Reggeization of External Particles*

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(Received 16 February 1968)

The possibility of determining the Regge trajectory of an external particle is raised by considering the constraints imposed by bootstrap conditions on the two-body scattering of particles of arbitrary spins and masses. Assuming that the trajectory is the solution of a certain system of equations, it is made plausible that a reduced system corresponding to the continuation in spin and mass of only one external particle can equivalently generate the trajectory if one point on the trajectory is known. The particular system of $\pi\pi$ scattering is taken up as a test case where one external pion has arbitrary mass and spin but otherwise carries the pion internal quantum numbers, and where the bootstrap conditions are based on the self-supporting mechanism of the ρ. Linear combinations of the two independent helicity amplitudes, having kinematic singularities that are uniformly factorizable for arbitrary J, are constructed and unitarized by a phase ND^{-1} method. An implicit solution of the bootstrap matching equations is found, yielding a trajectory which identically passes through the physical pion when $J\to 0$. Only the ratio of the Regge-pole energy-plane residues of the two helicity states is determinable. Finally, a test for uniqueness, largely based on the asymptotic behavior of this ratio as $|J|\to\infty$, is established.

1. INTRODUCTION

THE purpose of this paper is to determine whether crossing symmetry, implemented in the form of a bootstrap hypothesis, can generate the Regge trajectory of an external particle in a hadron scattering process. We first define what we mean by the Reggeization of an external particle. Consider a certain fixed set of strongly conserved internal quantum numbers excluding spin (parity, isotopic spin, G parity, etc.). The discrete family of hypothetically naturally occurring particles which share these quantum numbers will play a special role in what follows, in the sense that the masses and spins are not known at the outset. On the other hand, the masses and spins of all other naturally occurring particles will be regarded as empirical data which we may freely draw upon.

We make the following two suppositions:

(i) Suppose that we have at our disposal a dynamical recipe for computing the two-body scattering amplitude shown in Fig. 1, for particles of arbitrary spins J_i and masses M_i, $i=1,\cdots,4$, carrying the same fixed set of strongly conserved internal quantum numbers. This recipe is to be defined ordinarily for integral or half-integral $J_i\geq 0$ and for real M_i^2 subject to $0<M_i^2<M_t^2$, where M_t^2 is the threshold for instability and applies to all values of J_i of the same signature, which we also include among the set of quantum numbers labeling the special family of particles. Such arbitrary values of J_i and M_i we call "ordinary." Included in the recipe is the specification of the masses and spins of other particles and a set of coupling constants. Some of the coupling constants are to be regarded as empirical data, while others, involving the special family of particles, are to be regarded as unknown parameters a_k.

(ii) It is observed that for any given set of internal quantum numbers, very few values of spin and mass actually occur in nature. For example, there is only one stable particle of odd parity, isotopic spin$=1$, odd G parity, strangeness$=0$, baryon number$=0$, namely, the pion of mass $M=1$ and $J=0$. We suppose that this restriction to certain values of J_i and M_i is accomplished by including in the recipe a dynamical principle which is to be imposed on the scattering amplitude. The dynamical principle in mind is crossing symmetry in the form of bootstrap conditions which relate the amplitude to the masses and spins of other particles. After applying this additional dynamical principle, we must find that this restriction manifests itself by requiring each M_i to equal a certain function of J_i, say, $M_0(J_i)$. This supposition, coupled with the facts that J_i is integral or half-integral ≥ 0 and M_i^2 is real and positive, should yield the discrete family of naturally occurring particles, provided that the dynamical recipe and bootstrap conditions are correct.

In order to extend the above family of particles to include all the resonances sharing the same internal quantum numbers (but incapable of being prepared as asymptotic states), the complete dynamical recipe should admit of continuation to complex M_i^2. In fact, for many sets of internal quantum numbers it occurs that the observed values of M^2, for which J has physical values, will always be complex, since it is observed that there are many families of resonances which do not include any stable particles. We shall, however, suppose that there is at least one stable particle in the special family under consideration.

Now we note that if the inverse $J_i=J_0(M_i^2)$ of the function $M_i=M_0(J_i)$ were the Regge trajectory labeled by the internal quantum numbers under consideration, then the family of discrete particles and resonances would be just the Regge recurrences of the trajectory.

