theory, then the underlying idea proposed by the author[2,3] of treating the mesons and baryons on an equal basis would be directly relevant to the physics of elementary particles. In fact, in that case, the subdivision of particles into strange and nonstrange particles would be more fundamental than the subdivision into mesons and baryons which is commonly made. This point of view has already been presented in previous publications (in particular in Ref. 3, see the discussion on p. 360).

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Current-Algebra Calculation of the K_{e5} Vector Form Factors*

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From considerations of the current-commutation relations of Gell-Mann and the assumption of a partially conserved axial-vector current, the vector form factors for the rare K_{e5} decays are calculated. The presence of K-meson and π-meson pole terms in certain radiative amplitudes is responsible for the complicated momentum dependence of these form factors. The decay rates for the various K_{e5} modes are evaluated, and the results are compared to previous estimates based on various models.

I. INTRODUCTION

THE purpose of this paper is to determine the form factors and decay rates for the rare K-meson decays

$$K_{e5}:\quad K\to\pi+\pi+\pi+e+\nu\,.$$

Previous estimates of the rates for these decays have been based on a direct interaction model,[1] an η-pole model,[1] and a pion-pole model.[2] It would be of interest if one could make a model-independent prediction of these rates since such a prediction would serve as a test for the validity of the various models that have been proposed. The recent successes of the current-commutation relations of Gell-Mann[3] and the assumption of a partially conserved axial-vector current (PCAC) encourage one to believe that an application of these ideas to K_{e5} decay could lead to a correct and model-independent description of this process.[4]

By means of current algebra, we shall relate the K_{e5} form factors to those for the K_{e4}, K_{e3}, and K_{e2} decays. In particular, we shall adopt the technique developed by Weinberg.[5] In order to treat the pions on an equal footing, we shall take all of the pions off the mass shell. We will expand the decay amplitudes in powers of pion momenta. Our resulting expansions will give us *on-the-mass-shell* decay amplitudes up to lowest nonvanishing order in pion momenta. This expansion technique has been used successfully by Weinberg[5] for K_{e4} decay, by Abarbanel[6] for $K_{3\pi}$ decays, and by a number of other authors for various η decays.[7]

Although the K_{e5} modes have not yet been observed experimentally, they are of theoretical interest. In calculating the K_{e5} form factors, one encounters features that are not present in other decay modes. Furthermore, it is anticipated that the K_{e5} decays will be observed in the future. This calculation will then provide a basis for comparison with the experimental data.

II. DERIVATION OF THE THREE-PION-EMISSION FORMULA

We begin by considering the quantity

$$M_{\mu\nu\lambda}=\int dxdydz\, e^{i(q_a\cdot x+q_b\cdot y+q_c\cdot z)}\times\langle 0|T(A_\mu{}^a(x)A_\nu{}^b(y)A_\lambda{}^c(z)\mathcal{V}_\sigma{}^n(0))|K_{km}\rangle\,,\quad(1)$$

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† National Science Foundation predoctoral trainee.

[1] V. A. Kolkunov and I. V. Lyagin, Zh. Eksperim. i Teor. Fiz. **45**, 2009 (1963) [English transl.: Soviet Phys.—JETP **18**,1379 (1964)].

[2] G. W. Intemann and I. R. Lapidus, Nuovo Cimento **52A**, 432 (1967).

[3] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).

[4] While this work was being completed we received a paper by P. McNamee and R. J. Oakes [Phys. Rev. **168**, 1683 (1968)]. These authors also consider K_{e5} decays by current algebra, but their method differs somewhat from ours and their results for the form factors are quite different. The decay rates obtained are also somewhat larger than our calculated rates.

[5] S. Weinberg, Phys. Rev. Letters **16**, 879 (1966); **17**, 336 (1966).

[6] H. D. I. Abarbanel, Phys. Rev. **153**, 1547 (1967).

[7] J. Pasupathy and R. E. Marshak, Phys. Rev. Letters **17**, 888 (1966); J. Dreitlein and K. T. Mahanthappa, Phys. Rev. **160**, 1542 (1967); G. W. Intemann and I. R. Lapidus, *ibid.* **165**, 1650 (1968).

where $\mathcal{V}_\sigma^n$ is the $\Delta S=-1$, $\Delta I_3=-n$ vector current, A_μ^a is the $\Delta S=0$ axial-vector current, q_a, q_b, q_c are pion momenta and a, b, c are isotopic indices, k is the kaon momentum, and $m=\pm\frac{1}{2}$ is the I_3 value of the K meson.

Isolating the pion pole terms in Eq. (1) in the manner of Weinberg, we write

$$\begin{aligned}-iq_{a\mu}iq_{b\nu}iq_{c\lambda}N_{\mu\nu\lambda} &= -iq_{a\mu}iq_{b\nu}iq_{c\lambda}M_{\mu\nu\lambda} \\ &-\frac{F_\pi q_a^2 iq_{b\nu}iq_{c\lambda}M_{\nu\lambda}(q_b,q_c)}{(\mu^2-q_a^2)}+\text{permutations} \\ &-\frac{F_\pi^2 q_a^2 q_b^2 iq_{c\lambda}M_\lambda(q_c)}{(\mu^2-q_a^2)(\mu^2-q_b^2)}+\text{permutations} \\ &-\frac{F_\pi^3 q_a^2 q_b^2 q_c^2 M}{(\mu^2-q_a^2)(\mu^2-q_b^2)(\mu^2-q_c^2)}, \quad (2)\end{aligned}$$

with

$$M_{\mu\nu}(q_a,q_b)=(\mu^2-q_c^2)\int dxdydz\, e^{i(q_a\cdot x+q_b\cdot y+q_c\cdot z)} \times\langle 0|T(A_\mu^a(x)A_\nu^b(y)\phi_\pi^c(z)\mathcal{V}_\sigma^n(0))|K_{km}\rangle, \quad (3)$$

$$M_\mu(q_a)=(\mu^2-q_b^2)(\mu^2-q_c^2)\int dxdydz\, e^{i(q_a\cdot x+q_b\cdot y+q_c\cdot z)} \times\langle 0|T(A_\mu^a(x)\phi_\pi^b(y)\phi_\pi^c(z)\mathcal{V}_\sigma^n(0))|K_{km}\rangle, \quad (4)$$

$$M=(\mu^2-q_a^2)(\mu^2-q_b^2)(\mu^2-q_c^2)\int dxdydz \times e^{i(q_a\cdot x+q_b\cdot y+q_c\cdot z)}\langle 0|T(\phi_\pi^a(x)\phi_\pi^b(y)\phi_\pi^c(z) \times\mathcal{V}_\sigma^n(0))|K_{km}\rangle, \quad (5)$$

where F_π is the pion decay amplitude, ϕ_π is the pion field, and μ is the pion mass.

