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Some General Properties of Para-Fermi Field Theory

Y. Ohnuki*
Institute for Nuclear Study, University of Tokyo, Tanashi, Tokyo

AND

S. Kamefuchi†
Department of Physics, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin

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The nonrelativistic theory of a single para-Fermi field of order p is investigated. General properties of state vectors are studied in detail, and it is shown that the state-vector space can be spanned by what we shall call standard state vectors. A restriction on the form of interaction Hamiltonians is derived from the requirement that our formalism be described by local Lagrangian field theory. This restriction on interaction Hamiltonians gives rise to a conservation law of a physical quantity to be called A, which resembles the magnitude of angular momentum with respect to its rule of addition. The conservation law of A leads then to absolute selection rules for reactions, which are a generalization of those obtained elsewhere. The problem of bound states made up of our para-Fermi field is also studied, and all bound states are classified into $(p+1)$ categories according to their statistical behaviors. It is found that for $p\leqslant 3$ all bound states can be described by ordinary parafield theory, whereas for $p\geqslant 4$ such is no longer the case. Furthermore, it can be shown that in the theory of $p=2$ no fermion bound states are possible. In this sense it may be said that para-Fermi fields of $p=1$ and 3 occupy a very privileged position in para-Fermi theory in general. The main results in this paper are stated as 12 theorems. It is expected that the whole argument will be valid in a relativistic theory as well.

1. INTRODUCTION

As is well known, when one makes the simplest possible generalization of commutation relations between different parafields,[1,2] one can derive a rather severe selection rule, from which it follows that all the particles and resonances observed in nature are just ordinary fermions or bosons.[2] Thus, it appears at present that the only possible candidate to which parafield theory may be applicable is what is to be called the fundamental matter or field such as quarks, with which all observable particles are constructed. From the point of view of simplicity, the most interesting theory of this kind is the one in which such a fundamental field is described by a para-Fermi field with spin $\frac{1}{2}$.

Since in such a theory all of the observable particles are regarded as bound states of the fundamental particles, the question immediately arises as to what kinds of statistics are obeyed by such bound states in general. A detailed investigation into two-body bound states has recently been made by one of the present authors (Y. O.).[3] The purpose of this paper is to study some general properties of para-Fermi field theory with emphasis on statistical properties of bound states consisting of fundamental paraparticles, and furthermore on absolute selection rules for reactions involving these bound states, which follow as a consequence of the specific structure of parafield theory.

In the following sections, we shall restrict ourselves to a system consisting of a single para-Fermi field of order p (and, hence, of half-integral spin due to the spin-statistics theorem), which is described by $\phi(x,t)$ and its Hermitian conjugate $\phi^\dagger(x,t)$. Throughout this paper (unless otherwise stated) x is to be understood as representing the spatial coordinates and those corresponding to all internal degrees of freedom, such as particle-antiparticle property, spin and unitary spin, etc. For the sake of simplicity we assume that the field $\phi(x,t)$ is a nonrelativistic, Schrödinger field. The field operators ϕ and $\phi^\dagger$ satisfy the following equal-time commutation relations:

$$[\phi(x,t),[\phi^\dagger(y,t),\phi(z,t)]]=2\delta(x-y)\phi(z,t)\,, \tag{1.1a}$$

$$[\phi(x,t),[\phi^\dagger(y,t),\phi^\dagger(z,t)]]=2\delta(x-y)\phi^\dagger(z,t)-2\delta(x-z)\phi^\dagger(y,t)\,, \tag{1.1b}$$

$$[\phi(x,t),[\phi(y,t),\phi(z,t)]]=0\,, \tag{1.1c}$$

and their Hermitian-conjugate relations. Hereafter, we shall denote by ψ either ϕ or $\phi^\dagger$. We shall also introduce, on the right-hand side of the commutation relations, a function $\bar{\delta}$ such that $\bar{\delta}(x-y)\equiv\delta(x-y)$ or zero, depending on whether x and y come from the arguments of a pair of ϕ and $\phi^\dagger$ or otherwise. Then, all of the relations (1.1) and their Hermitian conjugates can be written in a unified form;

$$[\psi(x,t),[\psi(y,t),\psi(z,t)]]=2\bar{\delta}(x-y)\psi(z,t)-2\bar{\delta}(x-z)\psi(y,t)\,. \tag{1.1$'$}$$

* Permanent address: Department of Physics, Nagoya University, Nagoya, Japan.

† Permanent address: Department of Physics, Tokyo University of Education, Tokyo, Japan.

[1] References to earlier papers on parafield theory can be found in O. W. Greenberg and A. M. L. Messiah, Phys. Rev. **138**, B1155 (1965). We wish to add to the references of these authors one more paper: S. Kamefuchi and Y. Takahashi, Nucl. Phys. **36**, 177 (1962).

[2] O. W. Greenberg and A. M. L. Messiah, Ref. 1; S. Kamefuchi, Nuovo Cimento **36**, 1069 (1965).

[3] Y. Ohnuki, Progr. Theoret. Phys. (Kyoto) Suppl. **37** and **38**, 285 (1966). This paper will hereafter be referred to as (I).

For the vacuum state $|0\rangle$ of the state-vector space $\mathcal{A}$ of a para-Fermi field ϕ there exist the relations

$$\phi(x,t)|0\rangle=0, \tag{1.2}$$

$$\phi(x,t)\phi^\dagger(y,t)|0\rangle=p\delta(x-y)|0\rangle. \tag{1.3}$$

To decompose the field ϕ into the Green component fields ϕ_α ($\alpha=1, 2, \cdots, p$) is often very convenient.[4] The latter fields are defined by

$$\phi(x,t)=(\sum_{\alpha=1}^{p}\phi_\alpha(x,t))P \tag{1.4}$$

and

$$[\phi_\alpha(x,t),\phi_\beta(y,t)]_{\epsilon(\alpha,\beta)}=0,$$
$$[\phi_\alpha(x,t),\phi_\beta{}^\dagger(y,t)]_{\epsilon(\alpha,\beta)}=\delta_{\alpha\beta}\delta(x-y), \tag{1.5}$$

where $\epsilon(\alpha,\beta)=2\delta_{\alpha\beta}-1$ and P is the projection operator of the state-vector space $\mathcal{B}$ of the Green component fields ϕ_α onto its irreducible subspace of the parafield ϕ, i.e., $\mathcal{A}$. The vacuum state $|\bar{0}\rangle$ of the space $\mathcal{B}$ is defined by

$$\phi_\alpha(x,t)|\bar{0}\rangle=0, \quad (\alpha=1, 2, \cdots, p) \tag{1.6}$$

and has the property

$$|\bar{0}\rangle=P|\bar{0}\rangle=|0\rangle. \tag{1.7}$$

As for the operator P, there hold the relations

$$P=P^\dagger, \quad dP/dt=0, \quad [P,\phi(x,t)]=0. \tag{1.8}$$

Now, the space $\mathcal{A}$ consists of state vectors which are generated from the vacuum state $|0\rangle$ by applying polynomials of $\phi^\dagger(x,t)$, $\phi^\dagger(y,t)$, $\cdots$. Some of these states correspond to bound states. Thus, we start, in Sec. 2, with exploring general properties of state vectors. It is found there that an arbitrary n-particle state vector can be written as a superposition of the basic vectors which we call *standard state vectors*. Various properties of standard state vectors are studied, and the results are summarized as Theorems 1–4.

In Sec. 3 we study the restrictions imposed on interaction Hamiltonians from the requirement of locality. It is found that as a consequence of such restricted forms of interactions we have a certain conserved quantity which we shall call A. The rule of addition of this quantity and the selection rule arising from this will be studied in detail. The results obtained are summarized as Theorems 5–7. We then turn, in Sec. 4, to the problem of bound states. It is found there that statistical properties of bound states consisting of any number of particles can be classified into $(p+1)$ categories. Properties of each category and the relationship between different categories will also be discussed. The results are stated as Theorems 8–9′.

In Sec. 5, we consider the problem of whether, and in what cases, bound states belonging to the above-mentioned categories can be described by ordinary parafield theory. We shall find that only in the cases $p=1$, 2, 3, bound states belonging to all categories are ordinary paraparticles. That is, only in those three cases we say that the theory is *statistically closed*. This is the content of Theorem 10. On the other hand, however, Theorem 9′ states that a para-Fermi field of order $p=2$ cannot construct fermions as bound states. Consequently, as the fundamental field a para-Fermi field of $p=1$ or 3 occupies a very privileged position among para-Fermi fields in general. In Sec. 6 we shall give a few additional remarks. The first remark is concerned with the question of the statistically closed property of a system consisting of more than one field. One of the remarkable results is that two parafields of order 2 and of order 3 cannot coexist under the requirement of the system being statistically closed. The second remark deals with a formal relationship between two models of hadrons, i.e., three-triplet[5] and paraquark[6] models. The last remark is concerned with a possibility of generalizing our whole argument in this paper to the relativistic case. In fact, this generalization is obvious except for the treatment of many-particle bound states. In this connection we suggest a possible generalization of time-ordered products of parafield operators, which are closely connected with the definition of bound states in relativistic field theory.

2. PROPERTIES OF STATE VECTORS

Throughout the following sections we shall omit the argument t of the field operators $\phi(x,t)$ and $\phi^\dagger(y,t)$ as we shall be concerned only with those quantities at a common time t, say. Let us start with proving the following.

Theorem 1: Any monomial of degree n consisting of ϕ and $\phi^\dagger$, i.e., $\psi(x_1)\psi(x_2)\cdots\psi(x_n)$ can be written as a finite sum of terms with the *standard form*

$$\{\psi(x_{i_1}),\psi(x_{i_2}),\cdots,\psi(x_{i_a})\}[\psi(x_{j_1}),\psi(x_{k_1})]$$
$$\times[\psi(x_{j_2}),\psi(x_{k_2})]\cdots[\psi(x_{j_b}),\psi(x_{k_b})],$$

where $n\geqslant a+2b$ and $\{\psi(x_{i_1}),\psi(x_{i_2}),\cdots,\psi(x_{i_a})\}$ denotes the totally symmetrized product of those quantities in the bracket.

Proof: We assume that the statement holds true up to degree $n\leqslant m$. Then, we have

$$\psi(x_1)\psi(x_2)\cdots\psi(x_m)=a\{x_1,x_2,\cdots,x_m\}$$
$$+\sum b\{\psi,\psi,\cdots,\psi\}_{m-2}[\psi,\psi]$$
$$+\sum c\{\psi,\psi,\cdots,\psi\}_{m-4}[\psi,\psi][\psi,\psi]+\cdots, \tag{2.1}$$

where $\{x_1,x_2,\cdots,x_m\}$ is the shorthand notation for $\{\psi(x_1),\psi(x_2),\cdots,\psi(x_m)\}$ and the subscript attached to the curly bracket indicates the number of the operators

[4] H. S. Green, Phys. Rev. **90**, 270 (1953).

[5] M. Y. Han and Y. Nambu, Phys. Rev. **139**, B1006 (1965); Y. Miyamoto, Progr. Theoret. Phys. (Kyoto) Suppl. Extra No., 187 (1965). See also S. Hori, Progr. Theoret. Phys. (Kyoto) **36**, 131 (1966).

[6] O. W. Greenberg, Phys. Rev. Letters **13**, 598 (1964); A. N. Mitra, Phys. Rev. **151**, 1168 (1966).

