apply as well to a more general Low equation in which the single-meson scattering matrix is not diagonalized through conservation laws. We can easily handle the unitarity condition in matrix form, and several kinds of mesons with unequal masses. The generalization to relativistic equations with a finite number of partial waves also presents no great difficulty.

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Conspiracy or Evasion; the Electron's Way*

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The renormalizable perturbation expansion for the scattering of electrons and massive photons is studied, as a theorist's experiment, to test conspiracy theory for Regge trajectories. For convenience, all particles are assigned the same mass. It is shown that a nontrivial evasive solution is allowed kinematically, but that the electron trajectory does not choose it. Rather, at least to lowest order, there are three sequences of daughter trajectories, whose values differ by integers at $s=0$, and which all conspire in a complicated way with the electron to satisfy the kinematic constraints. This picture is greatly simplified using the $O(4)$ expansion of Freedman and Wang. There are only a small number of Lorentz poles. We reformulate the problem in this language and discuss some of the properties of these poles. The electron Lorentz pole factors and implies $M=\frac{1}{2}$.

I. INTRODUCTION

FOR the past year or two, we have all been interested in those kinematic constraints among helicity amplitudes which, as a consequence of crossing and angular momentum conservation, occur at $s=0$ in equal mass, elastic scattering amplitudes. If, furthermore, an amplitude has a Regge representation, i.e., is analytic in j except for moving poles to some useful approximation in some useful region of the complex j plane, the satisfaction of these constraints is not obviously trivial. "Conspiracies" among Regge trajectories may be required.[1,2]

Three ways of satisfying these constraints have been suggested: (i) They are satisfied by the contribution of each Regge trajectory alone. This solution is called evasion and has special predictions for forward scattering amplitudes. (ii) A few trajectories differ exactly by integers at $s=0$, and the residues adjust themselves to satisfy the constraints. This solution is called conspiracy and has been extensively studied for nucleon-nucleon scattering.[1,3] (iii) An infinite sequence of daughterlike trajectories arrange themselves in an organized way at $s=0$ to satisfy the constraints. In this case, the description of the amplitude may be much simpler in a representation related to $O(3,1)$ or $O(4)$, rather than the Regge representation.[3]

When attempting to guess general properties of scattering amplitudes, it is often instructive to look for a model to the perturbation expansion of a renormalizable field theory; this is the whole point of the present paper. The trouble is that such theories, while satisfying the kinematical constraints at $s=0$ (since Feynman diagrams have Lorentz invariance and

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