The $SU_3\otimes SU_3$ current-commutation relations of Gell-Mann[3] are given by[8]

$$[A_0^a(x),A_\nu^b(y)]\delta(x_0-y_0)=2i\epsilon_{abs}V_\nu^s(x)\delta^4(x-y), \quad (6)$$

$$[A_0^a(x),\mathcal{V}_\sigma^n(0)]\delta(x_0)=-(\tau_a)_{np}\mathcal{A}_\sigma^p(x)\delta^4(x), \quad (7)$$

$$[A_0^a(x),\mathcal{A}_\sigma^n(0)]\delta(x_0)=-(\tau_a)_{np}\mathcal{V}_\sigma^p(x)\delta^4(x), \quad (8)$$

$$[V_0^a(x),\mathcal{V}_\sigma^n(0)]\delta(x_0)=-(\tau_a)_{np}\mathcal{V}_\sigma^p(x)\delta^4(x). \quad (9)$$

We also make use of the additional commutation relations[9]

$$[A_0^a(x),\partial_\mu A_\mu^b(y)]\delta(x_0-y_0)=\sigma_{ab}(x)\delta^4(x-y), \quad (10)$$

$$[A_0^a(x),\sigma_{bc}(y)]\delta(x_0-y_0)=\delta_{bc}\partial_\mu A_\mu^a(x)\delta^4(x-y), \quad (11)$$

where $\mathcal{A}_\sigma^n$ is the $\Delta S=-1$, $\Delta I_3=-n$ axial-vector current, V_μ^s is the $\Delta S=0$, $\Delta I=1$ vector current, and $\sigma_{ab}(x)$ is a scalar density[10]; ϵ_{abs} is the totally antisymmetric symbol with $\epsilon_{123}=+1$, and the τ_a are Pauli matrices.

We will also make use of the conserved-vector-current (CVC) and PCAC hypotheses

$$\partial_\mu V_\mu^a(x)=0, \quad (12)$$

$$\partial_\mu A_\mu^a(x)=F_\pi\mu^2\phi_\pi^a(x). \quad (13)$$

The following identity for time-ordered products holds[6]:

$$\begin{aligned}(\partial/\partial x^\mu)T\{J_\mu(x)B_\nu(y)C_\lambda(z)D_\sigma(0)\} &= T\{\partial_\mu J_\mu(x)B_\nu(y)C_\lambda(z)D_\sigma(0)\} \\ &+\delta(x_0-y_0)T\{[J_0(x),B_\nu(y)]C_\lambda(z)D_\sigma(0)\} \\ &+\delta(x_0-z_0)T\{[J_0(x),C_\lambda(z)]B_\nu(y)D_\sigma(0)\} \\ &+\delta(x_0)T\{[J_0(x),D_\sigma(0)]B_\nu(y)C_\lambda(z)\}. \quad (14)\end{aligned}$$

Computing the various terms in Eq. (2) by partial integrations,[11] and making use of Eqs. (3)–(14), we find

$$\begin{aligned}-iq_{a\mu}iq_{b\nu}iq_{c\lambda}N_{\mu\nu\lambda} &= F_\pi^3(\mu^2-q_a^2)(\mu^2-q_b^2)(\mu^2-q_c^2)\int dxdydz\, e^{i(q_a\cdot x+q_b\cdot y+q_c\cdot z)}\langle 0|T(\phi_\pi^a(x)\phi_\pi^b(y)\phi_\pi^c(z)\mathcal{V}_\sigma^n(0))|K_{km}\rangle \\ &-F_\pi^2(\tau_a)_{np}(\mu^2-q_b^2)(\mu^2-q_c^2)\int dydz\, e^{i(q_b\cdot y+q_c\cdot z)}\langle 0|T(\phi_\pi^b(y)\phi_\pi^c(z)\mathcal{A}_\sigma^p(0))|K_{km}\rangle+\text{permutations} \\ &+F_\pi\delta_{ab}(\mu^2-q_c^2)\int dx\, e^{iq_c\cdot x}\langle 0|T(\phi_\pi^c(x)\mathcal{V}_\sigma^n(0))|K_{km}\rangle+\text{permutations}-F_\pi\epsilon_{bas}(q_b-q_a)_\mu(\mu^2-q_c^2)\int dxdz \\ &\times e^{i(q_a+q_b)\cdot x+iq_c\cdot z}\langle 0|T(\phi_\pi^c(z)V_\mu^s(x)\mathcal{V}_\sigma^n(0))|K_{km}\rangle+\text{permutations}-\tfrac{4}{3}i\epsilon_{bas}\epsilon_{cst}(q_b-q_a)_\mu\int dx\, e^{i(q_a+q_b+q_c)\cdot x} \\ &\times\langle 0|T(A_\mu^t(x)\mathcal{V}_\sigma^n(0))|K_{km}\rangle+\text{permutations}+\epsilon_{bas}(\tau_c)_{np}(q_b-q_a)_\mu\int dx\, e^{i(q_a+q_b)\cdot x}\langle 0|T(V_\mu^s(x)\mathcal{A}_\sigma^p(0))|K_{km}\rangle \\ &+\text{permutations}-\tfrac{1}{3}\{\delta_{ab}(\tau_c)_{np}+\delta_{ac}(\tau_b)_{np}+\delta_{bc}(\tau_a)_{np}\}\langle 0|\mathcal{A}_\sigma^p|K_{km}\rangle. \quad (15)\end{aligned}$$

[8] In these commutation relations we ignore the so-called "Schwinger terms."

[9] Strictly speaking, these additional commutation relations lie outside the algebra of $SU_3\otimes SU_3$. However, they may be derived within a quark model or a σ model.

[10] In the σ model the term $\sigma_{ab}(x)$ would be just $\delta_{ab}\sigma(x)$, with $\sigma(x)$ the σ-meson field. See M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

[11] We will neglect all surface terms arising from the partial integrations.