ψ contained in it. Multiplying (2.1) from the left by $\psi(y)$, we get the following product of degree $(m+1)$:

$$\psi(y)\psi(x_1)\cdots\psi(x_m)=a\psi(y)\{x_1,x_2,\cdots,x_m\}$$
$$+\sum b\psi(y)\{\psi,\psi,\cdots,\psi\}_{m-2}[\psi,\psi]$$
$$+\sum c\psi(y)\{\psi,\psi,\cdots,\psi\}_{m-4}[\psi,\psi][\psi,\psi]+\cdots. \quad (2.2)$$

The terms on the right-hand side, except the first one, contain the factors $\psi(y)\{\psi,\psi,\cdots,\psi\}_{m-2j}$ with $j\geqslant 1$, the degree of which is $m-2j+1<m$. Thus, by assumption, such a factor can be written as a linear combination of terms of the standard form. This means that the same is true also for the right-hand side of (2.2) except for the first term. Now, in order to study the first term $\psi(y)\{x_1,x_2,\cdots,x_m\}$, consider the product $\{x_1,x_2,\cdots,x_m,y\}$, which we rewrite in the form

$$\{x_1,x_2,\cdots,x_m,y\}=\psi(y)\{x_1,x_2,\cdots,x_m\}$$
$$+\sum_{(1,2,\cdots,m)}\sum_{j=1}^{m}\psi(x_m)\psi(x_{m-1})\cdots$$
$$\times\psi(x_j)\psi(y)\psi(x_{j-1})\cdots\psi(x_2)\psi(x_1), \quad (2.3)$$

where $\sum_{(1,2,\cdots,m)}$ means summation over all permutations of $(1,2,\cdots,m)$. In the second term on the right-hand side of (2.3), $\psi(y)$ can be shifted to the extreme left by means of commutators. In this way we arrive at the expression

$$(2.3)=(m+1)\psi(y)\{x_1,x_2,\cdots,x_m\}+\sum_{(1,2,\cdots,m)}\sum_{j=1}^{m}j\psi(x_m)\psi(x_{m-1})\cdots\psi(x_{j+1})[\psi(x_j),\psi(y)]\psi(x_{j-1})\cdots\psi(x_2)\psi(x_1)$$
$$=(m+1)\psi(y)\{x_1,x_2,\cdots,x_m\}+\sum_{(1,2,\cdots,m)}\sum_{j=1}^{m}j\psi(x_m)\psi(x_{m-1})\cdots\psi(x_{j+1})$$
$$\times(\psi(x_{j-1})\cdots\psi(x_2)\psi(x_1)[\psi(x_j),\psi(y)]+[[\psi(x_j),\psi(y)],\psi(x_{j-1})\cdots\psi(x_2)\psi(x_1)]). \quad (2.3')$$

From (2.3′) it follows that

$$\psi(y)\{x_1,x_2,\cdots,x_m\}=\frac{1}{m+1}\{x_1,x_2,\cdots,x_m,y\}+\tfrac{1}{2}m\sum_{j=1}^{m}\{x_1,x_2,\cdots,x_{j-1},x_{j+1},\cdots,x_m\}[\psi(y),\psi(x_j)]$$
$$+\frac{1}{m+1}\sum_{(1,2,\cdots,m)}\sum_{j=1}^{m}j\psi(x_m)\psi(x_{m-1})\cdots\psi(x_{j+1})[[\psi(y),\psi(x_j)],\psi(x_{j-1})\cdots\psi(x_2)\psi(x_1)]. \quad (2.4)$$

The commutators in the third term on the right-hand side of (2.4) can be evaluated by means of (1.1′), and consequently the third term as a whole becomes a polynomial of degree $(m-1)$, which by assumption can be written as a linear combination of terms of the standard form. Thus, we have proved that the monomial (2.2) of degree $(m+1)$ can be written as a linear combination of terms of the standard form. Now, obviously, for $m=1$ and 2 we have $\psi(x)=\{\psi(x)\}$ and $\psi(x_1)\psi(x_2)=\frac{1}{2}\{x_1,x_2\}+\frac{1}{2}[\psi(x_1),\psi(x_2)]\equiv\frac{1}{2}\{x_1,x_2\}+\frac{1}{2}[x_1,x_2]$, both of which are of the standard form. Thus, by mathematical induction we have proved Theorem 1.

Let us now turn to state vectors of the space $\mathfrak{A}$. The whole space is spanned by state vectors such as $\phi^\dagger(x_1)\phi^\dagger(x_2)\cdots\phi^\dagger(x_n)|0\rangle$, where the product of n operators acting on $|0\rangle$ consists of $\phi^\dagger$ operators only.[7] We can now apply to this product Theorem 1, in which all ψ are replaced by $\phi^\dagger$. In this way we get the following.

Theorem 1′: Any n-particle state vector can be written as a superposition of the *standard state vectors*

$$\{\phi^\dagger(x_{i_1}),\phi^\dagger(x_{i_2}),\cdots,\phi^\dagger(x_{i_a})\}[\phi^\dagger(x_{j_1}),\phi^\dagger(x_{k_1})]$$
$$\times[\phi^\dagger(x_{j_2}),\phi^\dagger(x_{k_2})]\cdots[\phi^\dagger(x_{j_b}),\phi^\dagger(x_{k_b})]|0\rangle$$
$$\equiv|\{i_1,i_2,\cdots,i_a\}^\dagger[j_1,k_1]^\dagger[j_2,k_2]^\dagger\cdots[j_b,k_b]^\dagger|0\rangle,$$

where

$$n=a+2b,\quad a\leqslant p,\quad i_1<i_2<\cdots<i_a,\quad j_1<j_2<\cdots<j_b,$$

and

$$j_l<k_l(l=1,\,2,\,\cdots,\,b).$$

Note in this connection that as a consequence of the Hermitian conjugate of (1.1c) the order of the factors $\{\ \}^\dagger$ and $[\ ,\ ]^\dagger$ in a standard state vector is immaterial.[8] Now, for given n and a, the number of possible standard state vectors is given by ${}_nC_a(2b-1)!!$. The question now arises as to whether or not all of the standard state vectors with common arguments x_i are linearly

[7] By repeated use of (1.3) and the Hermitian conjugate of (1.1a) it is shown that any state vector $\psi(x_1)\psi(x_2)\cdots\psi(x_n)|0\rangle$ can be written as a linear combination of state vectors $\phi^\dagger(x_{i_1})\phi^\dagger(x_{i_2})\cdots\phi^\dagger(x_{i_m})|0\rangle$, where $m\leqslant n$.

[8] When the arguments x_i and y_j are unimportant, the above standard state vector will simply be written as $|\{\ \}_a{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_b{}^\dagger|0\rangle$, where the subscripts a and b indicate the number of $\phi^\dagger$ in the bracket $\{\ \}^\dagger$ and the number of square brackets $[\ ]^\dagger$ contained in the state vector, respectively.

independent. Part of the answer to this question is given by the following.

Theorem 2: Two standard n-particle state vectors

$$|\{i_1,i_2,\cdots,i_a\}^\dagger[j_1,k_1]^\dagger[j_2,k_2]^\dagger\cdots[j_b,k_b]^\dagger|0\rangle$$

and

$$|\{i_1',i_2',\cdots,i_{a'}'\}^\dagger[j_1',k_1']^\dagger[j_2',k_2']^\dagger\cdots[j_{b'}',k_{b'}']^\dagger|0\rangle,$$

where $n=a+2b=a'+2b'$, are orthogonal to each other provided $a\neq a'$.

Proof: First consider the following expression:

$$\begin{aligned}[\phi(u),\phi(v)]|\{i_1,i_2,\cdots,i_a\}^\dagger[j_1,k_1]^\dagger[j_2,k_2]^\dagger\cdots\\ \times[j_b,k_b]^\dagger|0\rangle=[\phi(u),\phi(v)][\phi^\dagger(x_{j_1}),\phi^\dagger(x_{k_1})]\\ \times[\phi^\dagger(x_{j_2}),\phi^\dagger(x_{k_2})]\cdots[\phi^\dagger(x_{j_b}),\phi^\dagger(x_{k_b})]\\ \times\{\phi^\dagger(x_{i_1}),\phi^\dagger(x_{i_2}),\cdots,\phi^\dagger(x_{i_a})\}|0\rangle,\end{aligned} \quad (2.5)$$

where use is made of the Hermitian conjugate of (1.1c). Now, by use of (1.1), we can easily prove the following commutation relations:

$$\begin{aligned}\tfrac{1}{2}[[\phi(u),\phi(v)],[\phi^\dagger(x),\phi^\dagger(y)]]=\delta(v-x)[\phi(u),\phi^\dagger(y)]\\ -\delta(u-x)[\phi(v),\phi^\dagger(y)]+\delta(u-y)[\phi(v),\phi^\dagger(x)]\\ -\delta(v-y)[\phi(u),\phi^\dagger(x)]\end{aligned} \quad (2.6)$$

and

$$\begin{aligned}\tfrac{1}{2}[[\phi(u),\phi^\dagger(x)],[\phi^\dagger(y),\phi^\dagger(z)]]=\delta(u-z)[\phi^\dagger(x),\phi^\dagger(y)]\\ -\delta(u-y)[\phi^\dagger(x),\phi^\dagger(z)].\end{aligned} \quad (2.7)$$

By repeated use of (2.6), we can shift $[\phi(u),\phi(v)]$ in (2.5) to the right until it reaches the left of $\{i_1,i_2,\cdots,i_a\}^\dagger$. Every time $[\phi,\phi]$ is commuted with $[\phi^\dagger,\phi^\dagger]$, we are left with extra terms containing a bracket of the type $[\phi,\phi^\dagger]$, which can then be shifted to the right by means of (2.7). In this way we end up with a linear combination of three kinds of terms

$$[\phi^\dagger,\phi^\dagger][\phi^\dagger,\phi^\dagger]\cdots[\phi^\dagger,\phi^\dagger][\phi,\phi]|\{i_1,i_2,\cdots,i_a\}^\dagger|0\rangle,$$

$$[\phi^\dagger,\phi^\dagger][\phi^\dagger,\phi^\dagger]\cdots[\phi^\dagger,\phi^\dagger][\phi,\phi^\dagger]|\{i_1,i_2,\cdots,i_a\}^\dagger|0\rangle,$$

and

$$[\phi^\dagger,\phi^\dagger][\phi^\dagger,\phi^\dagger]\cdots[\phi^\dagger,\phi^\dagger]|\{i_1,i_2,\cdots,i_a\}^\dagger|0\rangle,$$

where the number of brackets $[\phi^\dagger,\phi^\dagger]$ in these expressions is b, $(b-1)$, and $(b-1)$, respectively.

We can now prove the following relations:

$$[\phi(u),\phi(v)]|\{i_1,i_2,\cdots,i_a\}^\dagger|0\rangle=0, \quad (2.8)$$

$$\begin{aligned}[\phi(u),\phi^\dagger(x)]|\{i_1,i_2,\cdots,i_a\}^\dagger|0\rangle=p\delta(u-x)\\ \times\{\phi^\dagger(x_{i_1}),\phi^\dagger(x_{i_2}),\cdots,\phi^\dagger(x_{i_a})\}|0\rangle\\ -2\sum_s\delta(u-x_{i_s})\{\phi^\dagger(x_{i_1}),\cdots,\\ \phi^\dagger(x_{i_{s-1}}),\phi^\dagger(x),\phi^\dagger(x_{i_{s+1}}),\cdots,\phi^\dagger(x_{i_a})\}|0\rangle.\end{aligned} \quad (2.9)$$

To prove (2.8) we first observe that $\phi(u)\phi(v)|\{i_1,i_2,\cdots,i_a\}^\dagger|0\rangle$ is a linear combination of terms such as $\delta(u-x_{l_1})\delta(v-x_{l_2})\phi^\dagger(x_{l_3})\cdots\phi^\dagger(x_{l_a})|0\rangle$, where $(l_1,l_2,\cdots,l_a)$ is a certain permutation of $(i_1,i_2,\cdots,i_a)$. Now (2.8) must be symmetric with respect to x_l's and, at the same time, antisymmetric with respect to u and v. Hence, it must vanish. The proof of (2.9) can easily be done by repeated use of the Hermitian conjugates of (1.1a) and (1.3). Combining the result of the preceding chapter with (2.8) and (2.9) we see that the state vector (2.5) is a linear combination of state vectors such as $|\{\ \}_a{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_{(b-1)}{}^\dagger|0\rangle$.

Suppose that $b>b'$ $(a<a')$. Then by repeating the above argument we see that evaluation of the inner product of the two state vectors given in the statement of Theorem 2 eventually reduces to evaluation of terms such as

$$\begin{aligned}\langle 0|\{i_1',i_2',\cdots,i_{a'}'\}\{\bar{i}_1,\bar{i}_2,\cdots,\bar{i}_a\}^\dagger\\ \times[\bar{j}_1,\bar{k}_1]^\dagger[\bar{j}_2,\bar{k}_2]^\dagger\cdots[\bar{j}_{b-b'},\bar{k}_{b-b'}]^\dagger|0\rangle.\end{aligned} \quad (2.10)$$

Now let $(\bar{l}_1,\bar{l}_2,\cdots,\bar{l}_{a'})$ be a permutation of

$$(\bar{i}_1,\cdots,\bar{i}_a,\bar{j}_1,\bar{k}_1,\cdots,\bar{j}_{b-b'},\bar{k}_{b-b'}).$$

Then, (2.10) is a linear combination of terms such as $\delta(i_1'-\bar{l}_1)\delta(i_2'-\bar{l}_2)\cdots\delta(i_{a'}'-\bar{l}_{a'})$. This last expression, however, is totally symmetric with respect to $(i_1',i_2',\cdots,i_{a'}')$. Therefore, (2.10) is totally symmetric with respect to $(\bar{l}_1,\bar{l}_2,\cdots,\bar{l}_{a'})$, and hence with respect to $(\bar{i}_1,\cdots,\bar{i}_a,\bar{j}_1,\bar{k}_1,\cdots,\bar{j}_{b-b'},\bar{k}_{b-b'})$ as well. However, (2.10) must be antisymmetric with respect to, for example, $\bar{j}_1$ and $\bar{k}_1$. Thus (2.10) must vanish identically. This completes the proof of Theorem 2.