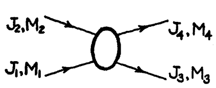

FIG. 1. Scattering amplitude for particles of arbitrary spins and masses.

* Research supported by the National Science Foundation.

This hypothesis requires that the dynamical model also admit of interpolation between integral values of J_i. The existence of a unique interpolating function in turn requires by Carlson's theorem[1] that the model be defined for all integers or half-integers $J_i \geq 0$ of definite signature for any given M_i^2 and that the amplitude satisfies certain other conditions of analyticity and boundedness in J_i.

This suggests that the dynamical recipe and bootstrap conditions could be said to generate the Regge trajectory; and hence by Reggeization of an external particle we mean the analytic continuation of the scattering amplitude from *ordinary* values of spin and mass, where the recipe is defined to values related by $J_i = J_0(M_i^2)$, where the recipe may not be defined.

In Sec. 2, the questions of completeness in the unitarity sum when there are recurrences and the role of different helicity states in external spin continuation are discussed. We also show how the specialized procedure of continuing in the mass and spin of only one external particle is related to the general and much more difficult procedure of continuing in all four.

The remaining sections are devoted to studying the concrete example of $\pi\pi$ scattering, in which we attempt to generate the trajectory with the pion as lowest member. In Sec. 3, the pion-pion amplitudes are constructed in a ρ-dominant model in terms of *a priori* unknown coupling constants that are assumed to depend on the spin and mass of an external "pion," and the Born amplitudes so found are unitarized. In Sec. 4, bootstrap equations are established and solved for the implicit trajectory equation, and in Sec. 5, the problem of uniqueness is taken up.

2. GENERAL REMARKS

A. Unitarity

An essential ingredient in the recipe for determining the scattering amplitude of the particles of arbitrary spin and mass must be the statement of unitarity for the transition matrix $T = -i(S-1)$. This is

$$\langle f|T^\dagger - T|i\rangle = i\langle f|T^\dagger|\sum_n |n\rangle\langle n|T|i\rangle,$$

where the sum extends over all states $|n\rangle$ of stable particles, assumed to make up a complete set of states, at least with respect to the initial and final states $|i\rangle$ and $|f\rangle$, when $J_i = J_0(M_i^2)$. Let us separate the sum into a part made up of states $|n_R\rangle$ containing the special particles and a part made up of states $|n'\rangle$ of all other particles which can participate in the unitarity equation

$$\sum_n |n\rangle\langle n| = \sum_{n_R} |n_R\rangle\langle n_R| + \sum_{n'} |n'\rangle\langle n'|.$$

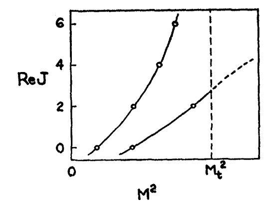

FIG. 2. Schematic plots of Re J as a function of real M^2: J_a and J_0 are examples of deformed and true trajectories, respectively. The dashed line at M_t^2 represents the threshold to the right of which the trajectories are complex-valued. The circles are even-signature recurrences which enter the unitarity sum.

Any stable recurrences of the special particles must of course be included among the states $|n_R\rangle$. In the following, we discuss alternate ways of introducing such resonant states in the unitarity sum:

(i) The recurrences may be introduced in a systematic way by tailoring the concept of arbitrary J_i and M_i as follows: Instead of arbitrary and independent values of J_i and M_i, let us consider an arbitrary trajectory $J_a(M_i^2)$, as in Fig. 2, which is a deformation of the naturally occurring (true) trajectory $J_0(M_i^2)$ and to which all the special particles are tentatively assigned.

Under arbitrary deformations of the trajectory, the mass of a recurrence may pass above the threshold; the particle becomes unstable and therefore seems to disappear discontinuously from the set $|n_R\rangle$. In fact, the state could be retained in the set $|n_R\rangle$ by using the isobar-model approximation,[2] but otherwise the effect of the state is not lost, since the contribution of any unstable particle may always be relegated to an asymptotic state containing a larger number of stable particles, with a corresponding redefinition of the T matrix.

