Khuri-Treiman–Type Equations for Three-Body Decay and Production Processes

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We discuss some aspects of a model for three-body decay (or production) processes based essentially on the elastic approximation in each subenergy channel. A simple method is given to reduce the dispersion integral equations into a single-variable form. In passing, some properties of the triangle-graph amplitude with respect to an internal mass are clarified. Finally we discuss the conditions under which the model can nearly satisfy three-body unitarity; in particular, it is shown that, by simulating high-energy effects in each subenergy variable by a convenient cutoff procedure, we can satisfy three-body unitarity, at least in the decay (or production) region. We restrict ourselves in this paper to the lowest possible angular momentum states.

I. INTRODUCTION

THE original application of Khuri-Treiman (KT) equations[1] was to the final-state two-body energy spectra in three-body decay (or production) processes [Fig. 1(a)]. But it has often been suggested that they may also be applied to the three-body spectrum itself.[2,3] For this purpose one needs to know the analytic properties and the discontinuities of KT amplitudes with respect to the total squared-energy variable [$s=m^2$ in Fig. 1(a)]. Two earlier works[4,5] have been devoted to this subject. Some aspects of the problem are reviewed and completed in the present one.

First we expose a simple procedure to derive the so-called[6] single-variable integral representation (SVR) of KT amplitudes and by doing so we get again some properties of the triangle Feynman graph with respect to an internal mass (Sec. III). Then we examine once again the problem of three-body unitarity in KT equations for the simplest case of a spinless decaying particle (or a production process in a total angular momentum $J=0$),[7] leading to three spinless particles interacting by pairs in S-wave states only (Sec. IV). We shall deal with the study of higher angular momenta in a forthcoming paper.

II. PRELIMINARY REMARKS

Let us first mention some interesting relations between the approach we consider and other well-known models: the usual models for low-energy scattering reactions—for this comparison we need the notion of analytic continuation in an external mass—(Sec. II A), the usual isobaric models (Sec. II B), and the three-body Bethe-Salpeter (BS) equations (Sec. II C).

A. Low-Energy Scattering Reactions

The idea of relating the dynamics of two-body scattering reactions and three-body decay by analytic continuation with respect to an external mass is not a new one.[3,8–10] It corresponds indeed to a general concept of S-matrix and perturbation theory.[11] In the present context this leads us to search for reliable and sufficiently simple representations for both the processes of Fig. 1(b) and those of Fig. 1(a), which correspond to small and to large values of s, respectively. A natural way to proceed is to take advantage of our knowledge of two-body scattering reactions and examine approximations to the Mandelstam representation of the process of Fig. 1(b). The most elaborate representation we are led to consider in this manner corresponds to the so-called strip approximation[12] and takes account of the whole two-body discontinuities in each channel of Fig. 1(b). After continuation up to decaying values of s, the representation correspondingly takes account of the lowest intermediate states allowed in each subchannel of Fig. 1(a) (see in Fig. 2 a typical diagram which enters in such a representation). For values of s

* Laboratoires associés au C.N.R.S.
[1] N. N. Khuri and S. B. Treiman, Phys. Rev. **119**, 1115 (1960).
[2] G. Bonnevay, in *Proceedings of the 1960 Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1960).
[3] G. Bonnevay, Nuovo Cimento **30**, 1325 (1963).
[4] R. Pasquier, Orsay Report IPNO/TH 31, 1965 (unpublished).
[5] I. J. R. Aitchison and R. Pasquier, Phys. Rev. **152**, 1274 (1966).
[6] I. J. R. Aitchison, Phys. Rev. **137**, B1070 (1965).
[7] Throughout this paper we use the word "decay" to speak of decay or production.
[8] J. B. Bronzan and C. Kacser, Phys. Rev. **132**, 2703 (1963); see also V. V. Anisovitch, A. A. Anselm, and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **42**, 224 (1962) [English transl.: Soviet Phys.—JETP **15**, 159 (1962)].
[9] D. Bessis and F. Pham, J. Math. Phys. **4**, 1253 (1963).
[10] G. Bonnevay, Proc. Roy. Soc. (London) **A226**, 68 (1962).
[11] R. C. Hwa, Phys. Rev. **134**, B1086 (1964). This work contains further references.
[12] S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963). This work contains numerous references on the usual approximation of the Mandelstam representation for a scattering process.

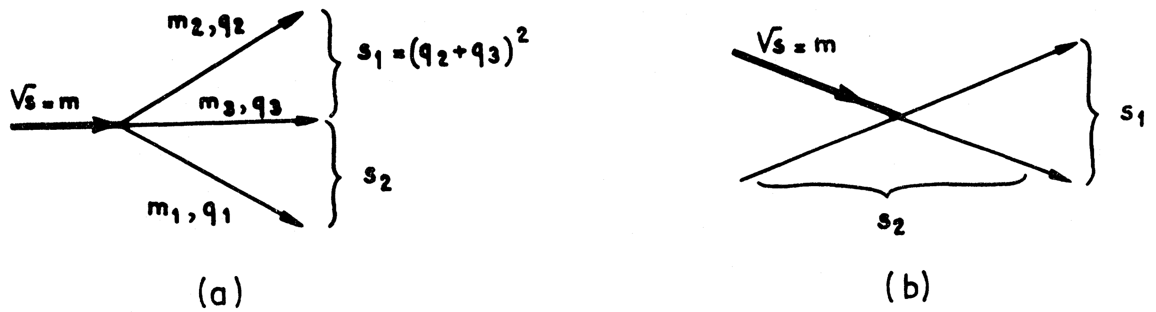

FIG. 1. (a) Three-particle decay (or production) process: $s=(\sum_{i=1}^{3} q_i)^2$ is the square of the total three-body mass, $s_1=(q_2+q_3)^2$ is a final-state subenergy variable; (b) "crossed" process of $2\rightarrow 2$ scattering.

which are not too great (below the four- and five-particle thresholds) the procedure can thus lead to a good picture of the process of Fig. 1(a). Unfortunately, the amplitudes involved in this approach depend upon *two* variables and the integral equations cannot be put into a single variable form, which excludes tractable theoretical and numerical investigations on them.

For this reason it is worthwhile, at least in a first step, to restrict oneself to a still cruder approximation by retaining only a few partial waves (denoted by l) in the two-body discontinuities. In the scattering region this assumption just leads to the so-called Cini-Fubini approach.[12,13] In the decay region the resulting equations have the same structure as the equations first used by Khuri and Treiman for describing final-state interactions in $K\rightarrow 3\pi$ decay. For this reason we refer to them here as KT-type equations, even though they have been used for the same purpose by many other authors.[14,15]

As is well known, the simplicity of a Cini-Fubini-type approximation is obtained at the cost of several drawbacks which do not appear in the more elaborate strip approximation.[12] But two arguments are generally put forward to justify the Cini-Fubini procedure in the low-energy scattering region. (i) The first takes into account the probable dominance in each channel of Fig. 1(b) of a few partial waves corresponding generally to resonant or bound states. (ii) The second is based on the properties of the support of the Mandelstam double-spectral function. This argument allows one to expand the amplitude in polynomials of at least one of the invariants, which suggests retention of a limited number of partial waves in the spectral function.

The latter argument no longer holds after continuation in s up to the decay region, since in that case the extension of the support makes doubtful the validity of a polynomial expansion[10,16] for a strip diagram as in Fig. 2. Correspondingly, the domain of convergence of the two-body discontinuity partial-wave expansion of this diagram is very restricted and, as one can show,[17] generally does not cover the whole range of integration needed in the dispersion relation. [Nevertheless, it contains, on the *principal*[11] sheet, the Dalitz-plot region where there is not the accumulation of leading Landau singularities considered in Ref. 10.]

However, it is possible that the effects of box diagrams like Fig. 2 are small in reality, since it is known that numerous three-body decay processes appear to be experimentally dominated by pairwise generally resonant final-state interactions occurring in a few angular momenta. This gives by far the most convincing argument[18] for attempting the KT approach which takes

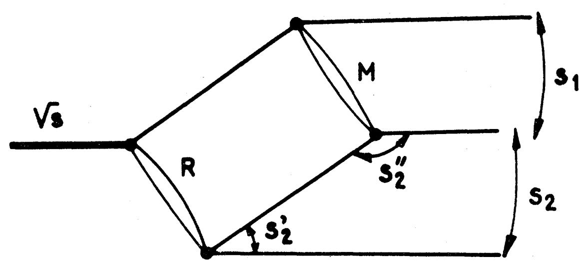

FIG. 2. Typical box diagram corresponding to the strip approximation.

[13] M. Cini and S. Fubini, Ann. Phys. (N. Y.) **10**, 352 (1960).

[14] R. F. Peierls and J. Tarski, Phys. Rev. **129**, 981 (1963); I. J. R. Aitchison, Nuovo Cimento **35**, 234 (1965).

[15] I. J. R. Aitchison, Nuovo Cimento **51A**, 249 (1967); see this work for further references.

[16] G. Barton and C. Kacser, Nuovo Cimento **21**, 988 (1961); V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 1215 (1961) [English transl.: Soviet Phys.—JETP **14**, 866 (1962)].

[17] The methods to be used are rather similar to those of Appendix A.

[18] It is nevertheless worth mentioning here an argument pointed to us by I. J. R. Aitchison, the essence of which he attributes to the late G. Bonnevay. The basic remark is that the diagrams giving the most important effects are those singular close to the physical boundary. Recall that this requires values of the external masses ($\sqrt{s}$ in particular) such that the associated processes are physically possible with all the intermediate states on the mass-shell (see Ref. 53). For diagrams based on the triangle-graph structure (which can be taken into account in the KT model), important effects can be obtained for $\sqrt{s}$ values just above the threshold for production of one two-body resonant state (see Ref. 15). On the contrary, only box-type diagrams involving two resonances in the total-energy intermediate state [see also P. Collas and R. E. Norton, Phys. Rev. **160**, 1346 (1967); V. V. Anisovitch, Serpukhov Report, 1967 (unpublished)] seem to be able to give both a singularity close to the physical region and important effects, but this is allowed only for $\sqrt{s}$ values above twice two-body resonant masses. This therefore justifies the KT model based on the triangle-graph structure, at least for not too large values of $\sqrt{s}$.

it for granted that such resonant interactions provide the basis of both the scattering reactions like Fig. 1(b) [justification (i) above] and the crossed decay process of Fig. 1(a). Notice that in this approach, because of the limited number of partial waves in each subenergy channel, the basic nontrivial analytic structure reduces essentially to that of the triangle instead of the box (Fig. 2) diagram.

B. Isobaric Models

The physical situation that the KT model is required to represent is therefore just the situation described by the usual isobaric models.[19] But in such models the amplitudes take the form of a sum of terms in which the angular dependences are made explicit. Can KT amplitudes be put in such a form? To answer this question it is worth mentioning that the basic assumptions which are recalled above determine only the analytic structure of these amplitudes (two-body cuts and associated discontinuities). Once this structure is given, various amplitudes can be reconstituted by means of Cauchy integrals. They differ from each other by functions that are known only to be free of the given two-body dynamical cuts. We shall take advantage of this indetermination essentially in the case of $l\neq 0$ two-body interactions and decaying particles with spin $J\neq 0$. We shall then choose the amplitude in such a way as to make the angular dependences explicit.

C. Three-Body Bethe-Salpeter Equations

The relationship of the KT model to three-body Bethe-Salpeter equations, from which the recent relativistic extensions of Faddeev equations originate,[20–23] also merits further comment. In the simplest case $J=0$, $l=0$—and constant two-body interactions—the comparison is simple. The KT dispersion terms can then be easily related to the corresponding Feynman terms summed up in three-body BS-type equations with constant zero-range "potentials." The lowest-order nontrivial term is indeed identical, in both approaches, to the triangle Feynman-graph amplitude of Fig. 3(a). But the next KT term stands for only one part of the corresponding Feynman graph pictured in Fig. 3(b); this is the part associated with the elastic discontunity—cut (A) of this figure—the remaining neglected part is that associated with the inelastic cut (B). A similar discrepancy occurs at each following iteration of KT equations. At each step the KT dispersion term takes into account only the elastic subenergy cut of the corresponding Feynman graph. It is reasonable to think that such a correspondence may be kept also in the case of more general two-body interactions and higher angular momentum states, and, to summarize, that KT-type integral equations will follow from BS equations by: (i) first, extracting from them the correct[24] elastic discontinuities in each subenergy variable (with a finite number of partial waves in them), and (ii) then, writing the Cauchy integrals associated with these discontinuities. (The nonuniqueness of the reconstituted amplitudes has to be exploited when choosing the appropriate form[25] of this amplitude.)