As for the linear independence of the standard state vectors for given n, a, and b $(n=a+2b)$ we have furthermore the following.

Theorem 3: The necessary and sufficient condition for all of the different standard state vectors with fixed a and b and common arguments $|\{i_1,i_2,\cdots,i_a\}^\dagger[j_1,k_1]^\dagger\times[j_2,k_2]^\dagger\cdots[j_b,k_b]^\dagger|0\rangle$ to be linearly independent of each other is that $p\geqslant a+b$.

The proof of this theorem is rather lengthy, and is relegated to Appendix A.

We have seen, through the proof of Theorem 2, that the state vector (2.5) and $[\phi,\phi^\dagger]|\{\ \}_a{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots\times[\ ]_b{}^\dagger|0\rangle$ can be written as a linear combination of state vectors of the type $|\{\ \}_a{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_{(b-1)}{}^\dagger|0\rangle$ and of those of the type $|\{\ \}_a{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_b{}^\dagger|0\rangle$, respectively. From the Hermitian conjugate of (1.1c) it is obvious that $[\phi^\dagger,\phi^\dagger]|\{\ \}_a{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_b{}^\dagger|0\rangle=|\{\ \}_a{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_{(b+1)}{}^\dagger|0\rangle$. These results can be stated as the following.

Theorem 4: When a standard state vector

$$|\{\ \}_a{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]^\dagger|0\rangle$$

is multiplied by any of the operators $[\phi,\phi]$, $[\phi,\phi^\dagger]$, or $[\phi^\dagger,\phi^\dagger]$, the resulting state vector is again a superposition of standard state vectors containing the bracket $\{\ \}_a{}^\dagger$ with the same value of a. Namely, the number of operators a in the curly bracket remains unchanged under this operation.

3. RESTRICTIONS OF INTERACTION HAMILTONIANS AND SELECTION RULES

It is most convenient to discuss interacting fields in the interaction representation. Let $H_I(x)$ be the interaction Hamiltonian (density) in the interaction representation, and a polynomial of the free-field variables $\phi(x,t)$ and $\phi^\dagger(x,t)$ taken at the same space-time point (x,t). First of all $H_I(\phi(x))$ is required to satisfy the locality (or integrability) condition

$$[H_I(\phi(x)),H_I(\phi(y))]=0 \quad \text{for} \quad x\sim y, \tag{3.1}$$

where $x\sim y$ means that the two points x and y are spatially distant from each other. In order that the theory be a local Lagrangian field theory, we must impose a further requirement that locality holds also between $\phi(x)$ and $H_I(\phi(y))$, too, i.e.,[9]

$$[\phi(x),H_I(\phi(y))]=0 \quad \text{for} \quad x\sim y. \tag{3.2}$$

In their paper on selection rules for parafields, Greenberg and Messiah[1] required the condition (3.1) only. However, it is known that a theory which satisfies (3.1) but not (3.2) leads to essentially nonlocal results.[10] In Appendix B we shall explicitly show such a feature by constructing a simple model field theory.

When we apply the Green decomposition (1.4) to each field ϕ in (3.1) and (3.2), these relations can be written as

$$\sum_{m,m'} [H_I{}^{(m)}(\phi_\beta(x)),H_I{}^{(m')}(\phi_\beta(y))]P=0$$

and

$$\sum_{\alpha=1}^{p}\sum_{m} [\phi_\alpha(x),H_I{}^{(m)}(\phi_\beta(y))]P=0,$$

respectively, where the superscript (m) labels those terms of H_I which are generated by the Green decomposition. However, according to a theorem due to Araki, Greenberg, and Toll,[11] the locality condition (in the space $\mathcal{A}$) implies actually the stronger paralocality condition (in the space $\mathcal{B}$) when the relation (1.7) is guaranteed. This means that we can omit the projection operator P from the above locality relations to obtain

$$\sum_{m,m'} [H_I{}^{(m)}(\phi_\beta(x)),H_I{}^{(m')}(\phi_\beta(y))]=0 \quad \text{for} \quad x\sim y, \tag{3.1$'$}$$

$$\sum_{\alpha=1}^{p}\sum_{m} [\phi_\alpha(x),H_I{}^{(m)}(\phi_\beta(y))]=0 \quad \text{for} \quad x\sim y. \tag{3.2$'$}$$

Since no cancellation is possible in (3.1′) and (3.2′) among the terms with different suffixes α, (m), and (m'), we are led to the following separate relations:

$$[H_I{}^{(m)}(\phi_\beta(x)),H_I{}^{(m')}(\phi_\beta(y))]=0 \quad \text{for} \quad x\sim y, \tag{3.1$''$}$$

$$[\phi_\alpha(x),H_I{}^{(m)}(\phi_\beta(y))]=0 \quad \text{for} \quad x\sim y, \tag{3.2$''$}$$

where $\alpha=1, 2, \cdots, p$ and $m,m'=1, 2, \cdots$.

Now, $H_I(x)$ in general is given by a polynomial of $\psi(x)$, and by virtue of Theorem 1 it can be written as a sum of monomials of the standard form. Thus, $H_I{}^{(m)}(\phi_\alpha(x))$ takes the form $\{\psi_{\alpha_1},\psi_{\alpha_2},\cdots,\psi_{\alpha_a}\}[\psi_{\beta_1},\psi_{\beta_1'}]\times[\psi_{\beta_2},\psi_{\beta_2'}]\cdots[\psi_{\beta_b},\psi_{\beta_b'}]$. Of great importance is the fact that as a consequence of (1.5) the Green suffixes $\alpha_1,\alpha_2,\cdots,\alpha_a$ of the operators contained in $\{\ \}_a$ are all different, whereas the suffixes β_i and β_i' of the operators in each bracket $[\ ]$ must be equal, i.e., $\beta_i=\beta_i'$. Consequently, the bracket $[\psi_\alpha(x),\psi_\alpha(x)]$ commutes with any quantity at another point y with $x\sim y$. In the following, such a quantity is said to be Bose-like. At any rate the existence of such Bose-like quantities in H_I does not give rise to any restrictions on H_I. We thus turn to the bracket $\{\ \}$. It is also easy to see by considering (1.5) that when we require the condition (3.1″) only, H_I can contain only one kind of curly bracket, i.e., $\{\ \}_p$ in the case of even p, whereas it cannot contain such brackets at all in the case of odd p. When taking into account the condition (3.2″), however, the bracket $\{\ \}_p$ for even p should also be excluded: This is because such a bracket always contains one ψ_α, and so $[\psi_\alpha,\{\psi_\alpha,\cdots\}_p]\neq 0$. Summarizing these results, we get the following.

Theorem 5: $H_I(x)$ consists only of terms of the type $[\psi(x),\psi(x)][\psi(x),\psi(x)]\cdots[\psi(x),\psi(x)]$, where, as before, $\psi(x)$ stands for either $\phi(x)$ or $\phi^\dagger(x)$.[12]

Combining Theorems 4 and 5, we can immediately see that if a state vector at $t=0$, say, is given by a standard state vector $|\{\ \}_a{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_b{}^\dagger|0\rangle$, then the state at $t>0$ will be a linear combination of standard state vectors of the type $|\{\ \}_a{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_{b'}{}^\dagger|0\rangle$ with the *same* value a (b' may take various values in general). Now, regarding a as values to be taken by a certain physical quantity A, we state the above result as the following.

Theorem 6: The quantity A, whose eigenvalue a is given by the number of field operators in the curly

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[9] Y. Takahashi and H. Umezawa, Progr. Theoret. Phys. (Kyoto) **9**, 50 (1953); **9**, 14 (1953).

[10] Since we regard our theory as the nonrelativistic limit of a relativistic theory, we impose the requirements (3.1) and (3.2), which are essentially those of relativistic field theory when $x\sim y$ is interpreted as spacelike separation. We note in this connection that in nonrelativistic theory proper, $H_I(x)$ may be an integral of a polynomial of ψ over a certain spatial domain. If, however, there exist no long-range forces, and if the spatial distance between x and y is taken to be sufficiently large, then the above requirements (3.1) and (3.2) must hold true in such a case also.

[11] H. Araki, O. W. Greenberg, and J. S. Toll, Phys. Rev. **142**, 1017 (1966).

[12] This is essentially the theorem of Umezawa *et al.* applied to field theory in the space $\mathcal{B}$ [H. Umezawa, J. Podolanski, and S. Oneda, Proc. Phys. Soc. (London) **A68**, 503 (1955)].

bracket of standard state vectors, is a constant of motion (absolute conservation law).[13]

We then ask ourselves the question, how the quantities A of two separate systems are to be added to give the same quantity A of the whole system. For this purpose, let us consider two spatially separated systems whose states are described by standard state vectors

$$|\{\ \}_{a_1}{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_{b_1}{}^\dagger|0\rangle$$

and

$$|\{\ \}_{a_2}{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_{b_2}{}^\dagger|0\rangle,$$

respectively. The state vector for the whole system is given by a linear combination of the direct products of these two state vectors, i.e.,

$$|\{\ \}_{a_1}{}^\dagger\{\ \}_{a_2}{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_{b_1+b_2}{}^\dagger|0\rangle$$

and

$$|\{\ \}_{a_2}{}^\dagger\{\ \}_{a_1}{}^\dagger[\ ]^\dagger[\ ]^\dagger\cdots[\ ]_{b_1+b_2}{}^\dagger|0\rangle.$$

Thus, our problem is reduced to finding a formula with which to express $\{\ \}_{a_1}\{\ \}_{a_2}$ as a sum of terms, each containing only one curly bracket. To this end we first prove the following relation:

$$\{x_1,x_2,\cdots,x_{a_1}\}^\dagger\{y_1,y_2,\cdots,y_{a_2}\}^\dagger=\sum_{b=0}^{\min(a_1,a_2)}\sum_{[l_1,l_2,\cdots,l_b;m_1,m_2,\cdots,m_b]}C_{a_1,a_2,b}\{x_1,x_2,\cdots,(x_{l_1}),\cdots,(x_{l_2}),\cdots,(x_{l_b}),\cdots,x_{a_1},y_1,y_2,\cdots,(y_{m_1}),\cdots,(y_{m_2}),\cdots,(y_{m_b}),\cdots,y_{a_2}\}^\dagger\prod_{i=1}^{b}[x_{l_i},y_{m_i'}]^\dagger,\quad(3.3)$$

the C's being constants. The notations (x_{l_i}) in the curly brackets mean that these letters x_{l_i} are to be omitted from the curly brackets, and $\sum_{[l_1,l_2,\cdots l_b;\ m_1,m_2,\cdots,m_b]}$ means the double summation, i.e., first one sums over all possible pairs (l_1,m_1'), (l_2,m_2'), $\cdots$, (l_b,m_b'), where l_i and m_i' are taken from the sets $(l_1,l_2,\cdots,l_b)$ and $(m_1,m_2,\cdots,m_b)$, respectively, and then one sums over all possible sets $(l_1,l_2,\cdots,l_b)$ and $(m_1,m_2,\cdots,m_b)$ which are chosen from among $(x_1,x_2,\cdots,x_{a_1})$ and $(y_1,y_2,\cdots,y_{a_2})$, respectively.[14] The proof of (3.3) runs as follows. Let us assume that (3.3) holds true up to certain values of a_1 and a_2, and then show that it holds true for a_1+1, a_2, and a_1,a_2+1. Now, by replacing all ψ in (2.4) with $\phi^\dagger$ and taking into account the Hermitian conjugate of (1.1c), we obtain the relation

$$\{x_1,x_2,\cdots,x_m,x_{m+1}\}^\dagger=(m+1)\phi^\dagger(x_{m+1})\{x_1,x_2,\cdots,x_m\}^\dagger+\tfrac{1}{2}m(m+1)\sum_{j=1}^{m}\{x_1,x_2,\cdots,(x_j),\cdots,x_m\}^\dagger[x_j,x_{m+1}]^\dagger.\quad(3.4)$$

By applying this relation to $\{x_1,\cdots\}_{a_1+1}{}^\dagger$, we obtain

$$\{x_1,x_2,\cdots,x_{a_1},x_{a_1+1}\}^\dagger\{y_1,y_2,\cdots,y_{a_2}\}^\dagger=(a_1+1)\phi^\dagger(x_{a_1+1})\{x_1,x_2,\cdots,x_{a_1}\}^\dagger\{y_1,y_2,\cdots,y_{a_2}\}^\dagger+\tfrac{1}{2}a_1(a_1+1)\sum_{j=1}^{a_1}\{x_1,x_2,\cdots,(x_j),\cdots,x_{a_1}\}^\dagger\{y_1,y_2,\cdots,y_{a_2}\}^\dagger[x_j,x_{a_1+1}]^\dagger.\quad(3.5)$$

The first term on the right-hand side of (3.5) can be transformed by our assumption as follows:

$$(a_1+1)\sum_{b=0}^{\min(a_1,a_2)}\sum_{[l_1,\cdots,l_b;\ m_1,\cdots,m_b]}C_{a_1,a_2,b}\phi^\dagger(x_{a_1+1})\{x_1,x_2,\cdots,(x_{l_1}),\cdots,(x_{l_b}),\cdots,x_{a_1},y_1,y_2,\cdots,(y_{m_1}),\cdots,(y_{m_b}),\cdots,y_{a_2}\}^\dagger{}_{a_1+a_2-2b}\prod_{i=1}^{b}[x_{l_i},y_{m_i'}]^\dagger$$

[13] We may apply to our theory of a single para-Fermi field the selection rules proposed by Greenberg and Messiah (Ref. 2), which are based on the condition (3.1) only, and which are expressed by $N_f\cong0$, $N^{(1)}\cong N^{(2)}\cong\cdots\cong N^{(p)}$, where $N_f(N^{(\alpha)})$ is the total number of external Fermi-like particles (that of external paraparticles with the Green suffix α) in any reactions, and the symbol $\cong$ means equality modulo 2. The conservation law of A, on the other hand, is based on the conditions (3.1) and (3.2). Thus, it may generally be expected that the latter is more restrictive than the former. It can be shown that this in fact is true for the case p=even: The condition (3.2) gives rise to additional restrictions. In the case p=odd, however, the conservation law of A is equivalent to the selection rules of Greenberg and Messiah; for it can be shown (for a single para-Fermi field) that the condition (3.2) is actually implied by the condition (3.1). It is also possible to give a direct proof of this equivalence.