In arriving at Eq. (15), we have neglected terms of the form

$$(\mu^2-q_c^2)\int dxdz\, e^{i(q_a+q_b)\cdot x+iq_c\cdot z} \times\langle 0|T(\sigma_{ab}(x)\phi_\pi{}^c(z)\mathcal{V}_\sigma{}^n(0))|K_{km}\rangle,$$

$$(\tau_c)_{np}\int dx\, e^{i(q_a+q_b)\cdot x}\langle 0|T(\sigma_{ab}(x)\mathcal{A}_\sigma{}^p(0))|K_{km}\rangle,$$

and

$$\delta_{ab}\int dx\, e^{i(q_a+q_b+q_c)\cdot x}\langle 0|T(\phi_\pi{}^c(x)\mathcal{V}_\sigma{}^n(0))|K_{km}\rangle.$$

These terms are typical "σ terms" which are generated from the commutation relations (10) and (11). There has been a great deal of speculation concerning the significance of such terms as well as their contribution to particular problems. In previous calculations, the procedure has been to neglect these "σ terms" in multipion decay processes.[5–7] In the Appendix we show, by considering soft-pion limits, that the above "σ terms" are of the order $\mu^2/M_K{}^2$ and can thus be neglected here since the K_{e5} form factors will only be calculated to order μ/M_K.

We also neglect terms cubic in pion momenta, i.e.,

$$iq_{a\mu}iq_{b\nu}iq_{c\lambda}N_{\mu\nu\lambda}\simeq 0.$$

In dropping this term we are making our expansion in pion momenta. We may safely neglect this cubic term since all of the remaining terms in Eq. (15) are of lower order in pion momenta. The error made here is approximately $O((k\cdot q_a)(k\cdot q_b)(k\cdot q_c)/(k\cdot k)^3)$ or $O(\mu^3/M_K{}^3)$. We may now safely go to the mass shell, i.e., $q_a{}^2=q_b{}^2=q_c{}^2=\mu^2$, in Eq. (15). We then obtain the desired three-pion emission formula:

$$\begin{aligned}
&(2\pi)^{9/2}(8q_a{}^0q_b{}^0q_c{}^0)^{1/2}\langle\pi^a\pi^b\pi^c|\mathcal{V}_\sigma{}^n|K_{km}\rangle\\
&=(1/F_\pi)(\tau_a)_{np}(2\pi)^3(4q_b{}^0q_c{}^0)^{1/2}\langle\pi^b\pi^c|\mathcal{A}_\sigma{}^p|K_{km}\rangle+\text{permutations}-(1/F_\pi{}^2)\delta_{ab}(2\pi)^{3/2}(2q_c{}^0)^{1/2}\langle\pi^c|\mathcal{V}_\sigma{}^n|K_{km}\rangle\\
&\quad+\text{permutations}+(1/3F_\pi{}^3)\{\delta_{ab}(\tau_c)_{np}+\delta_{ac}(\tau_b)_{np}+\delta_{bc}(\tau_a)_{np}\}\langle 0|\mathcal{A}_\sigma{}^p|K_{km}\rangle+(1/F_\pi{}^2)\epsilon_{bas}(q_b-q_a)_\mu(2\pi)^{3/2}(2q_c{}^0)^{1/2}\\
&\quad\times\int dx\, e^{i(q_a+q_b)\cdot x}\langle\pi^c|T(V_\mu{}^s(x)\mathcal{V}_\sigma{}^n(0))|K_{km}\rangle+\text{permutations}-(1/F_\pi{}^3)\epsilon_{bas}(\tau_c)_{np}(q_b-q_a)_\mu\int dx\, e^{i(q_a+q_c)\cdot x}\\
&\quad\times\langle 0|T(V_\mu{}^s(x)\mathcal{A}_\sigma{}^p(0))|K_{km}\rangle+\text{permutations}+(4i/3F_\pi{}^3)\epsilon_{bas}\epsilon_{cst}(q_b-q_a)_\mu\int dx\, e^{i(q_a+q_b+q_c)\cdot x}\\
&\quad\times\langle 0|T(A_\mu{}^t(x)\mathcal{V}_\sigma{}^n(0))|K_{km}\rangle+\text{permutations}. \qquad (16)
\end{aligned}$$

Equation (16) relates the K_{e5} decay amplitude to the K_{e4}, K_{e3}, and K_{e2} decay amplitudes as well as to certain kinds of radiative decay amplitudes. Once again, we stress that this formula is an *on-the-mass-shell* relation in which we have expanded in powers of the pion momenta.

III. EVALUATION OF RADIATION AMPLITUDES

In this section we will calculate the various radiation amplitudes appearing in Eq. (16). In order to evaluate these quantities we will make use of a method invented by Low.[12]

We define the first radiation amplitude $R_{\mu\lambda}{}^{(1)}$ by

$$(2\pi)^{3/2}(2q_a{}^0)^{1/2}\int dy\, e^{iq\cdot y}\langle\pi^a|T(V_\mu{}^s(y)\mathcal{V}_\lambda{}^n(0))|K_{km}\rangle \equiv(2\pi)^{-3/2}(2k_0)^{-1/2}(\tau_s\tau_a)_{nm}R_{\mu\lambda}{}^{(1)}(k,q_a,q). \quad (17)$$

Invoking CVC in Eq. (17), we obtain

$$q_\mu R_{\mu\lambda}(k,q_a,q)=(k+q_a)_\lambda f_+(0)+(k-q_a)_\lambda f_-(0), \quad (18)$$

where we define the K_{e3} form factors f_+ and f_- by

$$(2\pi)^{3/2}(2q_a{}^0)^{1/2}\langle\pi_a|\mathcal{V}_\lambda{}^p|K_{km}\rangle=-i(2\pi)^{-3/2}(2k_0)^{-1/2}\times(\tau_a)_{pm}[(k+q_a)_\lambda f_++(k-q_a)_\lambda f_-]. \quad (19)$$

Among the contributions to $R_{\mu\lambda}{}^{(1)}$ will be two different pole terms: One term representing the pole diagram in which the K meson first interacts with the V_μ current, continues as a K meson, and then disappears into the $\mathcal{V}_\lambda$ current in which a pion is emitted; a second term representing the diagram in which the K meson first disappears into the $\mathcal{V}_\lambda$ current producing a virtual π meson which then interacts with the V_μ current and continues as a real π meson. These diagrams are shown in Fig. 1.