By working so successively in the two-body subenergy channels only mass-shell amplitudes are involved, in contradistinction to what occurs in Faddeev equations obtained by a transformation of the three-body Green's function of BS equations. It is also reasonable to think that, by doing so, we can obtain equations involving a small number of integrations, since, at each step, we cut two particles instead of three.[5]

D. Khuri-Treiman Equations and Three-Body Unitarity

However, the KT equations will provide an interesting approach to the relativistic three-body problem only if they simultaneously satisfy three-body unitarity. At this point it is worth mentioning that in other S-matrix (mass-shell) approaches to the three-body problem, three-body unitarity provides *a priori* the basis of the treatment.[26,27] In the KT equations the inputs are two-body unitarity and continuation with respect to an external mass; we have thus to look *a posteriori* at whether the amplitudes generated in the model satisfy three-body unitarity. This was claimed in Refs. 4 and 5 to be effectively the case, but it is not quite exactly true, since in the equal-mass case for instance the $3\rightarrow 3$ amplitude involved in the model is not a symmetric function with respect to the initial and final subenergy variables, as it must be (indeed the analytic properties with respect to these variables are not quite the same). As a consequence it cannot be associated with a $3\rightarrow 3$ transition operator satisfying three-body unitarity and it can only be considered as a particular solution of the three-body discontinuity equations, the time-reversal requirement of symmetry for the physical amplitude being omitted.

The origin of this drawback may be looked for by

[19] S. J. Lindenbaum and R. M. Sternheimer, Phys. Rev. **105**, 1874 (1957); B. Deler and G. Valladas, Nuovo Cimento **45**, 559 (1966); (this work contains further references).

[20] R. Blankenbecler and R. Sugar, Princeton report, 1964 (unpublished); Phys. Rev. **142**, 1051 (1966).

[21] V. A. Alessandrini and R. L. Omnès, Phys. Rev. **139**, B167 (1965). One of us (R. P.) is indebted to Professor R. L. Omnès and Dr. J. L. Basdevant for a helpful discussion about Faddeev equations and the fact that the symmetry of these equations is broken when one wants to derive KT-type equations from them.

[22] D. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cimento **43**, 258 (1966).

[23] A. Ahmadzadeh and J. A. Tjon, Phys. Rev. **147**, 1111 (1966).

[24] By correct subenergy discontinuity we mean the expression obtained by analytic continuation from two-body unitarity, as done in Refs. 11, 4, and 5 for instance.

[25] See, for instance, J. M. Namyslowski, Phys. Rev. **160**, 1522 (1967), where such problems are considered.

[26] G. N. Fleming, Phys. Rev. **135**, B551 (1964); see also W. J. Holman, III, Phys. Rev. **138**, 1286 (1965).

[27] S. Mandelstam, Phys. Rev. **140**, B375 (1965).

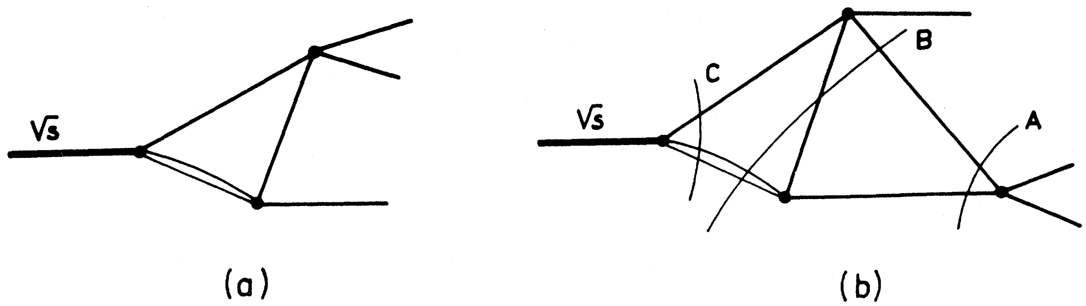

FIG. 3. Two lowest-order-scattering Feynman diagrams in the case of constant S-wave two-body interactions.

comparison with other relativistic equations where three-body unitarity is satisfied.[21] In fact the lack of symmetry already appears on the dispersion term associated with the Feynman graph of Fig. 3(b). The $3\to 3$ triangle amplitude appearing in its discontinuity with respect to s is not symmetric as it would be if the contribution of the cut (B) were taken into account [it would then be obtained by cutting Fig. 3(b) along (C)[28]].

This leads one to associate the lack of symmetry of the $3\to 3$ amplitude involved in the KT model (and thus the lack of three-body unitarity) with an irrelevancy of the elastic approximation for too great values of the subenergy variables. We shall return to this question later and try to construct amplitudes that satisfy two-body unitarity only in the lower range of the subenergy variables but three-body unitarity in a region containing at least the three-body decay region. The so-called SVR of KT amplitudes,[6] and the derivation we give below, are especially well adapted for such a discussion.

III. SINGLE-VARIABLE INTEGRAL REPRESENTATION OF KHURI-TREIMAN EQUATIONS IN THE CASE $J=0$, $l=0$

The case $J=0$, $l=0$ is the simplest one to be dealt with. It corresponds to the decay of a spinless particle of mass $\sqrt{s}$ into three spinless (isoscalar) particles with masses m_i and four-momenta q_i interacting pairwise in S waves only—this, for instance, is the situation customarily considered in the $K\to 3\pi$ decay. Let $\{s_i\}$ ($i=1, 2, 3$) be the set of the usual subenergy invariants [see Fig. 1(a)] and $\mathcal{R}(s,\{s_i\})$ the corresponding amplitude.

In the KT model, $\mathcal{R}(s,\{s_i\})$ essentially reduces to a sum of three terms, each one depending upon only one subenergy variable:

$$\mathcal{R}(s,\{s_i\})=\mathcal{R}_0(s,\{s_i\})+\sum_{i=1,2,3} R_i(s,s_i)\,, \qquad (3.1)$$

where $\mathcal{R}_0(s,\{s_i\})$ is an arbitrary function that we assume, for simplicity, to be free of singularities both in s_i and s.[29] For small s values the $R_i(s,s_i)$ are reconstituted from two-body unitarity by a dispersion Cauchy integral; elsewhere they are defined by analytic continuation[4,5] (the interest of considering directly the amplitudes instead of their S-wave projections[3] has been discussed in Ref. 6). The equations one gets in this way can be slightly condensed by summing up all the successive pairwise interactions occurring in the same subchannel by means of an Omnès inversion.[30] This leads one to introduce[6,31] the usual N_i and D_i parts of the S-wave scattering amplitude of particles (j) and (k), $M_i(s_i)=N_i(s_i)/D_i(s_i)$, and to work with the amplitudes $\varphi_i(s,s_i)=D_i(s_i)\times R_i(s,s_i)$ in place of the $R_i(s,s_i)$. For definiteness we consider the integral representation of $\varphi_1(s,s_1)$. Its general form is[6,5]

$$\varphi_1(s,s_1)=\varphi_{01}(s,s_1)+\frac{1}{\pi}\int_{(m_2+m_3)^2}^{\infty}\frac{ds_1'}{s_1'-s_1}\frac{N_1(s_1')}{k(s_1',s,m_1^2)}$$
$$\times\left\{\int_{s_{2-}(s,s_1')}^{s_{2+}(s,s_1')} ds_2'\frac{\varphi_2(s,s_2')}{D_2(s_2')}+\int_{s_{3-}(s,s_1')}^{s_{3+}(s,s_1')} ds_3'\frac{\varphi_3(s,s_3')}{D_3(s_3')}\right\}, \qquad (3.2)$$

where

$$\varphi_{01}(s,s_1)=R_0'(s,s_1)+\frac{1}{\pi}\int_{(m_2+m_3)^2}^{\infty}\frac{ds_1'}{s_1'-s_1}\times\rho(s_1')N_1(s_1')\mathcal{R}_0^0(s,s_1').$$

$\mathcal{R}_0^0(s,s_1')$ is the S-wave projection of $\mathcal{R}_0(s,\{s_i\})$ of Eq. (3.1) and $R_0'(s,s_1)$ is another arbitrary function which is also assumed to possess neither the cut $s_1\geq(m_2+m_3)^2$ nor the cut $s\geq(m_1+m_2+m_3)^2$. The expressions $s_{i\pm}(s,s_1')$ ($i=2, 3$) are solutions of the so-called Kibble equation[32]

$$s_1's_2's_3'=(m_1^2+m_2^2+m_3^2+s)(as_3'+bs_2'+cs_1') \qquad (3.3)$$

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[28] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

[29] Actually, if we were dealing with a production amplitude, $\mathcal{R}_0(s,\{s_i\})$ would stand for the $J=0$ projection of processes depending on momentum transfer variables. This is indeed the case of the $3\to 3$ amplitude considered in Sec. IV.

[30] R. Omnès, Nuovo Cimento 8, 1244 (1958).

[31] C. Kacser, Phys. Rev. 132, 2712 (1963).

[32] T. W. B. Kibble, Phys. Rev. 117, 1159 (1960).

with a, b, c as in Ref. 32 with $m_4=\sqrt{s}$; this equation limits all the physical regions associated with the four-leg processes of Fig. 1. We set $s_{i+}(s,s_1')>s_{i-}(s,s_1')$ for s_1' both on the path of integration and in the decay region, i.e., $(m_2+m_3)^2\le s_1'\le(\sqrt{s}-m_1)^2$ [elsewhere $s_{i+}(s,s_1')$ and $s_{i-}(s,s_1')$ mean the analytic continuation of these determinations]. Using the notation $k(a^2,b^2,c^2)$ for $[a^2-(b-c)^2]^{1/2}[a^2-(b+c)^2]^{1/2}$, we assume also that in the same region $k(s_1',s,m_1^2)\ge 0$ and $\rho(s_1')=k(s_1',m_2^2,m_3^2)/s_1'\ge 0$. The N and D functions are supposed to guarantee the convergence of the integrals.

Equation (3.2) provides what it is convenient to call the original form of the KT equations, as opposed to the SVR derived by Aitchison,[6] which has the advantage of involving the amplitude under a single (instead of a double) integral. The first step of Aitchison's procedure[33] clearly discloses the relation between KT amplitudes and the triangle Feynman graph. But its second step —the study of this graph as a function of an internal mass—has recourse to results external to the KT representation itself, in particular to the rather elaborate techniques of homological theory.[34] (Aitchison's work has been successively completed in Refs. 35 and 36.) For this reason, we develop here another method which is simply based on an inversion of the order of integration in Eq. (3.2).

To begin with it is necessary to recall the meaning of the paths of integration in this equation (or equivalently the determination of the integrands). For this purpose it is convenient to draw the path—call it (Γ)—followed by $s_{2\pm}(s,s_1')$ when s_1' is running along $[(m_2+m_3)^2,+\infty]$ [the results as regards $s_{3\pm}(s,s_1')$ follow by exchanging the indices 2 and 3]. (Γ) goes always from 0 to $-\infty$ but has a slightly different form depending on whether $m_3<m_1$ or $m_3>m_1$. This is shown in Fig. 4 for s values on the upper (physical) side of the three-body cut $s\ge(m_1+m_2+m_3)^2$.

In Eq. (3.2), the s_2' path of integration can always be chosen along (Γ) and may be cast into two parts such that

$$\int_{s_{2-}(s,s_1')}^{s_{2+}(s,s_1')}(\Gamma)\,ds_2'=\int_0^{s_{2+}(s,s_1')}(\Gamma)\,ds_2'-\int_0^{s_{2-}(s,s_1')}(\Gamma)\,ds_2'$$

(for simplicity, we drop the integrands). This allows us to write

$$\int_{(m_2+m_3)^2}^{+\infty}ds_1'\int_{s_{2-}(s,s_1')}^{s_{2+}(s,s_1')}(\Gamma)\,ds_2'=\int(C)ds_1'\int_0^{\tilde{s}_2(s,s_1')}(\Gamma)\,ds_2',$$

where (C) is shown in Fig. 5 and $\tilde{s}_2(s,s_1')$ stands for $s_{2\pm}(s,s_1')$, the plus or minus sign depending on the position of s_1' on (C). Now the variables of integration s_1' and s_2' may be defined by their curvilinear abscissas x and y on (C) and (Γ), respectively, and the new point of integration may be considered as running over the "linearized" domain shown in Fig. 6. x runs from $-\infty$ to $+\infty$ and y from 0 to $+\infty$. The origin $(x=0)$ on (C) is the point $s_1'=(m_2+m_3)^2$; $x>0$ $(x<0)$ corresponds to s_1' on (C) above (below) the real axis. As regards (Γ), $y=0$ $(y=+\infty)$ is associated with the point $s_2'=0$ $(s_2'=-\infty)$. Therefore one has the following:

(1) Given s_1' on (C), i.e., $x(s_1')$, the integration over $y(s_2')$ runs from 0 to $y(\tilde{s}_2)$ and thus the integration along (Γ) runs from 0 to $\tilde{s}_2$.