[14] The relation (3.3) is equivalent to the expansion of an outer product of two Young diagrams consisting of only one row with a_1 and a_2 boxes, respectively, in terms of those consisting of two rows with a_1+a_2-b and b boxes, respectively, where $b=0, 1, 2, \cdots$, $\min(a_1,a_2)$.

[by applying (3.4) again]

$$=(a_1+1)\sum_{b=0}^{\min(a_1,a_2)}\sum_{[l_1,\cdots,l_b;\ m_1,\cdots,m_b]} C_{a_1,a_2,b}\Big((a_1+a_2-2b+1)^{-1}\{x_1,x_2,\cdots,(x_{l_1}),\cdots,(x_{l_b}),\cdots,x_{a_1},x_{a_1+1},y_1,\cdots,(y_{m_1}),\cdots,(y_{m_b}),\cdots,y_{a_2}\}^\dagger{}_{a_1+a_2-2b+1}+\tfrac{1}{2}(a_1+a_2-2b)\sum_j\{x_1,\cdots,(x_{l_1}),\cdots,(x_j),\cdots,(x_{l_b}),\cdots,x_{a_1},y_1,\cdots,(y_{m_1}),\cdots,(y_{m_b}),\cdots,y_{a_2}\}^\dagger{}_{a_1+a_2-2b-1}[x_{a_1+1},x_j]^\dagger+\tfrac{1}{2}(a_1+a_2-2b)\sum_k\{x_1,\cdots,(x_{l_1}),\cdots,(x_{l_b}),\cdots,x_{a_1},y_1,\cdots,(y_{m_1}),\cdots,(y_k),\cdots,(y_{m_b}),\cdots,y_{a_2}\}^\dagger{}_{a_1+a_2-2b-1}[x_{a_1+1},y_k]^\dagger\Big)\prod_{i=1}^{b}[x_{l_i},y_{m_{i'}}]^\dagger. \quad (3.5')$$

Now, the sum of the second term on the right-hand side of (3.5) and the second term in the bracket ( ) of (3.5)′ vanishes, because the whole expression must be symmetrical with respect to x_j and x_{a_1+1}. Therefore, we are left with an expression for (3.5), which has the same form as (3.3) but with a_1 replaced by a_1+1. The proof that the relation (3.3) holds true for a_1 and a_2+1 can be performed in a similar manner. For $a_1=a_2=1$, the relation is trivially true; $\{x_1\}^\dagger\{y_1\}^\dagger=\frac{1}{2}\{x_1,y_1\}^\dagger+\frac{1}{2}[x_1,y_1]^\dagger$. Thus, by mathematical induction we conclude that (3.3) holds true in general.

Now, the right-hand side of (3.3) can be rewritten in the form

$$\sum_{b=0}^{\min(a_1,a_2)} C_{a_1,a_2,b}\frac{1}{(a_1-b)!(a_2-b)!(b!)^2}\Big(\sum_{\substack{\text{all perm. of }(x_1,x_2,\cdots,x_{a_1})\\ \text{and }(y_1,y_2,\cdots,y_{a_2})}}\{x_1,x_2,\cdots,x_{a_1-b},y_1,y_2,\cdots y_{a_2-b}\}^\dagger \times[x_{a_1-b+1},y_{a_2-b+1}]^\dagger[x_{a_1-b+2},y_{a_2-b+2}]^\dagger\cdots[x_{a_1},y_{a_2}]^\dagger\Big). \quad (3.3')$$

However, as will be proved in Appendix C, there holds, in general, the following relation:

$$\sum_{\substack{\text{all perm. of }(x_1,x_2,\cdots,x_{k+b})\\ \text{and }(y_1,y_2,\cdots,y_{m+b})}}\{x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_m\}^\dagger[x_{k+1},y_{m+1}]^\dagger[x_{k+2},y_{m+2}]^\dagger\cdots[x_{k+b},y_{m+b}]^\dagger=0, \quad (3.6)$$

where $p<(k+m)+b$. The relation (3.6) implies that the expression inside the bracket ( ) of (3.3′) vanishes for b such that $b<a_1+a_2-p$. Therefore, the lower bound of the summation over b in (3.3′) is given by $\max(a_1+a_2-p,\ 0)$, and the final expression for (3.3) reads

$$\{x_1,x_2,\cdots,x_{a_1}\}^\dagger\{y_1,y_2,\cdots,y_{a_2}\}^\dagger=\sum_{b=\max(a_1+a_2-p,0)}^{\min(a_1,a_2)} C'_{a_1,a_2,b}\Big(\sum_{\substack{\text{all perm. of }(x_1,x_2,\cdots,x_{a_1})\\ \text{and }(y_1,y_2,\cdots,y_{a_2})}}\{x_1,x_2,\cdots,x_{a_1-b},y_1,y_2,\cdots,y_{a_2-b}\}^\dagger \times[x_{a_1-b+1},y_{a_2-b+1}]^\dagger[x_{a_1-b+2},y_{a_2-b+2}]^\dagger\cdots[x_{a_1},y_{a_2}]^\dagger\Big). \quad (3.3'')$$

We remark here that (i) the expression inside the bracket ( ) of (3.3″) with $\min(a_1,a_2)\geqslant b\geqslant\max(a_1+a_2-p,\ 0)$ does not vanish thanks to Theorem 3, which holds true for monomials of $\phi^\dagger$ of standard form as well, and (ii) the p-independent, numerical coefficients $C'_{a_1,a_2,b}$ are all greater than zero, as is clear from the way of their construction. The relation (3.3″), together with the above two remarks supplies us with the required rule of addition of the quantity A. Since the number of operators inside the curly brackets of (3.3″) is given by a_1+a_2-2b with b as restricted above, we arrive at the following.

Theorem 7: The quantity A of the whole system consisting of two subsystems with $A=a_1$ and a_2, respectively, takes the values $|a_1-a_2|$, $|a_1-a_2|+2$, $|a_1-a_2|+4$, $\cdots$, $\min[(2p-a_1-a_2),\ (a_1+a_2)]$.

The above rule resembles that of addition of (twice the magnitudes of) angular momenta. Combining Theorems 6 and 7, we can derive absolute selection rules for reactions concerning the conservation of the quantity A. For example, in the para-Fermi quark theory ($p=3$) we see that the reactions $0+0\to0+0$, $1+1$, $2+2$, $3+3$ are all allowed, but $0+0\to3+1$, $2+0$ are absolutely forbidden, where the numbers on both sides of the arrow indicate the values of A for the respective systems.

4. STATISTICAL PROPERTIES OF BOUND STATES

We have seen in the preceding section that the quantity A is a constant of motion (Theorem 6). On the other hand, bound states are eigenstates of Hamiltonians. Therefore, the quantity A may be used as a parameter to label each bound state. Let $B^{(a,b)}$ be a bound state consisting of n $(=a+2b)$ particles such that

$$|B^{(a,b)}\rangle=\int dx\,dr_1dr_2\cdots dr_{n-1}f(x,r_1,r_2,\cdots,r_{n-1})\,|\{\phi^\dagger(x_1),\phi^\dagger(x_2),\cdots,\phi^\dagger(x_a)\}\prod_{i=1}^{b}[\phi^\dagger(y_i),\phi^\dagger(z_i)]|0\rangle, \quad (4.1)$$

where f is the wave function, and x and r_i ($i=1, 2, \cdots, n-1$) are the center-of-mass and relative coordinates, respectively.[15] Our problem now is to study the statistics obeyed by this bound state $B^{(a,b)}$. To this end, we consider the asymptotic field operator $B^{(a,b)}(x,t)^{\text{in}}$ which is given by

$$B^{(a,b)}(x,t)^{\text{in}\dagger}=\int dr_1dr_2\cdots dr_{n-1}(f(x,r_1,r_2,\cdots,r_{n-1},t)\{\phi^\dagger(x_1,t),\phi^\dagger(x_2,t),\cdots,\phi^\dagger(x_a,t)\}\prod_{i=1}^{b}[\phi^\dagger(y_i,t),\phi^\dagger(z_i,t)])^{\text{in}}. \quad (4.2)$$

The statistical properties are then determined by the commutation relations to be satisfied by the in-field operator $B^{(a,b)}(x,t)^{\text{in}}$.

We apply now the Green decomposition (1.4) to each field $\phi^\dagger$ in (4.2) to obtain

$$B^{(a,b)}(x,t)^{\text{in}\dagger}=\Big(\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_a\\ \text{(all different)}}} B^{(b)}{}_{(\alpha_1,\alpha_2,\cdots,\alpha_a)}(x,t)^{\text{in}\dagger}\Big)P, \quad (4.3)$$

where

$$B^{(b)}{}_{(\alpha_1,\alpha_2,\cdots,\alpha_a)}(x,t)^{\text{in}\dagger}\equiv\sum_{\beta_1,\beta_2,\cdots,\beta_b}\int dr_1dr_2\cdots dr_{n-1}(f(x,r_1,r_2,\cdots,r_{n-1},t)$$
$$\times\{\phi_{\alpha_1}{}^\dagger(x_1,t),\phi_{\alpha_2}{}^\dagger(x_2,t),\cdots,\phi_{\alpha_a}{}^\dagger(x_a,t)\}\prod_{i=1}^{b}[\phi_{\beta_i}{}^\dagger(y_i,t),\phi_{\beta_i}{}^\dagger(z_i,t)])^{\text{in}}. \quad (4.4)$$

Now, the field $B^{(b)}{}_{(\alpha_1,\alpha_2,\cdots,\alpha_a)}(x,t)^{\text{in}}$ is an in-field for a bound state consisting of ordinary fermions in the space $\mathfrak{G}$. Thus, as in (I), we can apply to this field the argument of Redmond and Uretsky,[16] which states in effect that when we choose a suitable normalization for this field, the commutation relations read

$$[B^{(b)}{}_{(\alpha_1,\alpha_2,\cdots,\alpha_a)}(x,t)^{\text{in}},B^{(b)}{}_{(\alpha_1',\alpha_2',\cdots,\alpha_a')}(y,t)^{\text{in}\dagger}]_{\epsilon(\alpha_1,\alpha_2,\cdots,\alpha_a;\ \alpha_1',\alpha_2',\cdots,\alpha_a')}=\delta(x-y)\sum_{\substack{k_1,k_2,\cdots,k_a\text{ over all}\\ \text{perm. of }(1,2,\cdots,a)}}\prod_{i=1}^{a}\delta(\alpha_i,\alpha_{k_i}'), \quad (4.5)$$

$$[B^{(b)}{}_{(\alpha_1,\alpha_2,\cdots,\alpha_a)}(x,t)^{\text{in}},B^{(b)}{}_{(\alpha_1',\alpha_2',\cdots,\alpha_a')}(y,t)^{\text{in}}]_{\epsilon(\alpha_1,\alpha_2,\cdots,\alpha_a';\alpha_1'\alpha_2',\cdots,\alpha_a')}=0, \quad (4.6)$$

where

$$\epsilon(\alpha_1,\alpha_2,\cdots,\alpha_a;\ \alpha_1',\alpha_2',\cdots,\alpha_a')\equiv-\prod_{i,j=1}^{a}(1-2\delta_{\alpha_i\alpha_j'}). \quad (4.7)$$

It is to be noted here that as seen from (4.7), the presence of b pairs of operators $[\phi^\dagger(y_i,t),\phi^\dagger(z_i,t)]$ in (4.2) and (4.4) does not affect the commutation properties (4.5) and (4.6). This is due to the Bose-like character of these pairs of operators. Since the commutation relations, which specify the statistical property of the fields $B^{(a,b)}(x,t)^{\text{in}}$, are essentially determined by the relations (4.3) and (4.5)–(4.7), it follows that the statistical property of the bound state $B^{(a,b)}$ is the same as that of the bound state $B^{(a,0)}$. This result therefore leads to the following.