By general covariance, we may write (exhibiting the pole terms explicitly)

$$\begin{aligned}
R_{\mu\lambda}{}^{(1)}=&\frac{c_1[2k_\mu+O(q)]}{2k\cdot q+O(q^2)}[(k-q+q_a)_\lambda f_+(0)\\
&+(k-q-q_a)_\lambda f_-(0)]+\frac{c_2(2q_{a\mu}+q_\mu)}{2q_{a\mu}\cdot q+q^2}\\
&\times[(k+q+q_a)_\lambda f_+(0)+(k-q-q_a)_\lambda f_-(0)]\\
&+c_3\delta_{\mu\lambda}+c_4k_\mu k_\lambda+O(q). \quad (20)
\end{aligned}$$

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[12] F. E. Low, Phys. Rev. 110, 974 (1958).

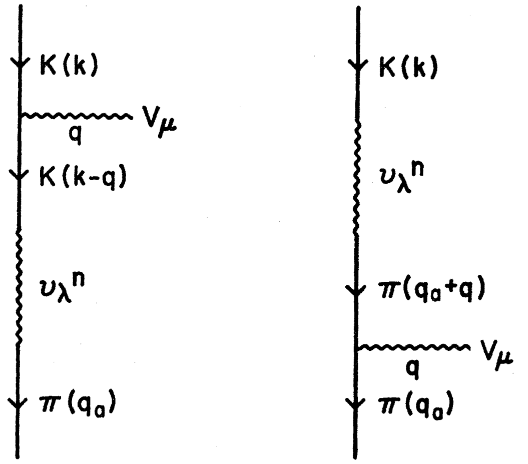

FIG. 1. K-meson and π-meson pole contributions to $R_{\mu\lambda}^{(1)}$.

From Eqs. (18) and (20) we have

$$c_1+c_2=1\,,\quad c_4=0\,,$$
$$c_2[f_+(0)-f_-(0)]-c_1[f_+(0)+f_-(0)]+c_3=0\,.$$

However, from the universal coupling of the ρ meson to the isospin current, we have[13]

$$c_2=2c_1\,.$$

Thus,

$$c_1=\tfrac{1}{3}\,,\quad c_2=\tfrac{2}{3}\,, \tag{21a}$$
$$c_3=f_-(0)-\tfrac{1}{3}f_+(0)\,, \tag{21b}$$
$$c_4=0\,. \tag{21c}$$

Hence, $R_{\mu\lambda}^{(1)}$ is given by

$$R_{\mu\lambda}^{(1)}(k,q_a,q)$$
$$=\frac{k_\mu}{3k\cdot q}[(k-q+q_a)_\lambda f_+(0)+(k-q-q_a)_\lambda f_-(0)]$$
$$+\frac{2}{3}\frac{(2q_{a\mu}+q_\mu)}{2q_a\cdot q+q^2}[(k+q+q_a)_\lambda f_+(0)$$
$$+(k-q-q_a)_\lambda f_-(0)]+(f_--\tfrac{1}{3}f_+)\delta_{\mu\lambda}\,. \tag{22}$$

We note that we have only been able to evaluate $R_{\mu\lambda}^{(1)}$ to zeroth order in pion momenta. Thus, we will make an error $O(\mu^2/M_K^2)$ in the final expressions for the K_{e5} form factors.

We define the second radiation amplitude $R_{\mu\lambda}^{(2)}$ by

$$\int dx\, e^{iq\cdot x}\langle 0|T(V_\mu^s(x)\mathcal{A}_\lambda^p(0))|K_{km}\rangle$$
$$\equiv(2\pi)^{-3/2}(2k_0)^{-1/2}(\tau_s)_{pm}R_{\mu\lambda}^{(2)}(k,q)\,. \tag{23}$$

This was the amplitude encountered by Weinberg[5] for K_{e4} decay. This amplitude also contains a K-meson pole term which is shown in Fig. 2. One can proceed in the same way as for the case of $R_{\mu\lambda}^{(1)}$ and we obtain to within the same approximation

$$R_{\mu\lambda}^{(2)}(k,q)=F_K(k_\mu/k\cdot q)(k-q)_\lambda+F_K\delta_{\mu\lambda}\,, \tag{24}$$

where F_K, the K_{e2} form factor, is defined by

$$\langle 0|\mathcal{A}_\lambda^p|K_{km}\rangle=iF_Kk_\lambda(2\pi)^{-3/2}(2k_0)^{-1/2}\delta_{pm}\,. \tag{25}$$

Finally, we consider the third radiation amplitude defined by

$$\int dx\, e^{iq\cdot x}\langle 0|T(A_\mu^t(x)\mathcal{V}_\lambda^n(0))|K_{km}\rangle$$
$$\equiv-(2\pi)^{-3/2}(2k_0)^{-1/2}(\tau_t)_{nm}R_{\mu\lambda}^{(3)}(k,q)\,. \tag{26}$$

Using PCAC we find in the limit $q^2\to 0$

$$q_\mu R_{\mu\lambda}^{(3)}=F_\pi[(k+q)_\lambda f_++(k-q)_\lambda f_-]-F_Kk_\lambda\,. \tag{27}$$

On grounds of general covariance, we write

$$R_{\mu\lambda}^{(3)}=[F_\pi q_\mu/(q^2-\mu^2)][(k+q)_\lambda f_++(k-q)_\lambda f_-]$$
$$+d_1\delta_{\mu\lambda}+d_2k_\mu k_\lambda+O(q)\,, \tag{28}$$

where we have explicitly exhibited the pole term represented by the diagram appearing in Fig. 3. Thus,

$$\lim_{q^2\to 0} q_\mu R_{\mu\lambda}^{(3)}=d_1q_\lambda+d_2(k\cdot q)k_\lambda\,, \tag{29}$$

and thus from Eq. (27) we obtain

$$d_1=F_\pi(f_+-f_-)\,, \tag{30a}$$
$$d_2=0\,. \tag{30b}$$

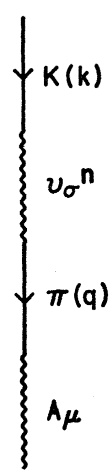

FIG. 2. K-meson pole contribution to $R_{\mu\lambda}^{(2)}$.