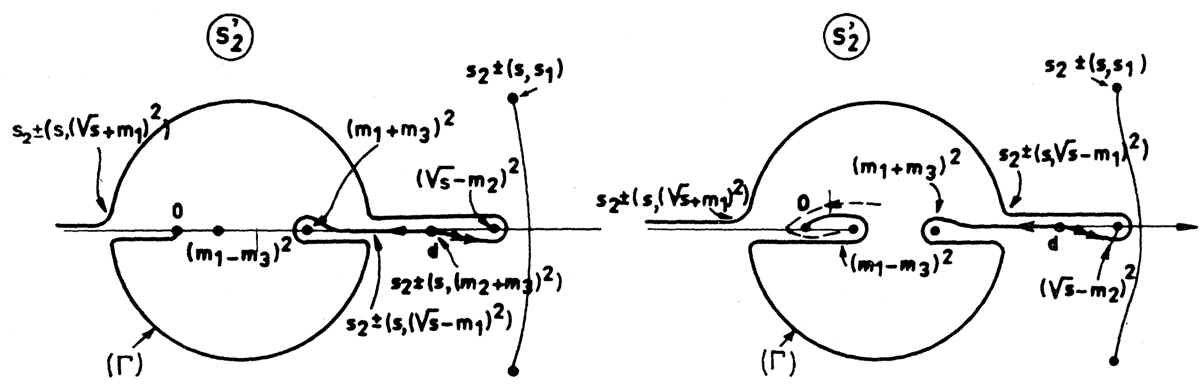

FIG. 4. Path (Γ) in the cases (a) $m_1>m_3$, (b) $m_1<m_3$. When s_1' in Eq. (3.2) goes from $(m_2+m_3)^2$ to $+\infty$, the points $s_{2-}(s,s_1')$ and $s_{2+}(s,s_1')$ go on (Γ) from (d) in the direction of the arrow and the double arrow, respectively. The dashed line in Fig. 4(b) shows the variation of $s_2'=(m_1-m_3)^2$ when m_1 is decreased from a value above m_3 to a value below.

[33] This procedure was suggested by V. V. Anisovitch, Zh. Eksperim. i Teor. Fiz. 44, 1593 (1963) [English transl.: Soviet Phys.—JETP 17, 1072 (1963)] in a work about the nonrelativistic Skornyakov-Ter-Martirosyan equations (see Ref. 45).

[34] D. Fotiadi, M. Froissart, J. Lascoux, and F. Pham, Ecole Polytechnique Report, France, 1965 (unpublished). See also R. C. Hwa and V. L. Teplitz, *Homology and Feynman Integrals* (W. A. Benjamin, Inc., New York, 1966). This work contains further references.

[35] I. J. R. Aitchison and C. Kacser, Nuovo Cimento 40, 576 (1965).

[36] C. Kacser, J. Math. Phys. 7, 2008 (1966).

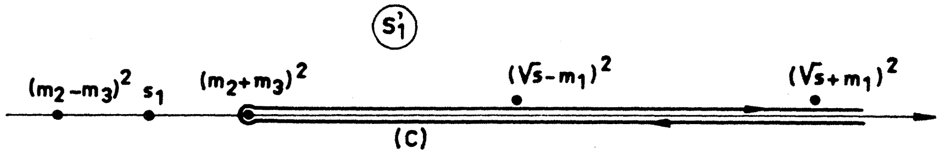

FIG. 5. Path (C) in the s_1' plane. When s_1' runs along (C) following the arrow, the point $s_2'=\tilde{s}_2\,(s,s_1')$ in Eq. (3.4) describes (Γ) from 0 to $-\infty$. For $s_1'\simeq(m_2+m_3)^2$, $\tilde{s}_2(s,s_1')=s_{2+}\,(s,s_1')$ lies in the vicinity of the point (d) on (Γ) (Fig. 4).

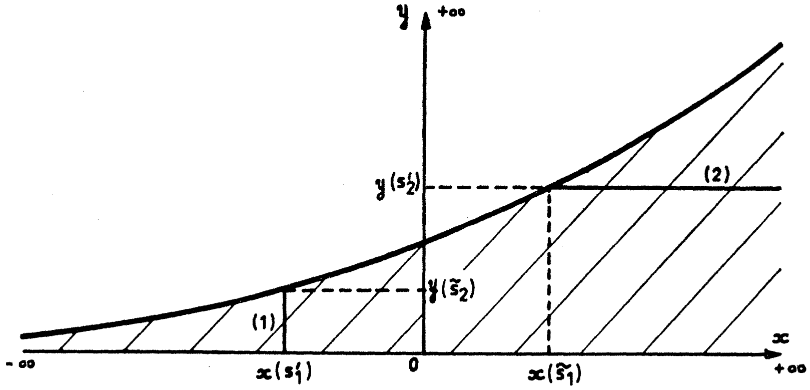

FIG. 6. Linearized domain of integration.

(2) Conversely, given s_2' on (Γ), i.e., $y(s_2')$, the integration over $x(s_1')$ runs from a value $x(\tilde{s}_1)$ to $+\infty$, and thus the integration along (C) runs from $\tilde{s}_1$ to the point $+\infty$ on the upper part of (C). Clearly $\tilde{s}_1=s_{1-}(s,s_2')$ $\times(s_{1-}<s_{1+})$ when $\tilde{s}_1$ lies in the neighborhood of $(m_2+m_3)^2$; elsewhere $\tilde{s}_1$ is the analytic continuation of this determination when s_2' runs along (Γ).

The inversion of the order of integration with the help of the curvilinear abscissas is thus a classical problem and, by setting up again the initial variables, yields finally

$$\int(C)\frac{ds_1'}{s_1'-s_1}\,\frac{N_1(s_1')}{k(s_1',s,m_1^2)}\int_0^{\tilde{s}_2(s,s_1')}(\Gamma)\;ds_2'\frac{\varphi_2(s,s_2')}{D_2(s_2')}$$
$$=\int(\Gamma)\,ds_2'\frac{\varphi_2(s,s_2')}{D_2(s_2')}\int_{\tilde{s}_1(s,s_2')}^{\infty}(C)\;\frac{ds_1'}{s_1'-s_1}$$
$$\times\frac{N_1(s_1')}{k(s_1',s,m_1^2)}. \quad (3.4)$$

By construction the determinations of the basic functions involved in the integrands of the left-hand and right-hand sides of Eq. (3.4) are the same (they are simply rearranged differently). Recall that the determinations of the left-hand side integrands are known[4,5]; they thus fix the s_2' path of integration of the right-hand side without ambiguity and in particular its position with respect to the singularities of $R_2(s,s_2')$ $=\varphi_2(s,s_2')/D(s_2')$ and

$$K(s_2',s,s_1)=\frac{1}{\pi}\int_{\tilde{s}_1(s,s_2')}^{\infty}(C)\;\frac{ds_1'}{s_1'-s_1}\,\frac{N_1(s_1')}{k(s_1',s,m_1^2)}. \quad (3.5)$$

As regards $R_2(s,s_2')$, only the singularity $s_2'=(m_1+m_3)^2$ has to be considered.[37] But $K(s_2',s,s_1)$ may be singular when $\tilde{s}_1(s,s_2')$ is, i.e., at the points $s_2'=0$, $(m_1\pm m_3)^2$, and $(\sqrt{s}\pm m_2)^2$, and when $\tilde{s}_1(s,s_2')=s_1$, i.e., at s_2' $=s_{2\pm}(s,s_1)$ [these last two values are complex for $(m_2-m_3)^2<s_1<(m_2+m_3)^2$, see Fig. 4]. $K(s_2',s,s_1)$ may be also singular at the end points $\tilde{s}_1(s,s_2')=\bar{s}_1$, i.e., $s_2'=s_{2\pm}(s,\bar{s}_1)$, where $\bar{s}_1$ is a left-hand singularity of $N_1(s_1')$; notice that the s_2' path (Γ) as it stands does not encounter such a singularity, since on the left-hand side of Eq. (3.4), the s_1' integration path does not encounter $\bar{s}_1$. Without loss of generality, we may restrict ourselves to the case $(m_2-m_3)^2<\bar{s}_1<(m_2+m_3)^2$, for which $s_{2\pm}(s,\bar{s}_1)$ are complex; the results for any other values of $\bar{s}_1$ will follow by analytic continuation.

All the abovementioned singularities of $R(s,s_2')$ and $K(s_2',s,s_1)$ have to be taken into account when we want to distort the path (Γ) from its original position. This is precisely what we need in order to compare our results with earlier works[6,35,36]: We have first to collapse (Γ) onto the real axis and then isolate the contributions of the singularities we encounter.[37] In this way (Γ) may be split into three paths (γ_1), (γ_2), and (γ_3) associated with the singularities $(\sqrt{s}-m_2)^2$, $(m_1-m_3)^2$, and 0,

[37] We omit here the singularities induced by the bound and resonant states poles of $D_2(s_2')$. Their inclusion would present no difficulty (see Ref. 5) but complicates the discussion. Note however a property which is interesting in practical applications: As is well known, singularities induced by bound-state poles in $D_2(s_2')$ ("anomalous singularities") need a distortion of the path $((m_2+m_3)^2,\infty)$ in the original form of KT equations (3.2). On the contrary, the paths of integration in the SVR form we derive below may be kept as they stand when D_2 has no pole. In particular, the path (γ_1) in Eq. (3.6) remains rectilinear; a pole of D_2 just lies below (above) it depending on whether s lies above (below) the two-body cut generated by the pole (see Ref. 5).

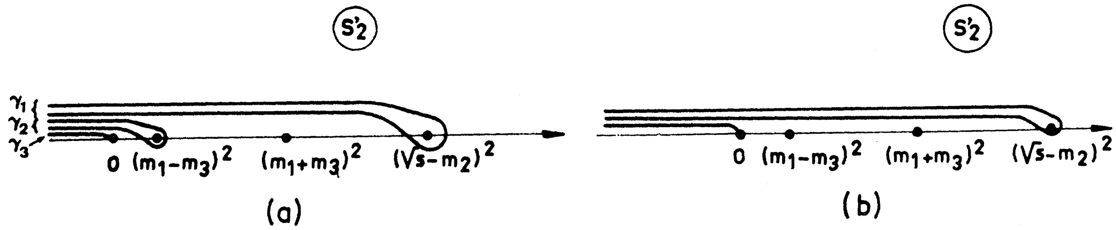

FIG. 7. Distortion of (Γ) allowing us to write the SVR Eq. (3.6). (a) Case $m_1>m_3$; (b) case $m_1<m_3$.

respectively [(γ_2) is absent if $m_1<m_3$]; this is shown in Figs. 7(a) and 7(b)—notice that no deformation arises from $s_{2\pm}(s,s_1)$ and $s_{2\pm}(s,\bar{s}_1)$ for the values of s_1 and $\bar{s}_1$ we have assumed. To these distortions of (Γ) there correspond distortions of (C). These are shown in Fig. 8 for, as an example, m_1, $m_2>m_3$. There also corresponds a decomposition of the right-hand side of Eq. (3.4) into three parts, each one being associated with one (γ_i). Indeed, the contributions of two parallel parts of (γ_1) and (γ_2) may be combined together; this leads to the introduction of new paths—call them also (γ_1) and (γ_2)—which run from $-\infty$ to the relevant singularity