Theorem 8: In a para-Fermi theory of order p, statistics which any bound states obey is classified into $(p+1)$ categories $B^{(a)}$ ($a=0, 1, 2, \cdots, p$), each of which is characterized by the statistical property of the bound states $B^{(a,0)}$.

Now, the commutation properties of the operator $B^{(a,0)\text{in}}$ are determined by the operators

$$B^{(0)}{}_{(\alpha_1,\alpha_2,\cdots,\alpha_a)}(x,t)^{\text{in}},$$

whose properties, in turn, are solely dependent upon the Green suffixes $(\alpha_1,\alpha_2,\cdots,\alpha_a)$, and not on specific forms of the wave function $f(x,r_1,r_2,\cdots,r_{n-1})$, as seen from (4.5) and (4.6). So we shall now pay attention only to the operator

$$\tilde{B}^{(a,0)\dagger}\equiv\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_a\\ \text{(all different)}}}\{\phi_{\alpha_1}{}^\dagger\phi_{\alpha_2}{}^\dagger\cdots\phi_{\alpha_a}{}^\dagger\}^{\text{in}}, \quad (4.8)$$

where we have omitted the arguments x_i of $\phi^\dagger$. Now, $\tilde{B}^{(a,0)}$ can be rewritten as

$$\tilde{B}^{(a,0)\dagger}=\sum_{\alpha_1<\alpha_2<\cdots<\alpha_a}(\phi_{\alpha_1}{}^\dagger\phi_{\alpha_2}{}^\dagger\cdots\phi_{\alpha_a}{}^\dagger)^{\text{in}}$$
$$=\sum_{\alpha_1<\alpha_2<\cdots<\alpha_{p-a}}\left(\frac{\delta^{p-a}}{\delta\phi_{\alpha_1}{}^\dagger\delta\phi_{\alpha_2}{}^\dagger\cdots\delta\phi_{\alpha_{p-a}}{}^\dagger}C\right)^{\text{in}}, \quad (4.9)$$

when the operator C is defined by

$$C\equiv(\phi_1{}^\dagger\phi_2{}^\dagger\cdots\phi_p{}^\dagger), \quad (4.10)$$

and the derivative operators satisfy the following commutation relations:

$$\left[\frac{\delta}{\delta\phi_\alpha{}^\dagger(x)},\frac{\delta}{\delta\phi_{\alpha'}{}^\dagger(y)}\right]_{\epsilon(\alpha,\alpha')}=0,$$
$$\left[\frac{\delta}{\delta\phi_\alpha{}^\dagger(x)},\phi_{\alpha'}{}^\dagger(y)\right]_{\epsilon(\alpha,\alpha')}=0\quad\text{for}\quad x\sim y. \quad (4.11)$$

Let $\phi_\alpha{}^\dagger$ ($\phi_\alpha{}^{\dagger\prime}$), $\delta/\delta\phi_\alpha{}^\dagger$ ($\delta/\delta\phi_\alpha{}^{\dagger\prime}$), and C (C') be the quantities referring to points in a spatial region V (V') and let V and V' be completely separated from each other. Then, we have the relations

$$[C,C']_{\mp}=0, \quad (4.12)$$

[15] Here, for simplicity, we are considering the case in which the particle number is conserved. However, for reasons which will become clear later the argument of this section can easily be extended to more general cases in which the particle number or the value b is not conserved.

[16] P. J. Redmond and J. L. Uretsky, Ann. Phys. (N. Y.) 9, 106 (1960).

where the upper and lower signs correspond to even and odd p, respectively. Multiplying (4.12) from the left by the operators $\delta^{p-a}/\delta\phi_{\alpha_1}{}^\dagger\delta\phi_{\alpha_2}{}^\dagger\cdots\delta\phi_{\alpha_{p-a}}{}^\dagger$ and $\delta^{p-a}/\delta\phi_{\alpha_1'}{}^{\dagger'}\delta\phi_{\alpha_2'}{}^{\dagger'}\cdots\delta\phi_{\alpha_{p-a}'}{}^{\dagger'}$, we obtain, by use of (4.11),

$$\left[\frac{\delta^{p-a}}{\delta\phi_{\alpha_1}{}^\dagger\delta\phi_{\alpha_2}{}^\dagger\cdots\delta\phi_{\alpha_{p-a}}{}^\dagger}C, \frac{\delta^{p-a}}{\delta\phi_{\alpha_1'}{}^{\dagger'}\delta\phi_{\alpha_2'}{}^{\dagger'}\cdots\delta\phi_{\alpha_{p-a}'}{}^{\dagger'}}C'\right]_\epsilon=0, \quad (4.13)$$

where

$$\epsilon=(-1)^p\epsilon(\alpha_1,\alpha_2,\cdots,\alpha_{p-a};\,\alpha_1',\alpha_2',\cdots,\alpha_{p-a}'). \quad (4.14)$$

From (4.9), (4.13), (4.14), (4.5), (4.6), and (4.7) we see that in the case of even p the commutation property of $\tilde{B}^{(a,0)}$ is the same as that of $\tilde{B}^{(p-a,0)}$, and in the case of odd p it is the same as that of $\tilde{B}^{(p-a)}F$, where F is an arbitrary Fermi field which is independent of and commutes (or anticommutes) with ϕ. This implies that the statistical property of the bound state $B^{(a,0)}$ is the same as that of $B^{(p-a,0)}$ $((B^{(p-a,0)}F))$ for even (odd) p, where $(B^{(p-a,0)}F)$ means the bound state consisting of $B^{(p-a,0)}$ and F.[17] We note that if the state $(B^{(a,0)}F)$ obeys para-Bose (para-Fermi) statistics of order $\tilde{p}$, then the state $B^{(a,0)}$ obeys the para-Fermi (para-Bose) statistics of order $\tilde{p}$. After all, among the $(p+1)$ categories $B^{(a)}$ stated in Theorem 8, about half of them are actually of the same statistical property as the other half, and for our present purpose it, therefore, suffices to consider the categories $B^{(a)}$ with $a=0, 1, 2, \cdots, q\leqslant\frac{1}{2}p$.

Now, the statistical property of the state $B^{(0,0)}$ is specified by the commutation property of the operator $\tilde{B}^{(0,0)}=1$ defined by (4.8). Thus we see that $B^{(0,0)}$ obeys Bose statistics, and $B^{(p,0)}$ obeys Bose or Fermi statistics according to whether p is even or odd. Next, we shall consider the case of $B^{(a,0)}$ with $a\neq 0$ or p. The expansion (4.3) of the in-field $B^{(a,0)\text{in}}$ takes the following form:

$$B^{(a,0)\text{in}}=\Big(\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_a\\ \text{(all different)}}} B^{(0)}{}_{(\alpha_1,\alpha_2,\cdots,\alpha_a)}{}^{\text{in}}\Big)P. \quad (4.15)$$

For the reason mentioned above, we have only to consider the case of $0\neq a\leqslant\frac{1}{2}p$. Then, we find that inside the bracket on the right-hand side of (4.15) there coexist mutually commuting components such as $B^{(0)}{}_{(1,2,\cdots,a)}{}^{\text{in}}$ and $B^{(0)}{}_{(p-a+1,\,p-a+2,\,\cdots,\,p-1,p)}{}^{\text{in}}$, and, at the same time, mutually anticommuting components such as $B^{(0)}{}_{(1,2,\cdots,a)}{}^{\text{in}}$ and $B^{(0)}{}_{(1,\,p-a+2,\,p-a+3,\,\cdots,\,p-1,p)}{}^{\text{in}}$. This means that the operator $B^{(a,0)\text{in}}$ does not have any simple, bilinear commutation or anticommutation relations. In other words, it obeys neither Bose nor Fermi statistics. Thus, it can be deduced that in the case of even p, there cannot exist any bound states which obey Fermi statistics.

All the results obtained above may be summarized in the following.

Theorem 9: Among the $(p+1)$ categories $B^{(a)}$ $(a=0, 1, 2, \cdots, p)$, the category $B^{(a)}$ has the same statistical property as that of $B^{(p-a)}$ $((B^{(p-a)}F))$ for even (odd) p, where the category $(B^{(p-a)}F)$ is characterized by the statistical property of the bound state consisting of $B^{(p-a,0)}$ and an arbitrary fermion F. The statistics of the category $B^{(0)}$ is the ordinary Bose statistics, and that of the category $B^{(p)}$ is the ordinary Bose or Fermi statistics, according to whether p is even or odd. The statistics of all other categories $B^{(a)}$ $(a\neq 0, p)$ must be something other than the ordinary Bose or Fermi statistics.

Theorem 9': In a para-Fermi theory of even p, no bound states are possible which obey the ordinary Fermi statistics.

According to Theorem 9 the value of the quantity A of a single fermion or boson is either 0 or p. An application of Theorem 7 shows that the value of the same quantity of a system consisting of an arbitrary number of fermions and bosons is again 0 or p (since the rule of addition of A is such that $0+0\rightarrow 0$, $p+0\rightarrow p$, and $p+p\rightarrow 0$). On the other hand, Theorem 9 says that the quantity A of any particle which obeys statistics other than the usual ones must take the value a such that $0<a<p$. When this a is added to 0 and p, the resultant values of A are a and $(p-a)$, respectively. By applying Theorem 6 we thus obtain a selection rule that reactions containing only one incoming or outgoing particle which obeys statistics other than Fermi or Bose statistics are absolutely forbidden. This is a generalization of the selection rule we have used before[2] in determining statistical types of presently known particles and resonances.

5. STATISTICALLY CLOSED PROPERTY OF PARA-FERMI FIELD THEORY

In a para-Fermi theory of order p, statistics of any bound states must fall in one of the categories $B^{(a)}$, where $a=0, 1, 2, \cdots, q\leqslant\frac{1}{2}p$ (Theorems 8 and 9). When the statistics of any of these categories corresponds to the well-known parastatistics of a certain order, i.e., when all possible bound states can be described by the ordinary parafields, then we say that such a theory is *statistically closed*. In this section, we shall study this property on the basis of the results of (I) and the theorems obtained in the preceding sections, and examine the case of each value of p separately.

(i) $p=1$. We have two categories $B^{(0)}$ and $B^{(1)}$, which correspond to Bose and Fermi statistics, respectively. In other words, bound states consisting of even (odd) numbers of particles obey Bose (Fermi) statistics. This is the well-known theorem of Ehrenfest

[17] This situation resembles that of hole theory: The state $C|0\rangle$ may be regarded as the *saturated* state, and such a state, but with $(p-a)$ holes in it, behaves in the same way as a state of a particles.

TABLE I. Statistical types of categories for $p=3$.

Category	$B^{(0)}$	$B^{(1)}$	$B^{(2)}$	$B^{(3)}$
Statistics	Bose	para-Fermi of order 3	para-Bose of order 3	Fermi

and Oppenheimer.[18] Hence, the theory is statistically closed.

(ii) $p=2$. From Theorem 9′, we see that there are no bound states obeying Fermi statistics. We have three categories $B^{(0)}$, $B^{(1)}$, and $B^{(2)}$. Owing to Theorem 8 the categories $B^{(0)}$ and $B^{(2)}$ correspond to Bose statistics, and $B^{(1)}$ to para-Fermi statistics of order 2. Furthermore, we can prove the following equal-time commutation relations:

$$[B^{(\alpha,0)}(x)^{\text{in}},B^{(1,0)}(y)^{\text{in}}]=[B^{(\alpha,0)}(x)^{\text{in}},B^{(1,0)}(y)^{\text{in}\dagger}]=0 \text{ for } x\sim y, \quad (5.1)$$

where $\alpha=0$, 2. The theory is thus closed in the above sense.

(iii) $p=3$. There are four categories $B^{(0)}$, $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$. The statistics corresponding to each category can be found by considering Eqs. (4.3)–(4.6), and the result is listed in Table I.