It is clear that for a given total energy the set of states $|n_R\rangle$ required by completeness depends only on the specification of the trajectory $J_a(M_i^2)$ and is independent of any particular choice of the four external masses, as long as they are on this same trajectory. Since $J_0(M_i^2)$ is an analytic function[3] with a branch point at the threshold M_t^2, it is natural to suppose that the deformed trajectory $J_a(M_i^2)$ is itself an analytic function with the same threshold.

However, the requirement that the bootstrap hypothesis and unitarity are to be satisfied for any values of J_i and M_i as long as each pair (J_i, M_i) is somewhere on the true trajectory suggests that the continuations of the four external particles from ordinary values of spin and mass to the true trajectory values should be carried out independently. Such relative independence is impossible if all four particles are always simultaneously assigned to the same trajectory.

(ii) The four external particles may be assigned to independent arbitrary external trajectories, each of which is a deformation of the true trajectory. In this case, there would be generally four different masses for each physical spin. In order to have completeness in

[1] E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, London, 1964), 2nd ed., p. 186.

[2] S. Mandelstam, J. E. Paton, R. F. Peierls, and A. Q. Sarker, Ann. Phys. (N. Y.) 18, 198 (1962).

[3] A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1962); J. R. Taylor, *ibid.* **127**, 2257 (1962).

in the unitarity sum, the states $|n_R\rangle$ would have to be made up of the physical points of four different trajectories. Since the true trajectory is independently approached for each external particle, the states $|n_R\rangle$ would be overweighted by a factor of 4, which of course is very unsatisfactory.

(iii) We could relax the strict notion of completeness as applied to arbitrarily deformed trajectories and instead suppose that there is a fifth independent trajectory to which the states $|n_R\rangle$ are assigned. Completeness would be achieved, however, when all five trajectories approach the same true trajectory.

(iv) If we adopt alternative (iii) above, we may note that there is now no necessary role played by the four arbitrarily specified trajectories. Instead, it is sufficient that the four external spins and masses are simply all independent variables, not assigned to any trajectories, while the particles of states $|n_R\rangle$, however, are still assigned to an arbitrary trajectory $J_a(M^2)$. This fourth alternative is the plan which we follow in the remainder of this section.

Since the continuations which must be admitted in the external spins and masses do not now apply to nor disturb the states $|n_R\rangle$ according to alternative (iv), the spins and masses of the special particles in the states $|n_R\rangle$ play the roles of discrete unknowns. In terms of the inverse function $M_a(J)$ of the arbitrary trajectory $J_a(M^2)$, the unknown spins and masses in $|n_R\rangle$ are j_k and $m_k=M_a(j_k)$, respectively, where k labels the finite set of spins on the stable part of the trajectory. Here we suppose that j_1 is the lowest spin value on the trajectory, j_2 is the next higher, etc.

B. Helicities

Within the framework of an angular momentum decomposition of the scattering amplitude where the external states $|i\rangle$ and $|f\rangle$ are helicity states of definite total angular momentum, we find that the independent continuations of the J_i must be carried out for fixed values of the helicities. We show this in the following by eliminating the only two reasonable alternatives:

(i) One procedure might be to continue the helicities along with the spins, e.g., let the helicity $\mu_1=J_1-\Delta_1$, where Δ_1 is some fixed non-negative integer. However, the requirement that the recipe for constructing the amplitude be defined for all ordinary values of J_1 can be fulfilled for the amplitude of total angular momentum L only for the finite number of values of J_1 of definite signature satisfying $-L\leq\mu_1-\mu_2\leq L$, or, equivalently,

$$J_2-\Delta_2-L\leq J_1-\Delta_1\leq J_2-\Delta_2+L.$$

(ii) An argument showing that the helicities μ_i cannot be continued independently of the spins and masses follows in a similar vein.

Thus, regarding the helicities as discrete-valued parameters having fixed values while the spins are varied, we accordingly expect that the recipe is not defined at the "nonsense" J values, and so we modify the definition of ordinary spin by requiring $|\mu_i|\leq J_i$ in place of $0\leq J_i$.