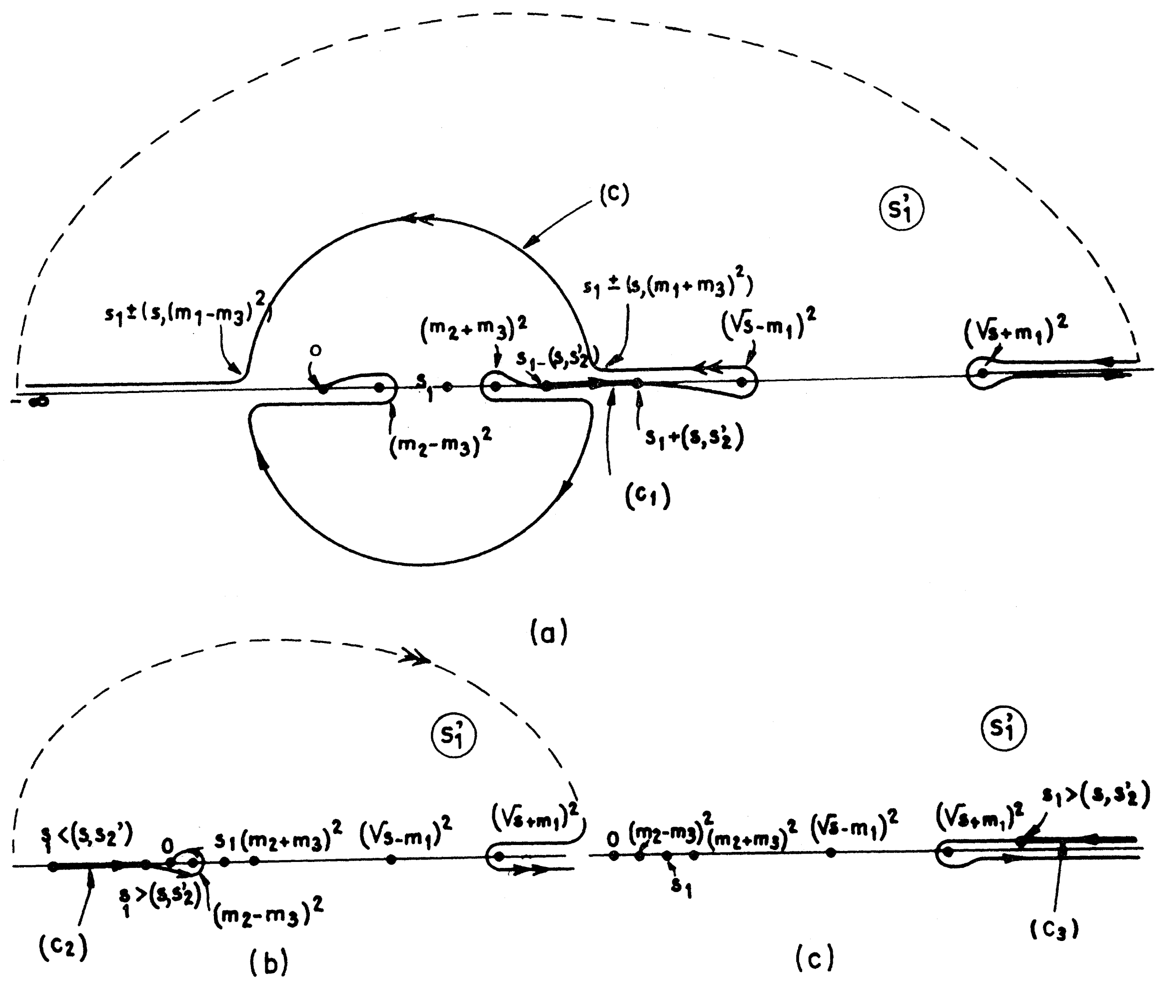

FIG. 8. Contours (C_i) involved in the $\Delta^{(i)}(s_2',s,s_1)$ for the case m_1, $m_2>m_3$. (a) Displacements of the points $s_{1<}(s,s_2')$ and $s_{1>}(s,s_2')$ when s_2' is decreased along (γ_1) are indicated by an arrow and a double arrow, respectively. (b) Same as above when s_2' is decreased along (γ_2). (c) Displacements of $s_{1>}(s,s_2')$ for s_2' decreasing along (γ_3). For completeness add that (1) the distortion induced by $(m_2-m_3)^2$ in the cases (a) and (b) is absent if $m_2\leqslant m_3$; (2) the point $s_{1\pm}(s,(m_1-m_3)^2)$ lies below or above the decay region depending on whether $m_1>m_3$ or $m_1<m_3$. In the intermediate case $m_1=m_3$, this point is removed at infinity and the associated "bubble" starting from $s_{1\pm}(s,(m_1+m_3)^2)$ reduces to the curve H of Ref. 31; see also Fig. 8 in Ref. 3.

of $K(s_2',s,s_1)$, as (γ_3) does. After all these transformations, we thus get[38]

$$\frac{1}{\pi}\int_{(m_2+m_3)^2}^{\infty}\frac{ds_1'}{s_1'-s_1}\frac{N_1(s_1')}{k(s_1',s,m_1^2)}\int_{s_{2-}(s,s_1')}^{s_{2+}(s,s_1')}ds_2'\frac{\varphi_2(s,s_2')}{D_2(s_2')}$$
$$=\int_{-\infty}^{(\sqrt{s}-m_2)^2}(\gamma_1)\quad ds_2'\frac{\varphi_2(s,s_2')}{D_2(s_2')}\Delta^{(1)}(s_2',s,s_1)$$
$$+\int_{-\infty}^{(m_1-m_3)^2}(\gamma_2)\quad ds_2'\frac{\varphi_2(s,s_2')}{D_2(s_2')}\Delta^{(2)}(s_2',s,s_1)$$
$$+\int_{-\infty}^{0}(\gamma_3)\,ds_2'\frac{\varphi_2(s,s_2')}{D_2(s_2')}\Delta^{(3)}(s_2',s,s_1),\quad(3.6)$$

the integral along (γ_2) being absent if $m_3>m_1$.

The kernels $\Delta^{(i)}(s_2',s,s_1)$ are precisely specified below, but let us first make some remarks about their meaning. If we set

$$\varphi_{01}(s,s_1)\equiv0,\quad\varphi_3(s,s_3')\equiv0,\quad\varphi_2(s,s_2')\equiv\pi,$$
$$N_1(s_1')=1,\quad D_2(s_2')=1/(\bar{s}_2-s_2'-i\epsilon),\qquad(3.7)$$

on the right-hand side of Eq. (3.2), we get just the Feynman triangle graph (Fig. 9)[39] with $\bar{s}_2-i\epsilon$ equal to an internal mass squared [for definiteness we choose $\bar{s}_2\geq(\sqrt{s}-m_2)^2$]. The same transformations as above apply to this particular case. On the resulting equations, it clearly appears that

(i) the points $(\sqrt{s}-m_2)^2$, $(m_1-m_3)^2$, and 0 are singular for the triangle graph in the plane of the internal mass $\bar{s}_2$; and

(ii) the quantities $\pi^2\Delta_\tau^{(i)}(s_2',s,s_1)$—the same as $\pi^2\Delta^{(i)}(s_2',s,s_1)$ with $N_1(s_1')=1$—are nothing but the discontinuities of the triangle amplitude across the associated cuts $\bar{s}_2\leq(\sqrt{s}-m_2)^2$, $\bar{s}_2\leq(m_1-m_3)^2$, $\bar{s}_2\leq0$.

We may now write the expressions of the $\Delta^{(i)}(s_2',s,s_1)$. The formulas below are in fact valid only on a restricted part of the s_2' path of integration (γ_i); the expressions elsewhere along the (γ_i) must be obtained by analytic continuation. The symbols $s_{1<}$ and $s_{1>}$ mean the lowest and greatest solutions in s_1 of Eq. (3.3).

We begin by the case $m_1>m_3$. The paths (C_i) below are pictured in Fig. 8.

For s_2' on (γ_1) and in the neighborhood of $s_2'=(\sqrt{s}-m_2)^2$, the first kernel reads

$$\Delta^{(1)}(s_2',s,s_1)=\frac{1}{\pi}\int_{s_{1<}(s,s_2')}^{s_{1>}(s,s_2')}(C_1)\quad\frac{ds_1'}{s_1'-s_1}\frac{N_1(s_1')}{k(s_1',s,m_1^2)}\qquad(3.8)$$

with $k(s_1',s,m_1^2)=|k(s_1',s,m_1^2)|$ on (C_1).

[38] As $\Delta^{(1)}(s_2',s,s_1)$ is not singular at $s_2'=(m_1-m_3)^2$ and 0, the position of (γ_1) with respect to these points is irrelevant. The same holds for the relative position of (γ_2) and $s_2'=0$.

[39] R. Karplus, C. M. Sommerfield, and F. H. Wichmann, Phys. Rev. 111, 1187 (1958).

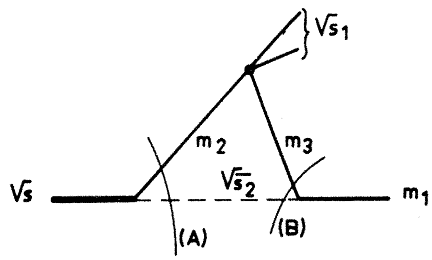

FIG. 9. Triangle Feynman graph with an internal mass $\sqrt{\bar{s}_2}$.

Similarly for s_2' on (γ_2) and in the neighborhood of $s_2'=(m_1-m_3)^2$

$$\Delta^{(2)}(s_2',s,s_1)=\frac{1}{\pi}\int_{s_{1<}(s,s_2')}^{s_{1>}(s,s_2')}(C_2)\quad\frac{ds_1'}{s_1'-s_1}\frac{N_1(s_1')}{k(s_1',s,m_1^2)}\qquad(3.9)$$

with $k(s_1',s,m_1^2)=|k(s_1',s,m_1^2)|$ on (C_2).

Finally, for s_2' on (γ_3) near 0, we have

$$\Delta^{(3)}(s_2',s,s_1)=\frac{-1}{\pi}\int_{s_{1>}(s,s_2')}^{\infty}(C_3)\quad\frac{ds_1'}{s_1'-s_1}\frac{N_1(s_1')}{k(s_1',s,m_1^2)}\qquad(3.10)$$

with $k(s_1',s,m_1^2)=-|k(s_1',s,m_1^2)|$ on (C_3).

From these integral representations, explicit evaluations of the $\pi^2\Delta_\tau^{(i)}(s_2',s,s_1)$ can be performed. The results are identical with the expressions given by Kacser.[36] We refer the reader to his work, in particular as concerns the analytic continuations in s_1. Let us, however, take note of the following properties:

(i) The expressions for the $\Delta^{(i)}(s_2',s,s_1)$ given by Eqs. (3.8)–(3.10) with $N_1=1$ clearly are real; they remain real on the s_2' and s_1 axes as long as no singularity $s_1=s_{1\pm}(s,s_2')$ is reached. This condition can only be fulfilled on restricted parts of the s_2' and s_1 real axes, which appear on Fig. 8.

(ii) The quantities $\pi s\Delta_\tau^{(1)}(s_2',s,s_1)/k(s_2',s,m_2^2)$ and $\pi s\Delta_\tau^{(2)}(s_2',s,s_1)/k(s_2',m_1^2,m_3^2)$ can be identified with the $J=0$ projections of the processes of Figs. 10(a) and 10(b) in the s and m_1^2 center-of-mass systems, respectively.[6,35] This indeed follows from Cutkosky's rules[28] applied to the graph of Fig. 9—cuts (A) and (B). As to $\Delta_\tau^{(3)}(s_2',s,s_1)$, it differs from a linear combination of $\Delta_\tau^{(1)}(s_2',s,s_1)$ and $\Delta_\tau^{(2)}(s_2',s,s_1)$ by a logarithmic term related to the $J=0$ projection in the s_1 center-of-mass system of the process of Fig. 9(c) [see Ref. 36, Eq. (4.8)].