By means of the same relations we can further show that the following equal-time commutation relations hold true between in-field operators of different categories:

$$[B^{(0,0)}(x)^{\text{in}},B^{(a,0)}(y)^{\text{in}}]=[B^{(0,0)}(x)^{\text{in}},B^{(a,0)}(y)^{\text{in}\dagger}]=0, \quad (a=1,2,3) \quad (5.2)$$

$$[B^{(3,0)}(x)^{\text{in}},B^{(2,0)}(y)^{\text{in}}]=[B^{(3,0)}(x)^{\text{in}},B^{(2,0)}(y)^{\text{in}\dagger}]=0, \quad (5.3)$$

$$\{B^{(3,0)}(x)^{\text{in}},B^{(1,0)}(y)^{\text{in}}\}=\{B^{(3,0)}(x)^{\text{in}},B^{(1,0)}(y)^{\text{in}\dagger}\}=0, \quad (5.4)$$

$$[B^{(2,0)}(x)^{\text{in}},[B^{(1,0)}(y)^{\text{in}},B^{(1,0)}(z)^{\text{in}}]]=[B^{(2,0)}(x)^{\text{in}},[B^{(1,0)}(y)^{\text{in}\dagger},B^{(1,0)}(z)^{\text{in}}]]=[B^{(2,0)}(x)^{\text{in}\dagger},[B^{(1,0)}(y)^{\text{in}},B^{(1,0)}(z)^{\text{in}}]]=0, \quad (5.5)$$

$$[B^{(1,0)}(x)^{\text{in}},\{B^{(2,0)}(y)^{\text{in}},B^{(2,0)}(z)^{\text{in}}\}]=[B^{(1,0)}(x)^{\text{in}},\{B^{(2,0)}(y)^{\text{in}\dagger},B^{(2,0)}(z)^{\text{in}}\}]=[B^{(1,0)}(x)^{\text{in}\dagger},\{B^{(2,0)}(y)^{\text{in}},B^{(2,0)}(z)^{\text{in}}\}]=0 \quad \text{for } x\sim y,z. \quad (5.6)$$

When there are two kinds of bound states $B^{(a,b)}$ and $B^{(a,b')}$, where $a=2$, 3, then it can be shown that both

$$[B^{(a,b)}(x)^{\text{in}},B^{(a,b)}(y)^{\text{in}}]_{\epsilon(a)}$$

and

$$[B^{(a,b)}(x)^{\text{in}\dagger},B^{(a,b)}(y)^{\text{in}}]_{\epsilon(a)}$$

are commutable with $B^{(a,b')}(z)^{\text{in}}$ or $B^{(a,b')}(z)^{\text{in}\dagger}$ for $x,y\sim z$, where $\epsilon(a)=(-1)^a$. At any rate the theory is statistically closed in this case also.

In this connection we shall make one remark on the asymptotic form of the Hamiltonian H^{in}. In the cases of $p\leqslant 3$, we can prove that when the in-field operators are suitably normalized, H^{in} takes the following form:

$$H^{\text{in}}=\frac{1}{2}\int d^3x \sum_{a,b}\sum_k \frac{1}{2M_k^{(a,b)}}\times[\nabla B_k^{(a,b)}(x,t)^{\text{in}\dagger},\nabla B_k^{(a,b)}(x,t)^{\text{in}}]_{\epsilon(a)}+c \text{ number}, \quad (5.7)$$

where k is the label to distinguish bound states of the kind $B^{(a,b)}$. In fact, as shown in (I), between the in-field and H^{in} there must hold the relation

$$i\dot{B}_k^{(a,b)}(x,t)^{\text{in}}=[B_k^{(a,b)}(x,t)^{\text{in}},H^{\text{in}}]=\frac{-1}{2M_k^{(a,b)}}\Delta B_k^{(a,b)}(x,t)^{\text{in}}. \quad (5.8)$$

Therefore, if we put

$$H^{\text{in}}=\frac{1}{2}\int d^3x \sum_{a,b}\sum_k \frac{1}{2M_k^{(a,b)}}\times[\nabla B_k^{(a,b)}(x,t)^{\text{in}\dagger},\nabla B_k^{(a,b)}(x,t)^{\text{in}}]_{\epsilon(a)}+R,$$

and use the commutation relations (5.1)–(5.6) and (5.8), we then obtain

$$[B_k^{(a,b)}(x,t)^{\text{in}},R]=0, \quad (5.9)$$

which implies that R is a c number. Hence, the expression (5.7) follows.

(iv) $p=4$. By using the results of (I) and Theorems 8 and 9, each of the five categories, when considered alone, has the statistical property shown in Table II. However, we can show that although $[B^{(1,0)}(x)^{\text{in}\dagger},B^{(1,0)}(y)^{\text{in}}]$ commutes with any $B^{(a,0)}(z)^{\text{in}}$, $\{B^{(2,0)}(x)^{\text{in}\dagger},B^{(2,0)}(y)^{\text{in}}\}$ does not commute with $B^{(1,0)}(z)^{\text{in}}$ [cf. (I)] for $x,y\sim z$. Therefore, when two bound states $B^{(1,0)}$ and $B^{(2,0)}$ coexist, something very unexpected happens. Let the asymptotic Hamiltonian H^{in} be written in the form

$$H^{\text{in}}=\frac{1}{2}\int d^3x\left(\frac{1}{2M^{(1)}}[\nabla B^{(1,0)}(x,t)^{\text{in}\dagger},\nabla B^{(1,0)}(x,t)^{\text{in}}]+\frac{1}{2M^{(2)}}\{\nabla B^{(2,0)}(x,t)^{\text{in}\dagger},\nabla B^{(2,0)}(x,t)^{\text{in}}\}\right)+R. \quad (5.10)$$

Then, R in this case is no longer a c number, but takes a very complicated form when expressed in terms of $B^{(1,0)\text{in}}$ and $B^{(2,0)\text{in}}$ and their Hermitian conjugates [see Eq. (39) of (I)].[19] More generally, when several bound states are possible, which belong to different categories, the description of their asymptotic fields will take a form quite different from that of the ordinary parafield theory. We conclude therefore that the theory with $p=4$ is not statistically closed.

[18] P. Ehrenfest and J. R. Oppenheimer, Phys. Rev. **37**, 333 (1931). A more rigorous proof of this theorem in relativistic quantum field theory can be given by employing the method of Nishijima and Zimmermann: K. Nishijima, Phys. Rev. **111**, 995 (1958); W. Zimmermann, Nuovo Cimento **10**, 597 (1958); R. Haag, Phys. Rev. **112**, 669 (1958).

[19] It is to be remarked that free Hamiltonians in parafield theory must, by definition, take the bilinear form such as given by (5.7).

(v) $p\geqslant 5$. Here, the situation becomes worse than the case $p=4$. For example, $B^{(2,0)\text{in}}$ in this case does not satisfy the trilinear commutation relation which is required for para-Bose fields;

$$[B^{(2,0)}(x)^{\text{int}\dagger},\{B^{(2,0)}(y)^{\text{int}\dagger},B^{(2,0)}(z)^{\text{int}\dagger}\}]\neq 0. \quad (5.11)$$

Now, by using (4.3) and (4.6), the left-hand side of (5.11) can be expanded in the form

$$4^2 \sum_{\substack{\alpha,\beta,\cdots=1\\ \text{(all different)}}}^{p} (B_{(\alpha\beta)}(x)^{\text{int}\dagger}B_{(\beta\gamma)}(y)^{\text{int}\dagger}B_{(\delta\rho)}(z)^{\text{int}\dagger} + B_{(\alpha\beta)}(x)^{\text{int}\dagger}B_{(\beta\gamma)}(z)^{\text{int}\dagger}B_{(\delta\rho)}(y)^{\text{int}\dagger})P. \quad (5.11')$$

The relation (5.11) can be proved by showing, for example, that the inner product of two state vectors

$$[B^{(2,0)}(x)^{\text{int}\dagger},\{B^{(2,0)}(y)^{\text{int}\dagger},B^{(2,0)}(z)^{\text{int}\dagger}\}]|0\rangle$$

and

$$P(B_{(12)}(x')^{\text{int}\dagger}B_{(23)}(y')^{\text{int}\dagger}B_{(45)}(z')^{\text{int}\dagger})|0\rangle$$

does not vanish, and this can be seen by using the expressions (5.11′), (4.5), and $[B^{(2,0)\text{int}\dagger},P]=0$. This result means that in the case $p\geqslant 5$, the category $B^{(2)}$ does not correspond to the usual parastatistics, or in other words, its asymptotic field $B^{(2,0)\text{in}}$ cannot be described within the framework of the ordinary parafield theory.[20] The theory in this case is not statistically closed. Hence, we have the following.

Theorem 10: The necessary and sufficient condition for a para-Fermi theory of order p to be statistically closed is

$$p\leqslant 3. \quad (5.12)$$

According to Theorem 9′, a para-Fermi theory of $p=2$ cannot supply fermions. Hence, such a theory is not applicable to the fundamental field which is to constitute all hadrons. In this sense a para-Fermi theory of $p=1$ or 3 occupies a very privileged position among para-Fermi theories in general.[21]

6. ADDITIONAL REMARKS

A. Case of More than One Parafield

So far we have been considering a system consisting of only one para-Fermi field of order p. Here, we shall briefly describe what happens when more than one parafield of different order coexist.[22] As we are interested in a statistically closed system, we shall restrict ourselves to $p\leqslant 3$. As has been proved elsewhere,[23] between parafields of different orders we must assume bilinear commutation or anticommutation relations. The case of coexistence of a field of $p=1$ and another field of order $p=2$ or 3 does not lead to any complication. For example, a bound state consisting of one fermion and one parafermion (paraboson) of order p obeys para-Bose (para-Fermi) statistics of the same order.

TABLE II. Statistical types of categories for $p=4$.

Category	$B^{(0)}$	$B^{(1)}$	$B^{(2)}$	$B^{(3)}$	$B^{(4)}$
Statistics	Bose	para-Fermi of order 4	para-Bose of order 3	para-Fermi of order 4	Bose

Thus, the most interesting case is the coexistence of two fields of order 2 and 3. Let χ and ϕ be mutually commuting para-Fermi fields of order 2 and of order 3, respectively. These fields can be expanded in the form

$$\chi=(\sum_{\alpha=1}^{2}\chi_\alpha)P, \quad \phi=(\sum_{i=1}^{3}\phi_i)P.$$

Now, consider a two-body bound state B consisting of χ and ϕ. Its asymptotic field $B(x)^{\text{in}}$ can be written as

$$B(x,t)^{\text{in}}=(\sum_{\alpha=1}^{2}\sum_{i=1}^{3}B_{\alpha i}(x,t)^{\text{in}})P, \quad (6.1)$$

and we have

$$[B(x,t)^{\text{in}},P]=0. \quad (6.2)$$

By arguing in a way similar to that in Sec. 4, we arrive at the following commutation relations:

$$[B_{\alpha i}(x,t)^{\text{in}},B_{\beta j}(y,t)^{\text{in}\dagger}]_{\epsilon(\alpha,\beta;\,i,j)}=\delta_{\alpha\beta}\delta_{ij}\delta(x-y), \quad (6.3)$$

$$[B_{\alpha i}(x,t)^{\text{in}},B_{\beta j}(y,t)^{\text{in}}]_{\epsilon(\alpha,\beta;\,i,j)}=0, \quad (6.4)$$

where $\alpha,\beta=1,2$, $i,j=1,2,3$, and

$$\epsilon(\alpha,\beta;\,i,j)=-(1-2\delta_{\alpha\beta})(1-2\delta_{ij}).$$

By using (6.4) we can easily show

$$[B(x,t)^{\text{in}},\{B(y,t)^{\text{in}},B(z,t)^{\text{in}}\}]\neq 0, \quad (6.5)$$

which implies that the field $B(x,t)^{\text{in}}$ cannot be described by the ordinary parafield theory. Exactly the same conclusion can be reached for any bound state consisting of two paraparticles of order 2 and 3 irrespective of whether each of them obeys para-Fermi or para-Bose statistics, and of whether these two fields commute or anticommute. Therefore, we can say that under the requirement of the system being statistically closed, coexistence of two parafields of order 2 and 3 is not allowed. It can further be shown in the same way that the system of two commutable or anticommutable parafields of the same order p ($p=2$ or 3) is not statistically closed, whether each of them obeys para-Frmie or para-Bose statistics. Thus, we see that for coexistence these two fields must satisfy the trilinear commutation

[20] For the possibility of a further generalization of parafield theory, see S. Kamefuchi and Y. Takahashi, Progr. Theoret. Phys. (Kyoto) Suppl. **37** and **38**, 244 (1966). This kind of generalized theory may be able to accommodate within its framework those bound states which do not obey the ordinary parastatistics.