[13] The constants c_1 and c_2 reflect the relative strengths of the interaction of the meson current with V_μ for each pole diagram. We observe, however, that V_μ is the conserved-isospin current. Thus, by Sakurai's theory, V_μ will be universally coupled to the ρ-meson field [see J. J. Sakurai, Ann. Phys. (N. Y.) 11, 1 (1960)]. The coupling of the ρ-meson field to the pion and kaon fields is described by the Lagrangian density

$$\mathcal{L}=g\boldsymbol{\rho}_\mu\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}+iK^\dagger\tfrac{1}{2}\boldsymbol{\tau}\partial_\mu K)\,.$$

It then follows that $c_2=2c_1$.

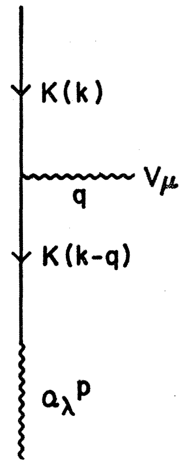

FIG. 3. π-meson pole contribution to $R_{\mu\lambda}^{(3)}$.

Hence we have

$$R_{\mu\lambda}^{(3)}(k,q)=[F_\pi q_\mu/(q^2-\mu^2)][(k+q)_\lambda f_+ + (k-q)_\lambda f_-] + F_\pi(f_+ - f_-)\delta_{\mu\lambda}. \quad (31)$$

IV. CALCULATION OF K_{e5} FORM FACTORS

We define the vector K_{e5} form factors G_i by[14]

$$(2\pi)^{9/2}(8q_a^0q_b^0q_c^0)^{1/2}\langle\pi^a\pi^b\pi^c|\mathcal{V}_\lambda^n|K_{km}\rangle = (i/M_K^2)(2\pi)^{-3/2}(2k_0)^{-1/2}\{G_1q_{a\lambda}+G_2q_{b\lambda} + G_3q_{c\lambda}+G_4(k-q_a-q_b-q_c)_\lambda\}, \quad (32)$$

where M_K is the K-meson mass.

We further define the K_{e4} axial-vector form factors F_i by

$$(2\pi)^3(4q_a^0q_b^0)^{1/2}\langle\pi^a\pi^b|\mathcal{A}_\lambda^p|K_{km}\rangle = (i/M_K)(2\pi)^{-3/2}(2k_0)^{-1/2}\{F_1(q_a+q_b)_\lambda + F_2(q_a-q_b)_\lambda+F_3(k-q_a-q_b)_\lambda\}. \quad (33)$$

These form factors have been calculated by Weinberg,[5] using current algebra. His results are

$$F_1=A\,\delta_{ab}\delta_{pm}, \quad (34a)$$

$$F_2=-iA\,\epsilon_{abs}(\tau_s)_{pm}, \quad (34b)$$

$$F_3=B\delta_{ab}\delta_{pm}+iB\,\epsilon_{abs}(\tau_s)_{pm} \times[k\cdot(q_b-q_a)/k\cdot(q_b+q_a)], \quad (34c)$$

where A and B are given by

$$A=2f_+M_K/F_\pi, \quad (35a)$$

$$B=(M_K/F_\pi)(f_++f_-). \quad (35b)$$

The K_{e3} and K_{e2} form factors are defined by Eqs. (19) and (25), respectively. The K_{e3} form factors satisfy the Callan-Treiman relation[15]

$$F_\pi(f_++f_-)=F_K. \quad (36)$$

If we now substitute Eqs. (17), (19), (22)–(26), (31)–(36) into Eq. (16) and equate coefficients of linearly independent terms, we obtain predictions for the K_{e5} form factors G_i. We shall give here only the results for the interesting K_{e5} modes. We find:

$K^+(k)\to\pi^+(q_1)+\pi^-(q_2)+\pi^0(q_3)+e^++\nu$:

$$G_1=\frac{4}{3}\left(\frac{M_K}{F_\pi}\right)^2 f_+ \times\left[1+\frac{4(q_1+q_2)^2-2(q_1+q_3)^2-2(q_2+q_3)^2}{(q_1+q_2+q_3)^2-\mu^2}\right], \quad (37a)$$

$$G_2=\frac{4}{3}\left(\frac{M_K}{F_\pi}\right)^2 f_+\left[1-\frac{k\cdot(q_3-q_1)}{k\cdot(q_3+q_1)} +\frac{4(q_1+q_2)^2-2(q_1+q_3)^2-2(q_2+q_3)^2}{(q_1+q_2+q_3)^2-\mu^2}\right], \quad (37b)$$

$$G_3=-\frac{2}{3}\left(\frac{M_K}{F_\pi}\right)^2 f_+\left[1-\frac{k\cdot(q_2-q_1)}{k\cdot(q_2+q_1)} -\frac{8(q_1+q_2)^2-4(q_1+q_3)^2-4(q_2+q_3)^2}{(q_1+q_2+q_3)^2-\mu^2}\right], \quad (37c)$$

$$G_4=\frac{1}{3}\left(\frac{M_K}{F_\pi}\right)^2 (f_++f_-)\left[1-2\frac{k\cdot(q_3-q_1)}{k\cdot(q_3+q_1)}+\frac{k\cdot(q_2-q_1)}{k\cdot(q_2+q_1)} +\frac{8(q_1+q_2)^2-4(q_1+q_3)^2-4(q_2+q_3)^2}{(q_1+q_2+q_3)^2-\mu^2}\right]. \quad (37d)$$

$K^+(k)\to\pi^0(q_1)+\pi^0(q_2)+\pi^0(q_3)+e^++\nu$:

$$G_1=G_2=G_3=2f_+(M_K/F_\pi)^2, \quad (38a)$$

$$G_4=(M_K/F_\pi)^2(f_++f_-). \quad (38b)$$

$K_1^0(k)\to\pi^-(q_1)+\pi^0(q_2)+\pi^0(q_3)+e^++\nu$:

$$G_1=-\frac{2}{3}\left(\frac{M_K}{F_\pi}\right)^2 f_+ \times\left[1-\frac{8(q_2+q_3)^2-4(q_1+q_3)^2-4(q_1+q_2)^2}{(q_1+q_2+q_3)^2-\mu^2}\right], \quad (39a)$$