The case $m_1<m_3$ can be studied in two ways. First, the same procedure as above applies without change, but here one has to take care of the different position of the singularity $s_2'=(m_1-m_3)^2$ of $K(s_2',s,s_1)$ with respect to the path (Γ). This difference may be understood within the context of the second way based on analytic continuation with respect to m_1. When m_1 decreases from values above m_3 to values below, $s_2'=(m_1-m_3)^2$ follows the dashed path of Fig. 4(b) and thus only the

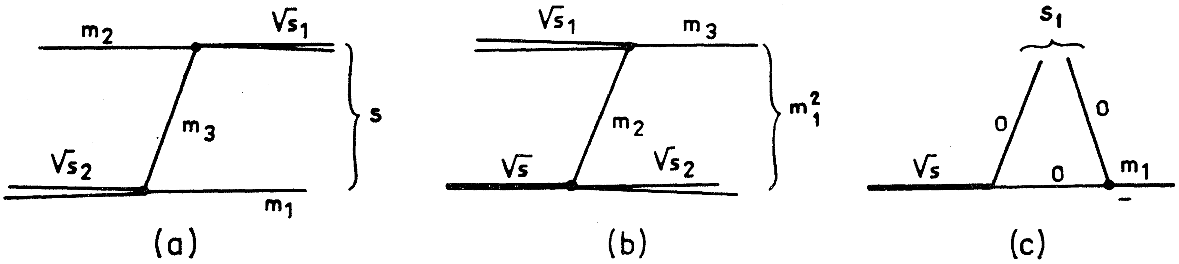

FIG. 10. The three graphs related to the kernels $\Delta^{(i)}(s_2',s,s_1)$ of the SVR Eq. (3.6).

two distortions of (Γ) shown in Fig. 7(b) have to be considered in this case; consequently, only two kernels $\Delta^{(i)}(s_2',s,s_1)$ [or $\Delta_\tau^{(i)}(s_2',s,s_1)$] have to be determined. While the first one is similar to $\Delta^{(1)}(s_2',s,s_1)$, the second one—say $\Delta'^{(3)}(s_2',s,s_1)$—reads

$$\Delta'^{(3)}(s_2',s,s_1)=-\frac{1}{\pi}\int_{-\infty}^{s_{1<}(s,s_2')}_{(C_3')}\frac{ds_1'}{s_1'-s_1}\frac{N_1(s_1')}{k(s_1',s,m_1^2)}\,,\tag{3.11}$$

at least for s_2' in the neighborhood of 0; (C_3') is shown in Fig. 11 and $k(s_1',s,m_1^2)=|k(s_1',s,m_1^2)|$ on (C_3'). As expected, Eq. (3.11) is the analytic continuation in m_1 and s_2' of the sum of $\Delta^{(2)}(s_2',s,s_1)$ and $\Delta^{(3)}(s_2',s,s_1)$ given by Eqs. (3.9) and (3.10).

The discussion about the relative values of m_1 and m_3 suggests interesting comments if one notes the symmetric roles played by m_1 and $\sqrt{s}$. An analytic continuation in s from $\sqrt{s}>m_2$ to $\sqrt{s}<m_2$ leads to a reduction in the number of kernels by one unit, as it does when m_1 is decreased from $m_1>m_3$ to $m_1<m_3$. Hence, for $m_1<m_3$ and $\sqrt{s}<m_2$, the SVR of KT equations, and thus also the triangle amplitude,[40] take the form of only one integral from $-\infty$ to zero with $-K(s_2',s,s_1)$ [Eq. (3.5)] as kernel [set $N_1(s_1')=1$ for the triangle]; the $\tilde{s}_1(s,s_2')$ of Eq. (3.5) stands here for $s_{1>}(s,s_2')$. [For this case, note that $-K(s_2',s,s_1)$ is then the sum of the three expressions of Eqs. (3.8)–(3.10) continued to $s_2'<0$ and small m_1 and s values.]

It is worth mentioning that the SVR obtained in this case provides a linear and bootstrapping form for nothing but the Cini-Fubini approximation for the scattering reaction of Fig. 1(b). In the case of the triangle graph, it is interesting to notice that for such small values of m_1 and $\sqrt{s}$, the points $\bar{s}_2=(m_1-m_3)^2$ and $\bar{s}_2=(\sqrt{s}-m_2)^2$ are singular only on an "unphysical" $\bar{s}_2$ sheet, reached from the value we have till now considered by looping around 0 (the cut delimiting this sheet runs from $-\infty$ to 0); when m_1 and $\sqrt{s}$ are increased, these singularities curl around zero and appear on the "physical" $\bar{s}_2$ sheet. This leads to the analytic structure considered elsewhere.[35,36] To conclude these remarks, note that all these properties of the triangle graph with respect to an internal mass are deduced from Eq. (3.2) and thus from two-body unitarity in the s_1 channel, i.e., from properties with respect to the external mass $\sqrt{s_1}$.

IV. THREE-BODY UNITARITY AND KT EQUATIONS IN THE CASE $J=0$, $l=0$

For simplicity, we restrict ourselves to the equal-mass symmetric case ($m_i=1$, $i=1, 2, 3$) in this section. Then, as discussed in Appendix B, the SVR corresponding to Eq. (3.2) can be reduced to a single integral equation:

$$\varphi(s,s_1)=\varphi_0(s,s_1)+2\int_{-\infty}^{(\sqrt{s}-1)^2}ds_2'\frac{\varphi(s,s_2')}{D(s_2')}\times\Delta^{(1)}(s_2',s,s_1)+2\int_{-\infty}^{0}ds_2'\frac{\varphi(s,s_2')}{D(s_2')}\times[\Delta^{(2)}(s_2',s,s_1)+\Delta^{(3)}(s_2',s,s_1)],\tag{4.1}$$

where the kernels $\Delta^{(1)}(s_2',s,s_1)$ and $\Delta^{(2)}(s_2',s,s_1)$

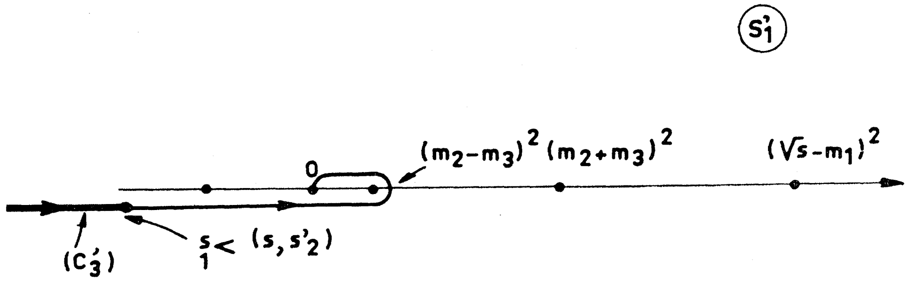

FIG. 11. Path (C_3') involved in Eq. (3.11) (case $m_1<m_3$); the displacement of $s_{1<}(s,s_2')$ when s_2' is decreased from 0 to $-\infty$ along the integration contour is indicated by an arrow [the loop around $s_1'=(m_2-m_3)^2$ is absent for $m_2<m_3$].

[40] Such a representation of the triangle graph can indeed be deduced, as one can verify, from its Feynman expression which involves the product of the three propagators associated with the three internal masses. We have first to perform all the integrations over the squared internal momenta except that associated with $\bar{s}_2$, and then to take the remaining path of integration along $(-\infty, 0)$.

$+\Delta^{(3)}(s_2',s,s_1)$ are given by Eqs. (3.8) and (3.11), adapted to the case $m_i=1$ ($i=1, 2, 3$).

The three-body and the over-all (i.e., the sum of the two-body and the three-body) discontinuities of $\varphi(s,s_1)$ can be determined from Eq. (4.1)[5] or from the associated original equation[4] (3.2) with $m_i=1$. Both methods lead to two possible equivalent forms for these discontinuities: (i) In the first one, the energy integration contour is forced to pass through a small (vanishing) gap introduced between the ends of the curve ($\mathcal{C}$) defined in Appendix A (see also Fig. 6 of Ref. 5). As a consequence the associated angular integration is not distorted and covers the usually considered Dalitz-plot phase-space region. (ii) In the second one no constraint affects the energy contour and the associated angular integration path is distorted by the threshold singularities of the crossed subchannels.

The first and second forms of the discontinuities involve $3\to3$ amplitudes which are denoted in Ref. 5 by $\bar{\psi}(s_2',s,s_1)$ and $\psi(s_2',s,s_1)$, respectively.

Clearly, the presence or the absence of a constraint on the energy path is important when one wants to continue these discontinuities in the s plane. Nevertheless, it is worth mentioning at this stage that only the first alternative seems to be retainable. On the one hand, as briefly discussed in Appendix A, it is only in the first form that the phase space integration path belongs entirely to the domain of convergence of the two-body subenergy partial-wave expansions of a production or decay amplitude. On the other hand, a careful investigation of the second form shows that on the energy integration path, $\psi(s_2',s,s_1)$ is indeed the sum of $\bar{\psi}(s_2',s,s_1)$ and of another function, which does not vanish only over a finite range.[41] Thus, to put together these two functions under the name of $\psi(s_2',s,s_1)$ we have to introduce a step, or Heaviside, function. Hence we have been unable to consider $\psi(s_2',s,s_1)$ as an analytic function of the initial subenergy variable s_2'. Besides, $\psi(s_2',s,s_1)$ does not possess the normal two-body singularity $s_2'=4$. As a consequence, we have not succeeded in writing for $\psi(s_2',s,s_1)$ relations similar to three-body discontinuity or unitarity relations.

Thus, the expression "$3\to3$ amplitude" refers in the following *only* to $\bar{\psi}(s_2',s,s_1)$ and other related amplitudes as the ${}_\lambda\bar{\psi}_i(s_\lambda',s,s_i)$ of Appendix B. The analytic properties of these amplitudes are determined from their integral representations which are themselves unambiguously defined from the derivation of the three-body discontinuity of $\varphi(s,s_1)$. Their study is rather similar to that of $\varphi(s,s_1)$: Eqs. (B10)–(B12) all have the same kernels as Eq. (4.1) but different inhomogeneous terms.[42] Complications nevertheless arise because of the singularities of $\Delta^{(1)}$—the so-called one-particle exchange (OPE) singularities—and the fact that we have one variable more, the initial subenergies (s_2' or s_λ'). In fact, the analytic structure generated by the OPE singularities—this relates initial and final subenergy states—needs to be considered only in the lowest iterations of the integral equations; in the higher-order terms, this structure is removed far from the physical boundary, so that the properties of the $3\to3$ amplitudes appear to be simply those of two disconnected production amplitudes for initial and final states. This expresses in terms of S-matrix language the fact that in the high-order terms, the properties of the initial and final states become rather independent from each other (compound model).

The various discontinuities of $\bar{\psi}(s_2',s,s_1)$ and of the ${}_\lambda\bar{\psi}_i(s_\lambda',s,s_i)$ can be calculated.[4] They have the same form as the discontinuity relations which can be derived, as done in Ref. 26, from three-body unitarity relations satsified by $J=0$ $3\to3$ amplitudes. For our purpose (three-body unitarity), it is convenient to consider the full $3\to3$ amplitude and its over-all discontinuity in the decay region—i.e., the sum of all its discontinuities across the various cuts present in this region. By the full $3\to3$ amplitude, we mean

$$
{}_3\mathcal{R}_3(\{s_\lambda'\},s,\{s_i\})={}_3\mathcal{R}_3{}^d(\{s_\lambda'\},s,\{s_i\})+{}_3\mathcal{R}_3{}^c(\{s_\lambda'\},s,\{s_i\}), \quad (4.2)
$$

where

$$
{}_3\mathcal{R}_3{}^c(\{s_\lambda'\},s,\{s_i\})=\sum_{\lambda,i}{}_\lambda\bar{\Psi}_i(s_\lambda',s,s_i), \quad (4.3)
$$

$$
{}_\lambda\bar{\Psi}_i(s_\lambda',s,s_i)=\frac{s}{k(s_\lambda',s,m_\lambda^2)}\frac{N(s_\lambda')}{D(s_\lambda')}{}_\lambda\bar{\psi}_i(s_\lambda',s,s_i)\frac{1}{D(s_i)}, \quad (4.4)
$$

and ${}_3\mathcal{R}_3{}^d(\{s_\lambda'\},s,\{s_i\})$ is the sum of the three disconnected $3\to3$ amplitudes. It is convenient to consider simultaneously the over-all discontinuity of the decay amplitude $\mathcal{R}(s,\{s_i\})$ [Eq. (3.1)]. This gives the system[43]

$$
\mathcal{R}(s_+,\{s_{i+}\})-\mathcal{R}(s_-,\{s_{i-}\})=\sum_3\mathcal{R}(s_\mp,\{s_\mp''\})\,{}_3\mathcal{R}_3(\{s_\pm''\},s_{\pm(\mp)},\{s_{i\pm}\}), \quad (4.5)
$$

$$
\begin{aligned}
{}_3\mathcal{R}_3(\{s_{\lambda+}'\},s_{+(-)},\{s_{i+}\})-{}_3\mathcal{R}_3(\{s_{\lambda-}'\},s_{-(+)},\{s_{i-}\})
&=\sum_3 {}_3\mathcal{R}_3(\{s_{\lambda\mp}\},s_{\mp(\pm)},\{s_\mp''\})\\
&\quad\times{}_3\mathcal{R}_3(\{s_\pm''\},s_{\pm(\mp)},\{s_{i\pm}\}), \quad (4.6)
\end{aligned}
$$

where $\sum_3$ stands formally for the usual three-body phase-space integration, with a constraint on the energy integration path as discussed above. The indices $\pm$ refer to the position of the variables with respect to the

[41] A closely related feature is that one cannot invert the order of integration in Eq. (A5) of Ref. 5 without splitting the energy integration path into two parts, each of which will involve in the end a different analytic function. So the integral representation given for ψ in the equation (A5) of Ref. 5 is only formally valid.