[21] By a similar argument we can also conclude that the necessary and sufficient condition for a para-Bose theory of order p to be statistically closed is given by (5.12). Of course such a theory is of no practical interest from the point of view of the fundamental field.

[22] It should be emphasized that Theorems 1–10 obtained in the preceding sections are specific to a system consisting of only one para-Fermi field. In cases where more than one parafield coexist these theorems should in general be modified. As for the restrictions on interaction Hamiltonians and the selection rules in the latter case, see Ref. 2.

[23] S. Kamefuchi and J. Strathdee, Nucl. Phys. **42**, 166 (1963).

relations. Summarizing, the cases in which two (or more) different parafields can coexist under our requirement are the following: (i) $p=1$, $p=2$; (ii) $p=1$, $p=3$; (iii) $p=2$, $p=2$; and (iv) $p=3$, $p=3$, where in cases (iii) and (iv) different fields must satisfy trilinear commutation relations.

B. Three-Triplet and Paraquark Models of Hadrons

When one assumes that all hadrons are made up of one fundamental field, and requires that the theory be statistically closed, then the field must necessarily be a para-Fermi field of order 1 or 3. The former possibility corresponds to the three triplet model of Nambu and Miyamoto,[5] and the latter to the paraquark model of Greenberg,[6] both of which were proposed in connection with the static $SU(6)$ theory. We wish here to remark on a formal relationship between these two models.

The three-triplet model is based on $SU(3)\times SU(3)$ invariance. Here, the first $SU(3)$ refers to the usual unitary symmetry and the second to a new symmetry. Three triplets $t_\alpha(x)$ together with their antifields $t^\alpha(x)$ are introduced, where the suffix $\alpha=1, 2, 3$ is related to the second $SU(3)$, and they satisfy the following (equal time) anticommutation relations:

$$\{t_\alpha^\dagger(x),t_\beta(y)\}=\{t^{\alpha\dagger}(x),t^\beta(y)\}=\delta_{\alpha\beta}\delta(x-y)\,,$$
$$\{t_\alpha(x),t_\beta(y)\}=\{t^\alpha(x),t^\beta(y)\}=\{t^\alpha(x),t_\beta(y)\}=\{t^{\alpha\dagger}(x),t_\beta(y)\}=0. \quad (6.6)$$

It is then assumed that hadrons are bound states made up of t which are singlet states with respect to the second $SU(3)$. Thus, the nonrelativistic three-body bound state

$$\sum_{\alpha,\beta,\gamma}\epsilon^{\alpha\beta\gamma}(t_\alpha t_\beta t_\gamma)^{\rm in} \text{ and two-body bound state } \sum_\alpha [t_\alpha,t^\alpha]^{\rm in}$$

are supposed to correspond to baryons and mesons, respectively. In the paraquark model, on the other hand, the quark and antiquark field, to be denoted by $q(x)$ and $\tilde{q}(x)$, respectively, can be expanded into the Green components as follows:

$$q(x)=(\sum_{\alpha=1}^{3}q_\alpha(x))P \quad\text{and}\quad \tilde{q}(x)=(\sum_{\alpha=1}^{3}q^\alpha(x))P.$$

The component fields q_α have the (equal-time) commutation relations

$$[q_\alpha^\dagger(x),q_\beta(y)]_{\epsilon(\alpha,\beta)}=[q^\alpha(x)^\dagger,q^\beta(y)]_{\epsilon(\alpha,\beta)}=\delta_{\alpha\beta}\delta(x-y)\,,$$
$$[q_\alpha(x),q_\beta(y)]_{\epsilon(\alpha,\beta)}=[q^\alpha(x),q^\beta(y)]_{\epsilon(\alpha,\beta)}=[q^\alpha(x),q_\beta(y)]_{\epsilon(\alpha,\beta)}=[q^\alpha(x)^\dagger,q_\beta(y)]_{\epsilon(\alpha,\beta)}=0. \quad (6.7)$$

In this model, bound states belonging to the categories $B^{(3)}$ and $B^{(0)}$ obey the ordinary Fermi and Bose statistics, respectively: They are given by

$$\{q,q,q\}^{\rm in}=\sum_{\alpha,\beta,\gamma}\sigma^{\alpha\beta\gamma}\{q_\alpha,q_\beta,q_\gamma\}^{\rm in}P$$

corresponding to baryons,

$$[q,\tilde{q}]^{\rm in}=\sum_\alpha[q_\alpha,q^\alpha]^{\rm in}P$$

corresponding to mesons,

$$[q,q]^{\rm in}=\sum_\alpha[q_\alpha,q_\alpha]^{\rm in}P$$

and their antiparticle states, where $\sigma^{\alpha\beta\gamma}=1$ or 0 according to whether α, β, γ are all different or otherwise.

Now, if we replace, in the former theory, the bracket $\{\ ,\ \}$ by $[\ ,\ ]_\epsilon$ and t_α (t^α) by q_α (q^α), then we will obtain the latter theory.[24] The state-vector space of the field t_α then corresponds to the space $\mathfrak{B}$ of the field q_α, whose subspace is the space $\mathfrak{A}$ of the parafield q. Thus, there is no one-to-one correspondence between state vectors in both theories. However, it is possible to make some special states, i.e., those which form singlets with respect to the second $SU(3)$ in the former theory, correspond to states of the space $\mathfrak{A}$ of the paraquark field q. To consider only singlet states in the former theory implies that the suffix α is unobservable, and this corresponds to the fact that in the latter theory, the Green component field does not show up by itself. In fact, the correspondence of baryon states

$$\sum\epsilon^{\alpha\beta\gamma}(t_\alpha t_\beta t_\gamma)^{\rm in}\rightarrow\sum\sigma^{\alpha\beta\gamma}\{q_\alpha,q_\beta,q_\gamma\}^{\rm in}P$$

and of meson states

$$\sum_\alpha[t_\alpha,t^\alpha]^{\rm in}\rightarrow\sum_\alpha[q_\alpha,q^\alpha]^{\rm in}P$$

is complete in the sense that each of them has the same property with respect to spin and unitary spin. It can be seen more generally that many particle bound states forming $SU(3)$ singlets in the former theory correspond to states of order 1 and of integral baryon number in the latter theory (assuming the baryon number of quarks to be $\frac{1}{3}$). However, the converse correspondence does not hold true: The bound state $\sum_\alpha[q_\alpha,q_\alpha]^{\rm in}P$ in the latter theory has the counterpart $\sum_\alpha[t_\alpha,t_\alpha]^{\rm in}$ in the former theory, which is one of the 6-fold degenerate states. Which model is better than the other cannot be judged on theoretical grounds only. We must await further experimental evidence, especially those concerning the existence and observability of quarks.

C. Relativistic Generalization

So far, we have been concerned with a nonrelativistic field theory. However, as can be easily seen, all the arguments presented in the preceding sections can be extended in a straightforward manner to the relativistic case, except for the definition of bound states.

Now, in order to define bound-state amplitudes in relativistic parafield theory we have to introduce something which corresponds to the T product in ordinary field theory. Since our field operator $\psi(x_1)$

[24] By means of a Klein transformation for the field variables q_α, we can change the commutation properties in such a way that the transformed variables have the same anticommutation relations as (6.6). But, this introduces, in general, nonlocal factors into interaction Hamiltonians.

does not simply commute or anticommute with $\psi(x_2)$ even for a spacelike separation $x_1 \sim x_2$, it is obvious that the ordinary chronologically ordered product does not work in the present case. (In this subsection, arguments $x_1, x_2, \cdots$ of ψ are to be understood as including time coordinates $t_1, t_2, \cdots$, also.) We thus propose, in analogy with the nonrelativistic case, to adopt the following generalized $\mathcal{T}$ product:

$$\mathcal{T}(\psi(x_1)\psi(x_2)\cdots\psi(x_n)) \equiv P[\sum_{\alpha_1,\alpha_2,\cdots,\alpha_n} T(\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\cdots\psi_{\alpha_n}(x_n))]P, \quad (6.8)$$

where the symbol T implies the usual T product. Since P is an operator defined independently of Lorentz frames, the above definition of $\mathcal{T}$ is relativistically invariant. Now, by arguing in a way similar to the derivation of Theorem 1, we can write the left-hand side of (6.8) as a sum of $\mathcal{T}$ products of the standard form:

$$\mathcal{T}(\{\psi(x_{i_1}),\psi(x_{i_2}),\cdots,\psi(x_{i_a})\}[\psi(x_{j_1}),\psi(x_{k_1})] \times[\psi(x_{j_2}),\psi(x_{k_2})]\cdots[\psi(x_{j_b}),\psi(x_{k_b})]). \quad (6.9)$$

By means of (6.8) the expression (6.9) can be rewritten as

$$(6.9)=P[\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_a\\ \text{(all different)}}} (\sum_{\beta_1,\beta_2,\cdots,\beta_b} T\{\psi_{\alpha_1}(x_{i_1}),\psi_{\alpha_2}(x_{i_2}),\cdots,\psi_{\alpha_a}(x_{i_a})\} \times[\psi_{\beta_1}(x_{j_1}),\psi_{\beta_1}(x_{k_1})][\psi_{\beta_2}(x_{j_2}),\psi_{\beta_2}(x_{k_2})]\cdots[\psi_{\beta_b}(x_{j_b}),\psi_{\beta_b}(x_{k_b})])]P. \quad (6.10)$$

Let $|B\rangle$ be a bound state which belongs to the state-vector space $\mathfrak{A}$. Then, the amplitudes

$$\langle 0| \sum_{\beta_1,\beta_2,\cdots,\beta_b} T(\{\psi_{\alpha_1}(x_{i_1}),\psi_{\alpha_2}(x_{i_2}),\cdots,\psi_{\alpha_a}(x_{i_a})\}[\psi_{\beta_1}(x_{j_1}),\psi_{\beta_1}(x_{k_1})][\psi_{\beta_2}(x_{j_2}),\psi_{\beta_2}(x_{k_2})]\cdots[\psi_{\beta_b}(x_{j_b}),\psi_{\beta_b}(x_{k_b})])|B\rangle \quad (6.11)$$

will have to satisfy the same Bethe-Salpeter equation, since the whole theory must remain invariant under any exchange of Green suffixes. From relation (6.10) we can thus obtain the Bethe-Salpeter equation to be satisfied by the amplitude $\langle 0|\mathcal{T}(\{\psi(x_{i_1}),\psi(x_{i_2}),\cdots,\psi(x_{i_a})\}[\psi(x_{j_1}),\psi(x_{k_1})][\psi(x_{j_2}),\psi(x_{k_2})]\cdots[\psi(x_{j_b}),\psi(x_{k_b})])|B\rangle$. Therefore, the in-field operator for (6.9) may be regarded as a free-field operator for the bound state $|B\rangle$.

Statistical properties of the state $|B\rangle$ will then be determined by commutation relations to be satisfied by the in-field operators for (6.9). Before trying to do this, however, we have to show for completeness of the arguments that (i) the quantity

$$\sum_{\substack{\alpha_1,\alpha_2,\cdots,\alpha_a\\ \text{(all different)}}} (\sum_{\beta_1,\beta_2,\cdots,\beta_b} T\{\psi_{\alpha_1},\psi_{\alpha_2},\cdots,\psi_{\alpha_a}\}[\psi_{\beta_1},\psi_{\beta_1}][\psi_{\beta_2},\psi_{\beta_2}]\cdots[\psi_{\beta_b},\psi_{\beta_b}])^{\text{in}}$$

is commutable with P, or in other words, it is a quantity defined in the space $\mathfrak{A}$, and (ii) all bound states in the space $\mathfrak{A}$ can be exhausted by taking in-field operators for all possible $\mathcal{T}$ products of the standard form (6.9). In the nonrelativistic theory the above conditions (i) and (ii) are both satisfied, whereas in the relativistic theory it is not obvious that such is also the case. We have not succeeded yet in giving rigorous proofs for (i) and (ii), but only conjecture that they are essentially correct, and therefore, statistical properties of bound states can be argued in exactly the same way as in the nonrelativistic case.