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[14] The total set of K_{e5} form factors is defined by

$$(2\pi)^6(16k^0q_a^0q_b^0q_c^0)^{1/2}\langle\pi^a\pi^b\pi^c|J_\lambda|K\rangle = (i/M_K^2)\{G_1q_{a\lambda}+G_2q_{b\lambda}+G_3q_{c\lambda}+G_4(k-q_a-q_b-q_c)_\lambda + (1/M_K^2)\epsilon_{\lambda\mu\nu\sigma}k_\mu(G_5q_{a\nu}q_{b\sigma}+G_6q_{a\nu}q_{c\sigma}+G_7q_{b\nu}q_{c\sigma}) + (1/M_K^2)G_8\epsilon_{\lambda\mu\nu\sigma}q_{a\mu}q_{b\nu}q_{c\sigma}\},$$

where J_λ is the total $\Delta I=\frac{1}{2}$, $\Delta S=-1$ weak hadron current. We shall neglect the axial-vector form factors $G_5, \cdots, G_8$ since they appear in terms which are quadratic or cubic in pion momenta and are expected to be small (if one assumes that the pions are in an S state, then such terms must be exactly zero according to Bose statistics).

[15] C. G. Callan and S. B. Treiman, Phys. Rev. Letters **16**, 153 (1966).

$$G_2=\frac{4}{3}\left(\frac{M_K}{F_\pi}\right)^2 f_+\left[1+\frac{1}{2}\frac{k\cdot(q_3-q_1)}{k\cdot(q_3+q_1)}+\frac{4(q_2+q_3)^2-2(q_1+q_3)^2-2(q_1+q_2)^2}{(q_1+q_2+q_3)^2-\mu^2}\right], \quad (39b)$$

$$G_3=\frac{4}{3}\left(\frac{M_K}{F_\pi}\right)^2 f_+\left[1+\frac{1}{2}\frac{k\cdot(q_2-q_1)}{k\cdot(q_2+q_1)}+\frac{4(q_2+q_3)^2-2(q_1+q_3)^2-2(q_1+q_2)^2}{(q_1+q_2+q_3)^2-\mu^2}\right], \quad (39c)$$

$$G_4=\frac{1}{3}\left(\frac{M_K}{F_\pi}\right)^2 (f_++f_-)\left[1+\frac{k\cdot(q_3-q_1)}{k\cdot(q_3+q_1)}+\frac{k\cdot(q_2-q_1)}{k\cdot(q_2+q_1)}+\frac{8(q_2+q_3)^2-4(q_1+q_3)^2-4(q_1+q_2)^2}{(q_1+q_2+q_3)^2-\mu^2}\right]. \quad (39d)$$

$K_1^0\rightarrow\pi^+(q_1)+\pi^-(q_2)+\pi^-(q_3)+e^++\nu$:

$$G_1=\frac{8}{3}\left(\frac{M_K}{F_\pi}\right)^2 f_+\times\left[1-\frac{2(q_2+q_3)^2-(q_1+q_3)^2-(q_1+q_2)^2}{(q_1+q_2+q_3)^2-\mu^2}\right], \quad (40a)$$

$$G_2=\frac{2}{3}\left(\frac{M_K}{F_\pi}\right)^2 f_+\left[1-\frac{k\cdot(q_3-q_1)}{k\cdot(q_3+q_1)}-\frac{8(q_2+q_3)^2-4(q_1+q_3)^2-4(q_1+q_2)^2}{(q_1+q_2+q_3)^2-\mu^2}\right], \quad (40b)$$

$$G_3=\frac{2}{3}\left(\frac{M_K}{F_\pi}\right)^2 f_+\left[1-\frac{k\cdot(q_2-q_1)}{k\cdot(q_2+q_1)}-\frac{8(q_2+q_3)^2-4(q_1+q_3)^2-4(q_1+q_2)^2}{(q_1+q_2+q_3)^2-\mu^2}\right], \quad (40c)$$

$$G_4=\frac{2}{3}\left(\frac{M_K}{F_\pi}\right)^2 (f_++f_-)\left[1-\frac{1}{2}\frac{k\cdot(q_3-q_1)}{k\cdot(q_3+q_1)}-\frac{1}{2}\frac{k\cdot(q_2-q_1)}{k\cdot(q_2+q_1)}-\frac{4(q_2+q_3)^2-2(q_1+q_3)^2-2(q_1+q_2)^2}{(q_1+q_2+q_3)^2-\mu^2}\right]. \quad (40d)$$

We defer discussion of these results until Sec. VI.

V. CALCULATION OF K_{e5} DECAY RATES

The matrix element for the K_{e5} decay is given by[16]

$$\mathfrak{M}=\frac{G\sin\theta}{\sqrt{2}M_K^2(2\pi)^5}\bar{u}_\nu(p_\nu)\gamma^\lambda(1+\gamma_5)u_e(p_e)\times\{G_1q_{1\lambda}+G_2q_{2\lambda}+G_3q_{3\lambda}\}\times\delta^4(k-p_e-p_\nu-q_1-q_2-q_3), \quad (41)$$

where G is the Fermi coupling constant, $G=1.0\times10^{-5}/M_p^2$, and θ is the Cabibbo angle, $\theta=15.4°$.

The decay rate is then obtained by squaring the matrix element $\mathfrak{M}$, summing over electron and neutrino spin states, and integrating over the five-body phase space, which yields[17]

$$T(Ke_5)=\frac{G^2\sin^2\theta}{16(2\pi)^{11}M_K^5n!}\int f(k,q_1,q_2,q_3,p_e,p_\nu)\times\frac{d^3p_e}{E_e}\frac{d^3p_\nu}{E_\nu}\frac{d^3q_1}{\omega_1}\frac{d^3q_2}{\omega_2}\frac{d^3q_3}{\omega_3}\times\delta^4(k-p_e-p_\nu-q_1-q_2-q_3), \quad (42)$$

where n is the number of identical pions in the decay mode and

$$f(k,q_1,q_2,q_3,p_e,p_\nu)=2[(G_1q_1+G_2q_2+G_3q_3)\cdot p_e]\times[(G_1q_1+G_2q_2+G_3q_3)\cdot p_\nu]-(G_1q_1+G_2q_2+G_3q_3)^2(p_e\cdot p_\nu). \quad (43)$$

E_e, E_ν, ω_1, ω_2, ω_3 are the electron, neutrino, and pion energies, respectively, and the G_i are given by Eqs. (37)-(40).