[42] Notice that the second formula (A5) of Ref. 5 contains an irrelevant star.

[43] Notice that the two forms of Eq. (4.6) can indeed be derived from the comparison of the two forms of Eq. (4.5) (see Ref. 4).

associated cuts (the second indices of s refer to the cuts induced by OPE singularities[4,26]).

From these results and thanks to the Schwarz reflection principle, it is clear that the above system expresses nothing but three-body unitarity, if the following two additional conditions are satisfied: (1) The amplitudes must be real for real values of the subenergy and total-energy variables just below the corresponding thresholds. (2) The $3\to 3$ amplitude must have good symmetry properties in the exchange of the initial and final subenergy variables.

The condition of reality (1) is discussed in Appendix C. The KT amplitudes are shown to be real if the input function φ_0 [the inhomogeneous term of Eq. (4.1)] is itself real. The condition of symmetry (2), as already stated in Sec. II, is fulfilled by none of the $3\to 3$ amplitudes involved in the model [there are extra cuts above $(\sqrt{s}+1)^2$ in the initial but not in the final subenergy variables. Besides, the singularities due to the N functions lie on different sheets.] To remain self-consistent it is worthwhile to search for the origin of this drawback within the KT model itself. Thanks to the particular aspect of the equations that relate the other $3\to 3$ amplitudes to $\psi(s_2',s,s_1)$ (see Appendix B) it is sufficient for this purpose to look at the basic elements, i.e., the $\Delta^{(i)}(s_2',s,s_1)$, which enter into the integral equation (B10). For $N\neq$constant, none of the kernels shows a particular symmetry. For $N=$constant, on the contrary (*until further notice we shall restrict ourselves to this case*), $\Delta^{(1)}(s_2',s,s_1)$ is a polar kernel, i.e., becomes symmetric with respect to s_2' and s_1 if it is multiplied by functions depending either on s_2' *or* s_1; indeed, the product of $\Delta^{(1)}(s_2',s,s_1)$ by $\pi s/k(s_2',s,m_2^2)$ is just the (symmetric) $J=0$ projection of a process like that in Fig. 10(a). No similar property arises for $\Delta^{(2)}$ or $\Delta^{(3)}$. From these results and from Appendix B it is easy to verify that either neglecting or "symmetrizing" $\Delta^{(2)}+\Delta^{(3)}$ in Eq. (B10) gives rise in the end to amplitudes ${}_\lambda\Psi_i(s_\lambda',s,s_i)$ [Eq. (4.4)] and ${}_3\mathfrak{R}_3(\{s_\lambda'\},s,\{s_i\})$ [Eq. (4.2)] symmetric with respect to the initial and final subenergy variables. But these new amplitudes do possess singularities and cuts absent in the original KT ones. One must therefore proceed in such a way that these new singularities and cuts are as far as possible from the physical region. One has also to take into account that the $\Delta^{(1)}$ part of Eq. (4.1) or Eq. (B10) is the more important as concerns the existence and the expression of the three-body discontinuity[5] (and also of the two-body discontinuities in the decay region). This leads us to examine equations of the type

$$\varphi_\alpha(s,s_1)=\varphi_0(s,s_1)+2\int_\alpha^{(\sqrt{s}-1)^2} ds_2'\Delta^{(1)}(s_2',s,s_1)\frac{\varphi_\alpha(s,s_2')}{D(s_2')}, \quad (4.7)$$

where the most appropriate value of α must still be found.

Obviously, such approaches do not appear to be satisfactory in the scattering region ($s\leq 1$), since then the form of Eq. (4.7) differs from the SVR which has been presented in Sec. III; the additional cuts of $\varphi_\alpha(s,s_1)$, which one can easily find by inverting once again the order of integration in Eq. (4.7), are then very close to or even cover the scattering region. These approaches are thus specific to the decay region ($s\geq 9$) where the three-body and over-all discontinuities of $\varphi_\alpha(s,s_1)$ can be evaluated from Eq. (4.7) by the same procedure as in Ref. 5 or Appendix B; the results are quite similar and lead to $3\to 3$ amplitudes Ψ_α which are in the end totally generated by OPE terms like[44]

$$\tilde{\Delta}_1(s_2',s,s_1)=\frac{1}{D(s_2')}\frac{s}{k(s_2',s,m_2^2)}\Delta^{(1)}(s_2',s,s_1)\frac{1}{D(s_1)}$$

(see Fig. 12) and which are thus symmetric by construction. But we shall have fully obtained three-body unitarity only if simultaneously the condition of reality holds. It is for this reason that we now examine different values of α separately.

Equation (4.7) with $\alpha=-\infty$ [like the KT equation (4.1) itself] reduces to one like the Skornyakov–Ter-Martirosian (STM)[45] equation in the nonrelativistic limit[6,46]; that is its main interest. On the other hand, if the convergence of this equation can be guaranteed, we can advantageously turn the path of integration over $[(\sqrt{s}-1)^2,+\infty]$ without distortion for physical values of s_1 greater than $1+\sqrt{s}$; this simply avoids the difficulties associated with the moving logarithmic singularities of the kernels.[15,47] Unfortunately, the convergence of Eq. (4.7) is poorer than that of Eq. (4.1). Given a value of s_1, $\Delta^{(1)}(s_2',s,s_1)$ behaves like a

[44] Notice that the amplitudes $1/D$ associated with the two bubbles of Fig. (12) are factorized, which is equivalent to taking the bubble functions just at the pole $t'=1$. This follows, as one can verify, from the elastic approximation in each subenergy channel; greater t' singularities in the bubble functions (among others, three-body singularities are given by the KT model) would have led to inelastic singularities in the subchannels. Notice also that since $\Delta^{(1)}$ is the $J=0$ projection of a $3\to 3$ OPE process, it is easy to write Ψ_α as the $J=0$ projection of a $3\to 3$ amplitude depending on the momentum transfer invariant t'. The integral equation satisfied by this unprojected amplitude looks rather like a two-body Bethe-Salpeter equation. For the study of equations of similar types, see for instance, L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962); (this work contains further references).

[45] G. V. Skornyakov and K. A. Ter-Martirosian, Zh. Eksperim. i Teor. Fiz. **31**, 775 (1956) [English transl.: Soviet Phys.—JETP **4**, 648 (1957)]. See also G. Flamand, in *Cargese Lectures in Theoretical Physics*, edited by F. Lurçat (Gordon and Breach Science Publishers, Inc., New York, 1967).

[46] By nonrelativistic limit of KT equations, we mean equations reconstituted from the nonrelativistic limit of the kinematical and phase-space functions involved in the two-body discontinuities. Our method of inverting the integrals may as well be applied to the nonrelativistic equations and thus provides a variant of the method used by Anisovitch (see Ref. 24).

[47] I. J. R. Aitchison, Cambridge Report, 1966 (unpublished). This work contains further references.

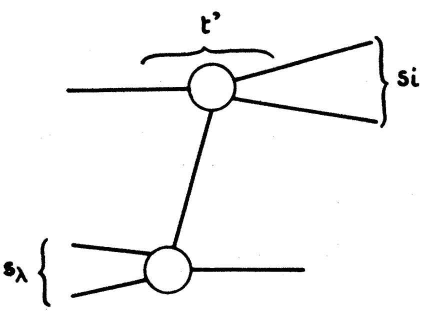

FIG. 12. One-particle exchange process involved in the $3\to3$ amplitudes.

constant when $s_2'\to-\infty$, while

$$-K(s_2',s,s_1)=\Delta^{(1)}(s_2',s,s_1)+\Delta^{(2)}(s_2',s,s_1)+\Delta^{(3)}(s_2',s,s_1)$$

behaves like $1/s_2'$. Hence the possibility of turning the contours in Eq. (4.7) can be investigated only with rather academic forms of the function $D(s_2')$. Furthermore, first, $\varphi_{\alpha=-\infty}(s,s_1)$ has a supplementary cut $s_1\leq0$; secondly, $\varphi_{\alpha=-\infty}$ is not real for $s_1\lesssim4$ (see footnote 48) and $\sqrt{s}\lesssim3$, even if $\varphi_0(s,s_1)$ is real. This is clear from the iterative expansion: the first nontrivial term has a cut along $s_1\leq0$, and thus an imaginary part, so that the second one which involves it all along $s_1\leq0$ is imaginary, and so on. A similar feature occurs for $\bar{\psi}_{\alpha=-\infty}(s_2',s,s_1)$. Correspondingly the quantity $\mathrm{Tr}[\Delta^{(1)}/D]$—which is important in three-body calculations[49]—is not real for $\sqrt{s}\lesssim3$, whereas in Eq. (4.1), the trace of the complete kernel was. This comes from the fact that, for $s'\leq1-\sqrt{s}$, $\Delta^{(2)}(s',s,s')+\Delta^{(3)}(s',s,s')$ [Eq. (3.11)] and $\Delta^{(1)}(s',s,s')$ [Eq. (3.8)] are both imaginary while the sum is real. Thence $\alpha=-\infty$ leads to various difficulties. It is important to note that these drawbacks disappear in the nonrelativistic limit since then the cut $s_1\leq0$ is absent.[50]

To return to the relativistic case, we have now to see whether the same difficulties arise or not in Eq. (4.7) for finite values of α. Equation (4.1) is then of Fredholm type (no matter what D is), and its resolvent is nothing but $\bar{\psi}_\alpha(s_2',s,s_1)/D(s_2')$.[5] Hence

$$\varphi_\alpha(s,s_1)=\varphi_0(s,s_1)+2\int_\alpha^{(\sqrt{s}-1)^2}ds_2'\frac{\bar{\psi}_\alpha(s_2',s,s_1)}{D(s_2')}\varphi_0(s,s_2')$$

and

$$\bar{\psi}_\alpha(s_2',s,s_1)=\mathfrak{N}(s_2',s,s_1)/\mathfrak{D}(s).$$

The eventual poles of $\bar{\psi}_\alpha(s_2',s,s_1)$ and $\varphi_\alpha(s,s_1)$, corresponding to the zeros of the Fredholm determinant $\mathfrak{D}(s)$, have to be interpreted as three-body bound or resonant states.

But only restricted choices of finite α can lead to an amplitude $\varphi_\alpha(s,s_1)$ real for $\sqrt{s}\lesssim3$, $s_1\lesssim4$ and such that $\mathrm{Tr}[\Delta^{(1)}/D]$ is also real for $\sqrt{s}\lesssim3$. This can be shown as in the preceding case $\alpha=-\infty$ by looking at the supplementary s_1 cuts of $\varphi_\alpha(s,s_1)$ and at their positions with respect to the integration path $(\alpha,(\sqrt{s}-1)^2)$. As one can verify, it is only for $-2<\alpha<4$ that these s_1 cuts do not encounter the range of integration. On the other hand, the cutoff at α induces additional s cuts [especially the cuts associated with the singularities $s=2\alpha+1$ and $s=(\alpha-1)^2$] in $\mathrm{Tr}[\Delta^{(1)}/D]$ and it is just over the same range of α values that there is a nonvanishing gap below $s\leq9$ on which $\mathrm{Tr}[\Delta^{(1)}/D]$ is real. The largest gap extends over $1\leq s\leq9$ and occurs for $\alpha=0$. For this value precisely (at least in the equal-mass case we consider) $\varphi_\alpha(s,s_1)$ just possesses an additional cut over $s_1\geq(\sqrt{s}+1)^2$.[51] In agreement with the remarks of Sec. II about the possible relations between the symmetry of the KT equations and high-subenergy states, we are tempted to consider that this cut simulates high-energy inelastic effects. One can even try, *a posteriori*—i.e., once the amplitude is calculated—to compare the contributions of this "inelastic" cut $[s_1\geq(\sqrt{s}+1)^2]$ and of the elastic one $(s_1\geq4)$; the smaller the first will be, the better the elastic approximation will hold. However, the interest of such considerations is rather limited; the solution of Eq. (4.7) $(\alpha=0)$ is stable with respect to the asymptotic behavior of D, while the contributions of each of the preceding cuts may indeed diverge.