C. Reduced Systems

Since the scattering amplitude decomposes into amplitudes of definite total angular momentum and helicities, the statement of unitarity decomposes correspondingly into a set of equations which we write symbolically as

$$U_\lambda(J_1M_1,\cdots,J_4M_4;\,j_1m_1',j_2m_2',\cdots;\,a_1,a_2,\cdots)=0,\quad(1)$$

where λ denotes the total angular momentum and helicity quantum numbers, and the entries j_km_k' appear in order of increasing spin. In similar symbolic form we write the bootstrap conditions which operate on the scattering amplitude as another set of equations:

$$B_\lambda(J_1M_1,\cdots,J_4M_4;\,a_1,a_2,\cdots)=0.\quad(2)$$

In a practical calculation, which inevitably entails approximations, it is not unreasonable to expect that we may be able to isolate a finite number of total angular momenta and to construct the dynamical model and its number of parameters a_k in such a way that the a_k can be eliminated among Eqs. (1) and (2), leaving a smaller set of such rank that the four trajectories are implicitly determined. Let us take this largely unjustified step forward for the sake of expedience. A more general justification is the subject of future work, but for the moment, let it suffice to observe that an explicit example of such a model exists and will be worked out in the following sections. Henceforth we assume that Eqs. (1) and (2) represent the smaller set. Eliminating the a_k, we obtain a set of equations of the form

$$C(J_1M_1,\cdots,J_4M_4;\,j_1m_1',j_2m_2',\cdots)=0.\quad(3)$$

If it is true that the reduced system of Eqs. (1) and (2) generates the true trajectory, then the only mass values that satisfy Eq. (3) must be $M_i=M_0(J_i)$ and $m_k'=M_0(j_k)$ for any values of J_i. Equivalently, these are the only values which can satisfy the reduced Eqs. (1) and (2) simultaneously. In particular, let us set three of the external spins and masses at the values j_1 and $m_1=M_0(j_1)$, the values of the lowest member of the true trajectory (assumed stable), and further suppose that the arbitrary trajectory for the states $|n_R\rangle$ also passes through the point j_1m_1. Thus, letting $J_2=J_3=J_4=j_1$ and $M_2=M_3=M_4=m_1'=m_1$, Eqs. (1) and (2) become, with an obvious simplification of notation,

$$U_\lambda(J_1M_1;\,j_1m_1;\,j_2m_2',j_2m_3',\cdots;\,a_1,a_2,\cdots)=0\quad(4)$$

and

$$B_\lambda(J_1M_1;\,j_1m_1;\,a_1,a_2,\cdots)=0.\quad(5)$$

This specialization of Eqs. (1) and (2) incidentally causes a further reduction because there are only a few possible helicity states for particles 2, 3, and 4. For example, the lowest member of an even-signature boson trajectory has both spin and helicity equal to zero.

The a_k can again be eliminated from this further reduced set of equations, yielding the single equation

$$C(J_1M_1;\, j_1m_1;\, j_2m_2',j_3m_3',\cdots)=0\,, \qquad (6)$$

which again must have the solution $M_1=M_0(J_1)$ and $m_k'=M_0(j_k)$. We can arrive at this solution systematically by generating a sequence of equations from Eq. (6) in which J_1M_1 are replaced by j_2m_2', j_3m_3', $\cdots$, solving these for the discrete unknown recurrent masses m_k', and then inserting these masses in Eq. (6) and solving for M_1 as a continuous function of J_1. Finally, substituting $M_1=M_0(J_1)$ in Eqs. (4) and (5), we can solve for the parameters a_k, or perhaps only for their ratios, in terms of the single independent variable J_1.

Thus, if we allow ourselves the luxury of regarding the spin and mass of the lowest member of the trajectory as part of the empirical data fed into the dynamical model, then it is sufficient to work with highly reduced Eqs. (4), (5), and (6), corresponding to the continuation of only *one* of the four external particles in spin and mass.

3. $\pi\pi$ SCATTERING

A. Introduction

To illustrate and test the above ideas in the framework of a realistic model, we consider the family of hadrons of which the pion is the lowest member. The model which we propose to study assumes that the scattering of such particles is accounted for in the same way as in the scattering of two pions, which in turn are assumed to interact primarily by the exchange of a ρ meson. The ρ, a dominant feature in $\pi\pi$ scattering, will be considered here for simplicity to be an elementary (non-Regge) particle with given properties: vector boson, odd parity, isotopic spin$=1$, fixed mass and width. Crossing symmetry, which, together with unitarity, we conjecture, will generate the pion trajectory, will be imposed in the approximate form of the self-supporting bootstrap of the ρ, i.e., the force due to the exchange of the ρ is such as to reproduce the ρ as a p-wave resonance in the $I=1$ scattering amplitude.[4]

In accordance with the suggested method of working with the reduced system of equations, we suppose also that we know the mass and spin of the pion (1 and 0, respectively), and so we consider the scattering process $\pi^*+\pi\rightarrow\pi+\pi$, where π^* is a particle of mass M and spin J, but otherwise carries the internal quantum numbers of the pion.