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APPENDIX A: PROOF OF THEOREM 3

For given n, a, b, and arguments x_i we mean by different standard state vectors those in which the arguments $i_1, i_2, \cdots, i_a; j_1, k_1, j_2, k_2, \cdots, j_b, k_b$, being subject to the conditions stated in Theorem 1′, appear in different combinations. We shall first prove that the condition $p \geqslant a+b$ is sufficient. When the Green decomposition (1.4) is performed for each field operator $\phi^\dagger$, a standard state vector, Ψ_1, say, defined by

$$\Psi_1 \equiv |\{i_1,i_2,\cdots,i_a\}^\dagger[j_1,k_1]^\dagger[j_2,k_2]^\dagger\cdots[j_b,k_b]^\dagger|0\rangle$$

can be expressed as a linear combination of state vectors in the space $\mathfrak{B}$. Now, the characteristic feature of Ψ_1 is that it contains, among others, the state vector

$$\Phi \equiv \phi_1^\dagger(i_1)\phi_2^\dagger(i_2)\cdots\phi_a^\dagger(i_a)\phi_{a+1}^\dagger(j_1)\phi_{a+1}^\dagger(k_1) \times\phi_{a+2}^\dagger(j_2)\phi_{a+2}^\dagger(k_2)\cdots\phi_{a+b}^\dagger(j_b)\phi_{a+b}^\dagger(k_b)|0\rangle.$$

If another standard state vector Ψ_2 ($\neq \Psi_1$), say, does not contain Φ in its expansion in terms of the Green component fields, we can then conclude that Ψ_2 is linearly independent of Ψ_1. Suppose that in Ψ_2 the letter i_1 appears in the square bracket $[i_1,s]^\dagger$, i.e.,

$$\Psi_2 = |\{\ \}_a^\dagger[\ ]^\dagger\cdots[i_1,s]^\dagger\cdots[\ ]_b^\dagger|0\rangle.$$

In order to see whether Ψ_2 contains Φ, we have to look for the terms in the Green expansion, which contain

$\phi_1^\dagger(i_1)$. However, such terms necessarily contain also $\phi_1^\dagger(s)$. Since Φ contains only one field operator with the Green suffix 1, we can conclude that Φ is not contained in Ψ_2. This means that Ψ_2 is linearly independent of Ψ_1. Next we consider another standard state vector Ψ_3 in which the letters j_1 and k_1 appear in a way different from Ψ_1, i.e., $\Psi_3=|\{\ \}_a{}^\dagger[\ ]^\dagger\cdots[j_1,s]^\dagger\cdots[k_1,t]^\dagger\cdots\times[\ ]_b{}^\dagger|0\rangle$. This time we have to look for the terms in its Green expansion which contain $\phi_{a+1}^\dagger(j_1)\phi_{a+1}^\dagger(k_1)$. However, such terms in Ψ_3 are necessarily of the form $\cdots\phi_{a+1}^\dagger(j_1)\phi_{a+1}^\dagger(s)\phi_{a+1}^\dagger(k_1)\phi_{a+1}^\dagger(t)\cdots$. This implies that Φ is not contained in Ψ_3, and Ψ_1 and Ψ_3 are linearly independent of each other.

To prove that the condition is necessary, it suffices to show that when $b=p-a+1$, some of the standard state vectors are linearly dependent, because the states with larger values of b can be obtained from the states with $b=p-a+1$ by multiplying a suitable number of brackets $[\ ]^\dagger$. Now, to this end let us show that the following relation holds true:

$$\sum_{((l_1,l_2,\cdots,l_s))}|\{i_1,i_2,\cdots,(i_{l_1}),\cdots,(i_{l_2}),\cdots,(i_{l_s}),\cdots,i_{p+1}\}^\dagger{}_{p-s+1}\prod_{m=1}^{s}[j_m,i_{l_m}]^\dagger|0\rangle=0, \tag{A1}$$

where the notation $\sum_{((l_1,l_2,\cdots,l_s))}$ means to first sum over all the permutations of $(l_1,l_2,\cdots,l_s)$, and then over all the possible sets $(l_1,l_2,\cdots,l_s)$ chosen from among $(1, 2, \cdots, p+1)$. We now consider the case $s=1$ of (A1). In the relation (3.4) we put $m=p+1$. Then the left-hand side and the first term on the right-hand side vanish, and we are left with

$$\sum_{l_1=1}^{p+1}\{i_1,i_2,\cdots,(i_{l_1}),\cdots,i_{p+1}\}^\dagger[j_1,i_{l_1}]^\dagger=0, \tag{A2}$$

which implies that (A1) holds true for $s=1$. Assume then that (A1) holds true for s. Multiplying (A1) by $\phi^\dagger(j_{s+1})$ from the left and then using (3.4), we obtain

$$\begin{aligned}0&=\phi^\dagger(j_{s+1})\sum_{((l_1,l_2,\cdots,l_s))}|\{i_1,i_2,\cdots,(i_{l_1}),\cdots,(i_{l_2}),\cdots,(i_{l_s}),\cdots,i_{p+1}\}^\dagger{}_{p-s+1}\prod_{m=1}^{s}[j_m,i_{l_m}]^\dagger|0\rangle\\&=\frac{1}{p-s+2}\sum_{((l_1,l_2,\cdots,l_s))}|\{i_1,i_2,\cdots,(i_{l_1}),\cdots,(i_{l_2}),\cdots,(i_{l_s}),\cdots,i_{p+1},j_{s+1}\}^\dagger{}_{p-s+2}\prod_{m=1}^{s}[j_m,i_{l_m}]^\dagger|0\rangle\\&+\frac{p-s+1}{2}\sum_{((l_1,l_2,\cdots,l_s,l_{s+1}))}|\{i_1,i_2,\cdots,(i_{l_1}),\cdots,(i_{l_2}),\cdots,(i_{l_s}),\cdots,(i_{l_{s+1}}),\cdots,i_{p+1}\}^\dagger{}_{p-s}\prod_{m=1}^{s+1}[j_m,i_{l_m}]^\dagger|0\rangle.\end{aligned} \tag{A3}$$

Now, according to Theorem 2 the two terms on the right-hand side of (A3) are linearly independent, and each of them must separately vanish. We have therefore

$$\sum_{((l_1,l_2,\cdots,l_s,l_{s+1}))}|\{i_1,i_2,\cdots,(i_{l_1}),\cdots,(i_{l_2}),\cdots,(i_{l_s}),\cdots,(i_{l_{s+1}}),\cdots,i_{p+1}\}^\dagger{}_{p-s}\prod_{m=1}^{s+1}[j_m,i_{l_m}]^\dagger|0\rangle=0. \tag{A4}$$

which implies that the relation (A1) holds true for $(s+1)$ as well. Hence, by mathematical induction we see that (A1) holds true in general, and this completes the proof of Theorem 3.

APPENDIX B: MODEL THEORY SATISFYING CONDITION (3.1) ONLY

As a simple example of field theory which satisfies the condition (3.1) but not (3.2), we consider the following static para-Fermi theory of $p=2$ with the interaction H_I which contains a term of the type $\{\ \}_2$:

$$H=\int d^3x\, H(x),\quad H(x)=H_0(x)+H_I(x), \tag{B1}$$

and

$$H_0(x)=\tfrac{1}{2}m{:}[\phi^\dagger(x),\phi(x)]{:},$$
$$H_I(x)=\tfrac{1}{2}g{:}\{\phi^\dagger(x),\phi(x)\}{:}, \tag{B2}$$

where

$$\begin{aligned}\tfrac{1}{2}{:}[\phi^\dagger(x),\phi(x)]{:}&\equiv\tfrac{1}{2}([\phi^\dagger(x),\phi(x)]-\langle[\phi^\dagger(x),\phi(x)]\rangle_0)\\&=(\phi_1^\dagger(x)\phi_1(x)+\phi_2^\dagger(x)\phi_2(x))P,\end{aligned} \tag{B3}$$

$$\begin{aligned}\tfrac{1}{2}{:}\{\phi^\dagger(x),\phi(x)\}{:}&\equiv\tfrac{1}{2}(\{\phi^\dagger(x),\phi(x)\}-\langle\{\phi^\dagger(x),\phi(x)\}\rangle_0)\\&=(\phi_1^\dagger(x)\phi_2(x)+\phi_2^\dagger(x)\phi_1(x))P.\end{aligned} \tag{B4}$$

We note that both $H(x)$ and $H_I(x)$ satisfy the locality condition (3.1), but the latter does not satisfy (3.2). By use of (1.5) we can prove

$$\frac{1}{2}\left\{{:}\int d^3y\,\{\phi^\dagger(y),\,\phi(y)\}{:},\,\phi^\dagger(x)\right\}=\phi^\dagger(x), \tag{B5}$$

$$\frac{1}{2}\left[{:}\int d^3y\,\{\phi^\dagger(y),\phi(y)\}{:},\,\phi^\dagger(x_1)\phi^\dagger(x_2)\right]=0, \tag{B6}$$

which can further be generalized, by mathematical induction, to

$$\frac{1}{2}\left\{:\int d^3y\,\{\phi^\dagger(y),\phi(y)\}:,\ \phi^\dagger(x_1)\phi^\dagger(x_2)\cdots\phi^\dagger(x_{2n+1})\right\}=\phi^\dagger(x_1)\phi^\dagger(x_2)\cdots\phi^\dagger(x_{2n+1}),\quad (B5')$$

$$\frac{1}{2}\left[:\int d^3y\,\{\phi^\dagger(y),\phi(y)\}:,\ \phi^\dagger(x_1)\phi^\dagger(x_2)\cdots\phi^\dagger(x_{2n})\right]=0.\quad (B6')$$

By using (B5′) and (B6′) we can easily see that eigenvalues of the Hamiltonian H are given by

$$H|2n+1\rangle=[(2n+1)m+g]|2n+1\rangle,\quad (B7)$$

$$H|2n\rangle=2nm|2n\rangle.\quad (B8)$$

where $|n\rangle\equiv\phi^\dagger(x_1)\phi^\dagger(x_2)\cdots\phi^\dagger(x_n)|0\rangle$ is the n-particle state.

The relations (B7) and (B8) show that the energy of the one-particle state is given by $(m+g)$. However, the energy of the n-particle state is not equal to n times $(m+g)$, even when the n particles are situated far apart from each other. This result implies that the theory is not "local" in the naïve sense of the word, and clearly we can not define the in-field ϕ^{in} in such a case. We encounter similar situations for the case of $p>2$, when the condition (3.2) is not fulfilled.

APPENDIX C: PROOF OF EQ. (3.6)

To prove (3.6) we start from (A1) which can be rewritten in the form (apart from a numerical factor):

$$\sum_{\substack{l_1,l_2,\cdots,l_{p+1}\text{ over all perm.}\\ \text{of }(1,2,\cdots,p,p+1)}}\{i_{l_1},i_{l_2},\cdots,i_{l_{p-s+1}}\}^\dagger\times[j_1,i_{l_{p-s+2}}]^\dagger[j_2,i_{l_{p-s+3}}]^\dagger\cdots[j_s,i_{l_{p+1}}]^\dagger=0.\quad (C1)$$

Changing the variables $i_{l_1},i_{l_2},\cdots,i_{l_{p+1}}$ into $x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_{p+1-k}$ and $j_1,j_2,\cdots,j_s$ into $x_{k+1},x_{k+2},\cdots,x_{p+1-m}$, respectively, where $s=p+1-(k+m)$, (C1) is cast into the form

$$\sum_{\substack{\text{all perm. of }(x_1,x_2,\cdots,x_k;\\ y_1,y_2,\cdots,y_{p+1-k})}}\{x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_m\}^\dagger\times[x_{k+1},y_{m+1}]^\dagger[x_{k+2},y_{m+2}]^\dagger\cdots\times[x_{p+1-m},y_{p+1-k}]^\dagger=0.\quad (C1')$$

We then sum (C.1′) over all permutations of $x_1,x_2,\cdots,x_{p+1-m}$ to obtain

$$\sum_{\substack{\text{all perm. of }(x_1,x_2,\cdots,x_{p+1-m})\\ \text{and }(y_1,y_2,\cdots,y_{p+1-k})}}\{x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_m\}^\dagger\times[x_{k+1},y_{m+1}]^\dagger[x_{k+2},y_{m+2}]^\dagger\cdots\times[x_{p+1-m},y_{p+1-k}]^\dagger=0.\quad (C2)$$

Multiplying (C2) by

$$[x_{p+2-m},y_{p+2-k}]^\dagger[x_{p+3-m},y_{p+3-k}]^\dagger\cdots[x_{k+b},y_{m+b}]^\dagger$$

and then summing the resulting expression over all permutations of $(x_1,x_2,\cdots,x_{k+b})$ and of $(y_1,y_2,\cdots,y_{m+b})$, where $b\geqslant p+1-(k+m)$, we obtain

$$\sum_{\substack{\text{all perm. of }(x_1,x_2,\cdots,x_{k+b})\\ \text{and }(y_1,y_2,\cdots,y_{m+b})}}\{x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_m\}^\dagger\times[x_{k+1},y_{m+1}]^\dagger[x_{k+2},y_{m+2}]^\dagger\cdots\times[x_{k+b},y_{m+b}]^\dagger=0,\quad (C3)$$

where $p<(k+m)+b$.