We have thus shown that it is possible to conveniently truncate KT equations and get amplitudes which, as one can check, satisfy three-body unitarity in the decay region; what remains to be done is to consider the KT amplitude Eq. (4.1) itself. Remember that the difference between the two is essentially related to the nonsymmetric part $(-\infty,0)$ of Eq. (4.1). Hence in the calculation of $\bar{\psi}$, as long as the contribution associated with this part is small enough compared to that of the symmetric one $[0,(\sqrt{s}-1)^2]$, we obtain an amplitude nearly symmetric and nearly satisfying three-body unitarity.

It is reasonable to think that the above requirement is fulfilled when the two-body resonance poles in $M=1/D$ sufficiently enhance the contribution of the integration part covering the decay region $[4,(\sqrt{s}-1)^2]$. We have also reason to believe that, under the same conditions, our preceding results, although no longer valid, apply nearly as well in the case of nonzero-range two-body interactions $N\neq$constant; indeed the singularities of $N(s_1')$ lie only on $s_1'\leq0$, so that $N(s_1')$ may be smoothly varying in the more important (but limited) part of the integration, and in the end $\Delta^{(1)}(s_2',s,s_1)$

[48] By $x\lesssim y$ we mean x in the neighborhood of, but less than, y.

[49] J. L. Basdevant and R. L. Omnès, Phys. Rev. Letters **17**, 775 (1966), (further references are given in this work); J. L. Basdevant, thesis, Université de Strasbourg, France (unpublished).

[50] Remember that STM equations provide a particular case of nonrelativistic Faddeev equations {see Ref. 45 and R. A. Minlos and L. D. Faddeev, Dokl. Akad. Nauk. SSSR **141**, 1335 (1961) [English transl.: Soviet Phys.—Doklady **6**, 1072 (1962)]}, where three-body unitarity is satisfied.

[51] This point is indeed generally singular in the true $J=0$ projection of a $3\to3$ amplitude; it then corresponds to a non-Landau (see Ref. 54) singularity induced by cuts in the transfer channel.

may be as nearly symmetric. On the other hand, the presence of the factor $N(s_1')$—just as the form factors in Faddeev equations—ensures the equations a better convergence[52] and thus can decrease the importance of the part $(-\infty, 0)$. Note that we would have also a better convergence if instead of Eq. (3.8) we started from its subtracted form at a point $\hat{s}_1$ [neglecting as usual the s dependence of $\varphi(s,\hat{s}_1)$ and thus considering this function as a part of a new $\varphi_0(s,s_1)$]; but still in this case the symmetry of the $\Delta^{(1)}$ part would be broken [unless we do the subtraction only on the nonsymmetric part $(-\infty, 0)$].

V. CONCLUSION

In conclusion, the KT model essentially offers us different possibilities for treating the three-body relativistic problem. One can work (1) either with the complete KT equations and thus with the elastic approximation in each subchannel. Then, under well-defined conditions (N nearly constant, dominance of two-body resonances) we are dealing with amplitudes nearly satisfying three-body unitarity. Or (2), one can work with truncated KT equations and thus introduce fictitious subenergy cuts besides the elastic one. Then we know that for $N=$constant, we are dealing with amplitudes satisfying exact three-body unitary, at least in the decay region. Recall that to take $N=$constant is simply equivalent to neglecting the effects of the two-body forces compared to those of two-body unitarity (indeed, we implicitly assume that the two-body forces have generated the resonant and bound states of D). It is reasonable to think[49] that this approximation is valid for intermediate values of the energy $\sqrt{s}$—not too low, because in the KT equations the effects of N or of the "nonsymmetric" part can then become important; and not too large either, because the effects of more-than-two-body subenergy states need then to be considered.

Nevertheless, it seems *a priori* difficult to prefer one of these alternatives to the other and to easily understand the effects of N functions without carrying out numerical calculations (research of three-body resonances for instance) on definite examples. For this purpose, in a forthcoming paper we consider the case $J=1$, $l=1$ which is often encountered in problems of practical interest. In this study we encounter new complications due to the divergence of the equations and the kinematical singularities of helicity amplitudes; nevertheless, in the end, the preceding conclusions mainly hold.

ACKNOWLEDGMENTS

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APPENDIX A: CONVERGENCE OF THE SUBENERGY PARTIAL-WAVE EXPANSIONS OF A KT AMPLITUDE

We consider the decay amplitude $\mathcal{R}(s,\{s_i\})$ pictured in Fig. 1(a) and its partial-wave expansion in the s_1 subenergy center-of-mass system—the associated angular variable z_1 is linearly related to the invariants s_2 and s_3. For simplicity, we assume that all particles are spinless and all $m_i=1$.

As is well known, if $\mathcal{R}(s,\{s_i\})$, considered as a function of z_1, is analytic in a domain containing the segment $(-1, +1)$, its partial-wave expansion converges in an ellipse whose foci are -1 and $+1$, and is limited by the nearest z_1 singularities. Correspondingly, in the s_2 plane, given any s_1 value, we have an ellipse of convergence $E(s_1)$, whose foci are now $s_{2\pm}(s,s_1)$ [Eq. (3.3)], limited by the nearest s_2 singularities (similar considerations hold in the s_3 plane). Indeed, as noted elsewhere,[3–5,9,11,31] for s_1 in the physical decay region, the precise positions of the $s_{2\pm}(s,s_1)$ must be determined by analytic continuation from regions where there are no ambiguities in defining them. This leads us to consider the crossed process Fig. 1(b) corresponding to small values of s, where the preceding partial-wave expansion is nothing but the usual Legendre expansion of a $2\rightarrow 2$ amplitude. Then, given any $s_1\geq 4$ and s small, the segment $\sigma=(s_{2-}(s,s_1),s_{2+}(s,s_1))$ lies on the real negative axis of the s_2 physical sheet, where the amplitude has no singularity. (We exclude the possibility of bound states without great loss of generality.)

When s is increased, on the one hand, the point $s_{2+}(s,s_1)$ [or $s_{2-}(s,s_1)$] describes a curve in the complex plane, which jumps above (or below) $s_2=4$; on the other hand, new singularities, especially generated by resonant poles, appear on the physical s_2 sheet; one has thus to see whether all these singularities can encounter σ or not. It happens that the last type of singularities appear only in the vicinity of the physical boundary if they correspond to physically possible processes where all the intermediate states may be real.[53] For this reason we assume a configuration of masses such that only the threshold singularity $s_2=4$ has to be considered. In this case, denote by $(\mathcal{C})$ the s_1 curve on which σ meets $s_2=4$. It is easy to show that $(\mathcal{C})$ is a curve beginning and ending at $s_1=\frac{1}{2}(s-1)$ and

[52] $N(s_1')$ introduces also, in a certain sense, a part of inelasticity in each subenergy channel; $N(s_1')$ is indeed related in the box diagram (Fig. 2) to singularities in the channel s_2'' of M, which leads to (inelastic) singularities in the channel s_2. It is thus convenient to consider the case $N(s_1')=$constant as the "purely elastic" one.

[53] J. B. Bronzan, Phys. Rev. 134, B687 (1964); C. Kacser, Phys. Letters 12, 269 (1964); S. Coleman and R. E. Norton, Nuovo Cimento 38, 438 (1965).

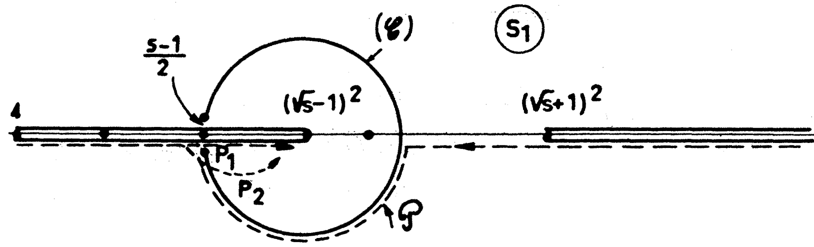

FIG. 13. Different elements involved in the discussion of Appendix A; the path P_1 (P_2) corresponds to continuation on the "principal" ("nonprincipal") sheet.

enclosing $s_1=(\sqrt{s}-1)^2$; thus, when s is curled around (and above) 9 up to a value on the upper side of the cut $s\geq 9$, ($\mathcal{C}$) follows a similar path around $s_1=4$ and "falls" on the upper side of the real axis $s_1\geq 4$.[3]

Hence, while it is possible for small s to define ellipses $E(s_1)$ along all $s_1\geq 4$, after s has been increased this becomes possible only along a path $\mathcal{P}$ (Fig. 13) avoiding ($\mathcal{C}$) from below. By following this path from the scattering region $[s_1\geq(\sqrt{s}+1)^2]$ till the decay region, we arrive at contours σ lying on the upper side of the cut $s_2\geq 4$ for $4\leq s_1\leq\frac{1}{2}(s-1)$. Obviously the ellipse $E(s_1)$ then covers a domain larger than or equal to σ. But to continue to $s_1\geq\frac{1}{2}(s-1)$ we have two possible paths: (1) One goes through a little gap we can introduce by a limiting procedure in the masses at the point $\frac{1}{2}(s-1)$, so that σ does not curl around $s_2=4$ and remains rectilinear. $E(s_1)$ may then always be defined. (2) Another cuts ($\mathcal{C}$), so that σ is distorted by $s_2=4$ (see, for instance, Ref. 31); for $s_1=\frac{1}{2}(s-1)$, $E(s_1)$ is just the segment σ itself and for s_1 above this value, $E(s_1)$ becomes meaningless, so that our partial-wave expansion nowhere converges [indeed it is possible to define another convergent partial-wave expansion in this region,[3,9] but its sum differs from the analytic continuation of the amplitude of Fig. 1(b)].

These two situations are those encountered in Sec. IV in the two possible forms (i) and (ii) of the three-body discontinuity of a KT amplitude. Clearly in the first one, the path of the angular integration which is identical to σ always belongs to the domain of convergence of the partial-wave expansion of the decay KT amplitude (and, of course, of the 3-3 KT amplitude minus the lowest-order terms which possess OPE singularities in the physical region).

Finally, note that the two above analytic continuations—through or avoiding $\frac{1}{2}(s-1)$—correspond to the distinction (see Hwa, Ref. 11) between continuation on the "principal" and the "nonprincipal" sheets of a two-body discontinuity: ($\mathcal{C}$) is the "natural" cut which separates them. We can therefore reformulate the distinction between the two forms of the three-body discontinuity of the KT amplitude by saying that:

(1) In the first form, the two-body subenergy discontinuities are always involved on their principal sheet.

(2) On the contrary, in the second form, the part $[\frac{1}{2}(s-1),(\sqrt{s}-1)^2]$ of the energy path belongs to a nonprincipal sheet, where the subenergy discontinuities are indeed singular for $s_1=(\sqrt{s}-1)^2$ (non-Landau singularity[54]).

APPENDIX B: REDUCTION OF THE SYSTEM OF KT EQUATIONS IN THE EQUAL-MASS CASE

In the unequal-mass case the SVR associated with Eq. (3.2) may be written in matrix form as

$$(\varphi)=(\varphi_0)+(\mathbf{K})(\varphi/D),\tag{B1}$$

where (φ/D) and (φ_0) are column matrices of elements $\varphi_i(s,s_i)/D_i(s_i)=R_i(s,s_i)$ and $\varphi_{0i}(s,s_i)$, respectively. $(\mathbf{K})$ is a 3×3 matrix operator such that the ith element of the product $(\mathbf{K})(R)$ is

$$\{(\mathbf{K})(R)\}_i=\sum_{j\neq i}\int ds_j'\mathbf{K}(s_j',s,s_i)R_j(s,s_j')\tag{B2}$$

with i, $j=1$, 2, 3, and

$$\begin{aligned}\mathbf{K}(s_j',s,s_i)=&\,\theta[(\sqrt{s}-m_j)^2-s_j']\Delta^{(1)}(s_j',s,s_i)\\&+\theta(s_j')\theta(m_i-m_k)\theta[(m_i-m_k)^2-s_j']\Delta^{(2)}(s_j',s,s_i)\\&+\theta(-s_j')[\Delta^{(2)}(s_j',s,s_i)+\Delta^{(3)}(s_j',s,s_i)]\end{aligned}$$

$k\neq i$, j; $\theta(x)=0$ if $x\leq 0$, 1 if $x>0$. The $\Delta^{(i)}(s_j',s,s_i)$ are given by Eqs. (3.8)–(3.10).