Due to the extremely complicated momentum dependence in the form factors G_i, it will not, in general, be possible to analytically integrate Eq. (42) (except for the trivial δ-function integrations). Consequently, we have calculated the rates numerically by means of a Monte Carlo technique. The results are[18]

$$\Gamma(K^+\rightarrow\pi^+\pi^-\pi^0e^+\nu)=3.20\times10^{-5}\text{ sec}^{-1},$$
$$\Gamma(K^+\rightarrow\pi^0\pi^0\pi^0e^+\nu)=3.56\times10^{-5}\text{ sec}^{-1},$$
$$\Gamma(K_1^0\rightarrow\pi^-\pi^0\pi^0e^+\nu)=1.45\times10^{-5}\text{ sec}^{-1},$$
$$\Gamma(K_1^0\rightarrow\pi^+\pi^-\pi^-e^+\nu)=5.25\times10^{-5}\text{ sec}^{-1}.$$

We have also calculated various spectra and correlation functions for K_{e5} decays based on our current-algebra predictions of the form factors. We shall present these results in a subsequent paper.

VI. DISCUSSION OF RESULTS

From consideration of the current-commutation relations of Gell-Mann and the assumption of a partially conserved axial-vector current we have calculated, to lowest order in pion momenta, the K_{e5} vector form factors in terms of the K_{e3} form factors. By consistently taking all of the pions off the mass shell and expanding amplitudes in powers of pion momenta, we have

[16] We have neglected the term containing the form factor G_4 since it is proportional to the mass of the electron. We shall always neglect the electron's mass.

[17] We have neglected the electromagnetic mass differences of the pions.

[18] We obtain the value of F_π from the Goldberger-Treiman relation, $F_\pi=2g_AM_N/g_r$.

encountered various radiation amplitudes which we are able to calculate to zeroth order in pion momenta. The presence of K-meson and π-meson pole terms in these amplitudes accounts for the complicated momentum dependence of the K_{e5} form factors.

From Eqs. (37)–(40) it can be seen that there are terms in the K_{e5} form factors that vary greatly when different soft-pion limits are taken. It is of particular interest to note that certain soft-multipion limits do not even exist. For example, for the case of $K^+ \to \pi^+\pi^-\pi^0 e^+\nu$,

$$\lim_{q_2\to 0,\, q_3\to 0} G_1 = \infty\,.$$

Thus, in dealing with multipion systems great care should be taken if one is going to consider soft-pion limits or else ambiguities could arise. In this paper we have never gone to a zero four-momentum limit but rather we have only neglected higher powers of pion momenta.

Recently, McNamee and Oakes[19] have also estimated the K_{e5} form factors using current algebra and PCAC. However, they do not treat the pions on an equal footing since they only take one pion off the mass shell and consider its zero four-momentum limit. Furthermore, they insert K-meson and π-meson pole contributions to the K_{e5} amplitude since the momentum dependence of these pole terms cannot be neglected in passing to the soft-pion limit. In evaluating these pole contributions, McNamee and Oaks make use of Weinberg's estimates for the K-π and π-π scattering amplitudes based on current algebra.[20] Although there is a great deal of evidence, especially from K_{e4} decay, to support the validity of using Weinberg's prediction of the K-π scattering amplitude, there is no reason to expect or demand that his results for the π-π scattering amplitude be valid for K_{e5} decay.

In the calculations of the pion-pion scattering amplitude by Weinberg[20] and also by Khuri,[21] a power series expansion in the variables s, t, u is assumed for the amplitude[22]:

$$\begin{aligned}\langle\pi^d(q_4)\pi^b(q_2)|T|\pi^c(q_3)\pi^a(q_1)\rangle &= \delta_{ab}\delta_{cd}[A+B(s+u)+Ct+\cdots]\\ &\quad+\delta_{ad}\delta_{cb}[A+B(s+t)+Cu+\cdots]\\ &\quad+\delta_{ac}\delta_{bd}[A+B(u+t)+Cs+\cdots],\end{aligned}$$

where A, B, C, $\cdots$ are constant coefficients and $s=(q_3+q_1)^2$, $t=(q_1-q_2)^2$, $u=(q_3-q_2)^2$. However, their result for the π-π scattering amplitude is valid only in the domain $0\leq s$, t, $u\leq\mu^2$. On the other hand, in the case of K_{e5} decay, the π-π amplitude is far off the mass shell and $4\mu^2\leq s$, t, $u\leq(M_K-\mu)^2$. The Weinberg-Khuri power series expansion need not be valid up to these values of s, t, u, and their resulting π-π amplitude need not correctly describe the amplitude in K_{e5} decay. Thus, the validity of McNamee and Oakes's evaluation of the pion pole contribution is open to discussion.

In our calculation of the K_{e5} form factors, we do *not* assume a particular form for the K-π or π-π scattering amplitudes. These amplitudes emerge naturally from our expansion in pion momenta and, in fact, whereas our K-π scattering amplitude agrees with Weinberg's prediction, the corresponding π-π amplitude does not. As a result of these differences, the form factors calculated by McNamee and Oakes are noticeably different from ours. Their decay rates are also about one order of magnitude larger than ours.

The K_{e5} rates that we have calculated may serve as a test for the validity of the various models that have been proposed to describe K_{e5} decay. The current-algebra decay rates are of the same order of magnitude as those of the direct interaction model. On the other hand, the current-algebra decay rates are about three orders of magnitude smaller than those of the pion-pole model and about three orders of magnitude larger than the predictions based on the η model. However, it is difficult to compare the current-algebra results with those of the pion-pole model since the current-algebra approach does not take account of possible strong final-state interactions.[23]

APPENDIX

In this Appendix we discuss the "σ terms" which we neglect in Eq. (15). We present two arguments to justify the neglect of these terms. The "σ terms" are given by

$$\Sigma_1 = (1/F_\pi^2)(2\pi)^{3/2}(2q_c^0)^{1/2}\int dx\, e^{i(q_a+q_b)\cdot x} \times\langle\pi^c|T(\sigma_{ab}(x)\mathcal{V}_\sigma^n(0))|K_{km}\rangle\,, \quad \text{(A1)}$$

$$\Sigma_2 = (1/F_\pi^3)(\tau_c)_{np}\int dx\, e^{i(q_a+q_b)\cdot x} \times\langle 0|T(\sigma_{ab}(x)\mathcal{A}_\sigma^p(0))|K_{km}\rangle\,, \quad \text{(A2)}$$

$$\Sigma_3 = -(\mu^2/F_\pi^2)\delta_{ab}\int dx\, e^{i(q_a+q_b+q_c)\cdot x} \times\langle 0|T(\phi_\pi^c(x)\mathcal{V}_\sigma^n(0))|K_{km}\rangle\,. \quad \text{(A3)}$$