The three-body discontinuity of (φ)—say $([\varphi]_3)$—may be evaluated as in Ref. 5. Two possible equivalent forms can be obtained; we restrict ourselves to the form (i) of Sec. IV. We first get

$$([\varphi]_3)=(\square^{(1)})([R]_2)+(\mathbf{K})([\varphi]_3/D)\tag{B3}$$

[see Eq. (12) in Ref. 5], where $([R]_2)$ is a column matrix whose elements are the two-body s_i discontinuities of the $R_i(s,s_i)$'s (and then of the $\mathcal{R}$'s) taken on their "principal" sheet (cf. Appendix A), and $(\square^{(1)})$ is a 3×3 matrix operator such that

$$\{(\square^{(1)})([R]_2)\}_i=\sum_{j\neq i}\int_{s_j^0}^{(\sqrt{s}-m_j)^2}ds_j'\Delta^{(1)}(s_j',s,s_i)\times[R_j(s,s_j')]_2,\tag{B4}$$

where s_j^0 stands for the normal two-body singularity of $R_j(s,s_j)$. On writing Eq. (B3) out in iterated form and inverting the order of integration, we get

$$([\varphi]_3)=(\Psi)([R]_2),\tag{B5}$$

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[54] D. B. Fairlie, P. V. Landshoff, J. Nuttal, and J. C. Polkinghorne, J. Math. Phys. 3, 594 (1962).

where (Ψ) is a 3×3 matrix operator such that

$$\{(\Psi)([R]_2)\}_i=\sum_j\int_{s_j^0}^{(\sqrt{s}-m_j)^2}ds_j'\,{}_j\bar{\psi}_i(s_j',s,s_i)\times[R_j(s,s_j')]_2 \quad \text{(B6)}$$

and the ${}_j\bar{\psi}_i$ satisfy the following system of integral equations:

$${}_j\bar{\psi}_i(s_j',s,s_i)=(1-\delta_{ij})\Delta^{(1)}(s_j',s,s_i)+\sum_{k\neq i}\int ds_k''\mathbf{K}(s_k'',s,s_i)\,{}_j\bar{\psi}_k(s_j',s,s_k'')/D_k(s_k'') \quad \text{(B7)}$$

or in matrix notation

$$(\bar{\psi})=(\Delta^{(1)})+(\mathbf{K})(\bar{\psi}/D). \quad \text{(B7}')$$

The functions ${}_j\bar{\psi}_k(s_j',s,s_k)$ are the same as in Ref. 4, where their integral representation was written under the original KT form, which can be deduced from Eq. (B7) by inverting the order of integration.

Now in the equal-mass symmetric case (all $m_i=1$), Eq. (B1) reduces immediately to the single equation (4.1), since all the functions φ_i (φ_{0i}) may be considered as the same function φ (φ_0)—each one depending upon a different variable. This is not the case as regards ${}_j\bar{\psi}_i(s_j',s,s_i)$ because of the presence of δ_{ij} in the inhomogeneous term of Eq. (B7). However, we can identify all the ${}_j\bar{\psi}_i$, $i\neq j$, with ${}_2\bar{\psi}_1$ for instance, and all the ${}_i\bar{\psi}_i$ with ${}_1\bar{\psi}_1$ for instance. Then Eq. (B5) yields

$$[\varphi(s,s_1)]_3=\int_4^{(\sqrt{s}-1)^2}ds'[{}_1\bar{\psi}_1(s',s,s_1)+2\,{}_2\bar{\psi}_1(s',s,s_1)][R(s,s')]_2. \quad \text{(B8)}$$

If we set

$$2\bar{\psi}(s',s,s_1)={}_1\bar{\psi}_1(s',s,s_1)+2\,{}_2\bar{\psi}_1(s',s,s_1) \quad \text{(B9)}$$

this gives just the form of the three-body discontinuity one can obtain directly from Eq. (4.1). [Equation (B8) is nothing but the equation following Eq. (12) in Ref. 5.]

Now Eq. (B7) reduces to

$${}_1\bar{\psi}_1(s',s,s_1)=2\int ds''\mathbf{K}(s'',s,s_1)\,{}_2\bar{\psi}_1(s',s,s'')/D(s''),$$

$${}_2\bar{\psi}_1(s',s,s_1)=\Delta^{(1)}(s',s,s_1)+\int ds''\mathbf{K}(s'',s,s_1)\times[{}_1\bar{\psi}_1(s',s,s'')+{}_2\bar{\psi}_1(s',s,s'')]/D(s''),$$

from which, taking account of Eq. (B9), we get

$$\bar{\psi}(s',s,s_1)=\Delta^{(1)}(s',s,s_1)+2\int ds''\mathbf{K}(s'',s,s_1)\bar{\psi}(s',s,s'')/D(s''), \quad \text{(B10)}$$

$${}_2\bar{\psi}_1(s',s,s_1)=\bar{\psi}(s',s,s_1)-\int ds''\mathbf{K}(s'',s,s_1)\,{}_2\bar{\psi}_1(s',s,s'')/D(s''), \quad \text{(B11)}$$

$${}_1\bar{\psi}_1(s',s,s_1)=\bar{\psi}(s',s,s_1)-\Delta^{(1)}(s',s,s_1)-\int ds''\mathbf{K}(s'',s,s_1)\,{}_1\bar{\psi}_1(s',s,s'')/D(s''). \quad \text{(B12)}$$

Let us end with the following familiar remark. The reduction we have set up above is closely related to the diagonalization of the 3×3 matrix $(\kappa)_{ij}=1-\delta_{ij}$ that we can factor out of $(\mathbf{K})$ in the equal-mass case. Its eigenvalues are -1 (twice) and 2. Accordingly the system of Eq. (B1) may be split into three independent integral equations, each one being associated with an eigenvalue of (κ) and involving a well-defined combination of the φ_i's. Indeed, if all the functions φ_i are equal to the same function φ (Bose-Einstein symmetry in the exchange of two particles), only the combination of the φ_i's associated with the eigenvalue 2 does not vanish and the corresponding equation is just Eq. (4.1). Similar considerations apply to the functions ${}_j\bar{\psi}_i(s_j',s,s_i)$; if the initial subenergies are made equal in all the ${}_j\bar{\psi}_i$ (as in three-body unitarity equations) the transformed matrix of $(\bar{\psi})$ is diagonal under the transformation that diagonalizes (κ), and the element associated with the eigenvalue 2 of (κ) is nothing but ${}_1\bar{\psi}_1+2\,{}_2\bar{\psi}_1$.

APPENDIX C: CONDITION OF REALITY FOR THE AMPLITUDES

We consider Eq. (4.1) for $\sqrt{s}\lesssim3$, $(\sqrt{s}-1)^2\lesssim s_1\lesssim4$ (see footnote 48) and write it as

$$\varphi(s,s_1)=\varphi_0(s,s_1)-2\int_{-\infty}^0 ds_2'K(s_2',s,s_1)\frac{\varphi(s,s_2')}{D(s_2')}+2\int_0^{(\sqrt{s}-1)^2}ds_2'\Delta^{(1)}(s_2',s,s_1)\frac{\varphi(s,s_2')}{D(s_2')}. \quad \text{(C1)}$$

First, one can verify that the kernels $K(s_2',s,s_1)$ and $\Delta^{(1)}(s_2',s,s_1)$ are real on the needed range of integration, if the function $N(s_1')$ has singularities and cuts only for $s_1'\leq0$, as we may assume. As regards $K(s_2',s,s_1)$ this clearly follows from its definition [Eq. (3.5)]: the s_1' path of integration runs on the real axis over a range of values $[s_1'\geq(\sqrt{s}+1)^2]$ for which the integrand is real—remember that in our conventions $k(s_1',s,1)$ is imaginary only for $(\sqrt{s}-1)^2\leq s_1'\leq(\sqrt{s}+1)^2$. On the other hand, $\Delta^{(1)}(s_2',s,s_1)/k(s_2',s,1)$ is real for s_2' real just above $(\sqrt{s}-1)^2$; in Eq. (3.8) $s_{1>}(s,s_2')$ and $s_{1<}(s,s_2')$ lie then on the real axis in the vicinity of $s_1'=1+\sqrt{s}>s_1$ whereas $k(s_2',s,1)$ and $k(s_1',s,1)$ are both pure imaginary. The same quantity remains real on the real axis as long as no singularity is encountered. It is thus real for $0 \leq s_2' \leq (\sqrt{s}-1)^2$; as $k(s_2',s,1)$ is also real in this range, so is $\Delta^{(1)}(s_2',s,s_1)$.

From these results one can now show that $\varphi(s,s_1)$ is real for the values of $\sqrt{s}$ and s_1 previously considered if $\varphi_0(s,s_1)$ is real in the range $s_1 \leq 4$. Of course, one has also to assume that Eq. (4.1) possesses at least one solution.[55] This is the case, in particular when the iterative expansion of the equation converges. The first iteration is then real—recall that $1/D(s_2')$ is real for $s_2' \leq 4$—and remains real along all $s_1 \leq 0$ since there is no cut below $s_1 = 4$; hence the second iteration is real, and so on. The same property holds also if Eq. (4.1) is of Fredholm type [this requires $D(s_2') \sim \mathcal{O}(s_2'^{\epsilon})$, $\epsilon > 0$, for $s_2' \to \infty$, if $N(s_2') = \text{constant}$]. Then $\mathrm{Im}\,\varphi(s,s_1)$ satisfies a homogeneous Fredholm-type equation, and is thus zero except at the eigenvalues in s of this equation.

As regards the $3 \to 3$ amplitudes, it is convenient to begin with the function $\bar{\psi}(s_2',s,s_1)$, Eq. (B10). The inhomogeneous term of this equation is $\Delta^{(1)}(s_2',s,s_1)$; this is pure imaginary for $\sqrt{s} \lesssim 3$, $(\sqrt{s}-1)^2 \lesssim s_2' < 4$, and $s_1 < s_{1<}(s,s_2')$ $(\sim 1+\sqrt{s})$, as already noted. So, if we can guarantee the existence of a solution $\bar{\psi}(s_2',s,s_1)$ as in the two cases considered above for $\varphi(s,s_1)$, we can conclude at the same time that this solution is pure imaginary for $\sqrt{s}$ and s_2' as above and $(\sqrt{s}-1)^2 \lesssim s_1 < s_{1<}(s,s_2')$. Under similar considerations, the $3 \to 3$ amplitudes ${}_\lambda\Psi_i(s_\lambda',s,s_i)$ [Eqs. (B11), (B12), (4.4)] are thus real and remain real on $s_\lambda' \leq 4$ and $s_i \leq 4$, as long as an OPE singularity $s_\lambda' = s_{\lambda\pm}(s,s_i)$ of $\Delta^{(1)}(s_\lambda',s,s_i)$ (and eventually a singularity generated by N) is not reached. Finally, the full decay amplitude of Eq. (3.1) and the $3 \to 3$ amplitudes of Eqs. (4.3) and (4.2) are also real under the same conditions.

[55] For general considerations about the existence and uniqueness of the solutions, we refer the reader to works about Skornyakov–Ter-Martirosian equations {G. S. Danilov and V. I. Lebedev, Zh. Eksperim. i Teor. Fiz. 44, 1509 (1963) [English transl.: Soviet Phys.—JETP 17, 1015 (1963)]; N. N. Beloozerov, Yadern. Fiz. 2, 552 (1965) [English transl.: Soviet J. Nucl. Phys. 2, 395 (1966)]} or about singular bootstrapping equations [D. Atkinson and A. P. Contogouris, Nuovo Cimento 39, 1082; 39, 1102 (1965)]. The same methods may be applied to the KT model without great change.