[19] P. McNamee and R. J. Oakes, Phys. Letters **24B**, 629 (1967).
[20] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).
[21] N. N. Khuri, Phys. Rev. **153**, 1477 (1967).
[22] Recently, this expansion has been criticized on grounds that at the physical threshold, such an expansion neglects the unitarity branch cut [see J. Iliopoulos, Nuovo Cimento **52A**, 192 (1967)].
[23] There has been some recent speculation that the "σ terms," arising from the current-commutation relations when more than one pion is taken off the mass shell, correspond to final-state interactions. See L. S. Kisslinger, Phys. Rev. Letters **18**, 861 (1967).

We consider the quantity

$$B_\mu=\int dxdz\, e^{i(q_a+q_b)\cdot x+iq_c\cdot z}\times\langle 0|T(A_\mu{}^c(z)\sigma_{ab}(x)\mathcal{V}_\sigma{}^n(0))|K_{km}\rangle. \quad (A4)$$

If we extract out the pion-pole term, we can write

$$-iq_{c\mu}C_\mu=-iq_{c\mu}B_\mu-[F_\pi q_c^2D/(\mu^2-q_c^2)], \quad (A5)$$

where

$$D=(\mu^2-q_c^2)\int dxdz\, e^{i(q_a+q_b)\cdot x+iq_c\cdot z}\times\langle 0|T(\phi_\pi{}^c(z)\sigma_{ab}(x)\mathcal{V}_\sigma{}^n(0))|K_{km}\rangle. \quad (A6)$$

Computing $iq_{c\mu}B_\mu$, we obtain

$$-iq_{c\mu}C_\mu=F_\pi(2\pi)^{3/2}(2q_c{}^0)^{1/2}\int dx\, e^{i(q_a+q_b)\cdot x}\langle\pi^c|T(\sigma_{ab}(x)\mathcal{V}_\sigma{}^n(0))|K_{km}\rangle-(\tau_c)_{np}\int dx\, e^{i(q_a+q_b)\cdot x}\times\langle 0|T(\sigma_{ab}(x)\mathcal{A}_\sigma{}^p(0))|K_{km}\rangle+F_\pi\mu^2\delta_{ab}\int dx\, e^{i(q_a+q_b+q_c)\cdot x}\langle 0|T(\phi_\pi{}^c(x)\mathcal{V}_\sigma{}^n(0))|K_{km}\rangle. \quad (A7)$$

We observe that the right-hand side of Eq. (A7) has precisely the "σ terms" we are interested in studying. If we now proceed to the soft-pion limit $q_c\to 0$ we obtain the relation

$$\Sigma_1-\Sigma_2=\Sigma_3 \quad (q_c\to 0). \quad (A8)$$

In the spirit of PCAC we may argue that the Σ_i are slowly varying functions of pion mass and that Eq. (A8) will also hold approximately on the mass shell $q_c^2=\mu^2$. Furthermore, we can neglect Σ_2 on the basis of the arguments given by Weinberg.[5] Thus we have on the mass shell

$$\Sigma_1\approx\Sigma_3 \quad (q_c^2=\mu^2). \quad (A9)$$

Σ_3 can be evaluated in terms of the K_{e3} form factors, keeping only the pion-pole contribution. One finds

$$\Sigma_3=-\frac{1}{F_\pi{}^2}\frac{\mu^2}{(q_a+q_b+q_c)^2-\mu^2}i(2\pi)^{-3/2}(2k_0)^{-1/2}\times[(k+q_a+q_b+q_c)_\sigma f_++(k-q_a-q_b-q_c)_\sigma f_-]. \quad (A10)$$

From Eq. (A10) it is evident that Σ_3 is of the order μ^2/M_K^2 since $9\mu^2\leq(q_a+q_b+q_c)^2\leq M_K^2$. Furthermore, from Eq. (A9), Σ_1 will also be of this same order. However, in our approximation scheme, we have consistently neglected terms of this order. We thus conclude that we may safely neglect all of the "σ terms" in our calculation.

It is possible to give a somewhat different argument for dropping the "σ terms." Once again neglecting Σ_2 as Weinberg did and also nelgecting Σ_3 since it is of order μ^2/M_K^2, we consider Σ_1. This term contains a pion-pole contribution involving the matrix element $\langle\pi|\sigma|\pi\rangle$. More precisely, keeping only the one-pion pole contribution, Eq. (A1) yields

$$\Sigma_1=\frac{\delta_{ab}}{F_\pi{}^2}(2\pi)^{3/2}(2q_c{}^0)^{1/2}\frac{\langle\pi^c(q_c)|\sigma|\pi^c(q_a+q_b+q_c)\rangle}{(q_a+q_b+q_c)^2-\mu^2}\times\langle\pi^c(q_a+q_b+q_c)|\mathcal{V}_\sigma{}^n|K_{km}\rangle, \quad (A11)$$

where $\sigma_{ab}(x)=\delta_{ab}\sigma(x)$. There has been some study of the matrix element $\langle\pi|\sigma|\pi\rangle$ which defines the $\sigma\pi\pi$ vertex. In particular, Khuri[21] has shown using current algebra that

$$\lim_{p\to 0,\, q^2\to\mu^2}(2\pi)^3(4p_0q_0)^{1/2}\langle\pi(p)|\sigma|\pi(q)\rangle\equiv f^\sigma(\mu^2,0,\mu^2)=-\mu^2, \quad (A12)$$

where $f^\sigma(q^2,p^2,(q-p)^2)$ is the σ form factor. From Eq. (A11) it is seen that we are dealing with the σ form factor $f^\sigma(q^2,\mu^2,\Delta^2)$, where $9\mu^2\leq q^2\leq M_K^2$, $4\mu^2\leq\Delta^2\leq(M_K-\mu)^2$. Thus, in order to make use of Khuri's result, we must make a large extrapolation in q^2 and Δ^2. However, if we do this then we find that, once again, $\Sigma_1=O(\mu^2/M_K^2)$ and can thus be neglected.