Of fundamental importance in the model is the $\pi\pi^*\rho$ vertex. In general, there are three types of couplings

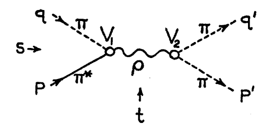

FIG. 3. The direct amplitude for $\pi^*\pi\rightarrow\pi\pi$ containing the ρ pole in the s channel.

between particles of the specified spins, but parity conservation reduces these to two. In an angular momentum tensor representation of the spin wave functions, the direct product of the spin wave functions is a tensor of rank $J+1$. To obtain a scalar vertex function, we must contract this tensor into a tensor of rank $J+1$ formed from the one linearly independent momentum vector in the $\pi\pi^*\rho$ system. Taking this to be the momentum $\mathbf{q}$ of the π in the rest frame of the π^*, we find three possible tensor types: $(\mathbf{q}^{J+1})_{ijk\cdots mn}$, $\delta_{ij}(\mathbf{q}^{J-1})_{k\cdots mn}$, and $(\mathbf{q}^{J})_{ijk\cdots l}\epsilon_{lmn}$. These tensors have parities $(-)^{J+1}$, $(-)^{J-1}$, and $(-)^{J}$, respectively. Since the product of the intrinsic parities of the three particles is *odd*, J must be *even* for the first and second types and *odd* for the third type. But the family of particles π^* has positive signature, and therefore we can only have couplings of the first and second types. Furthermore, when $\pi^*\rightarrow\pi$, the second type of coupling cannot exist.

B. Direct Amplitude

We calculate the direct s-channel amplitude of Fig. 3, where initial particles of momenta p and q have spin and mass (J,M) and $(0,1)$, respectively. The intermediate particle ρ has spin and mass $(1,m)$, and the final particles of momenta p' and q' both have spin and mass equal to $(0,1)$. The amplitude for π^* helicity $=\lambda$ is

$$A_\lambda{}^{(s)}=V_2{}^\delta P_\delta{}^\epsilon V_{1\epsilon\eta\mu\cdots\nu}(\Phi_\lambda{}^J)^{\eta\mu\cdots\nu}\,, \qquad (7)$$

where $\Phi_\lambda{}^J$ is the tensor-represented spin wave function for spin J and helicity λ, V_1 is the rank $J+1$ tensor associated with the first vertex, $P_\delta{}^\epsilon$ is the vector-boson propagator, and $V_2{}^\delta$ is the rank-1 tensor associated with the second vertex.

The momentum tensor of the first vertex is taken to be

$$V_1=a'(q-p)_\epsilon(q-p)_\eta\cdots(q-p)_\nu + b'g_{\epsilon\eta}(q-p)_\mu\cdots(q-p)_\nu\,, \qquad (8)$$

carrying $J+1$ indices, with $b'=0$ when $J=0$. The combination $(q-p)$ is one choice of many possibilities. It is chosen here because the interaction is then formally antisymmetric under exchange of q and p when the signature is even. Since the initial particles are in the antisymmetric $I=1$ isotopic-spin state, under exchange of both isotopic spin and momenta, the total state is symmetric, as it should be for bosons when $J=0$. For J different from zero we no longer have a state of identical bosons, and the meaning of the symmetry is obscure. However, since $p_\eta(\Phi_\lambda{}^J)^{\eta\mu\cdots\nu}=0$, Eq. (8) is

[4] F. Zachariasen and C. Zemach, Phys. Rev. 128, 849 (1962).

indistinguishable from

$$V_1 = a'(q-p)_\epsilon q_\eta q_\mu \cdots q_\nu + b' g_{\epsilon\eta} q_\mu \cdots q_\nu , \quad (9)$$

which no longer has any formal symmetry, except in term $(q-p)_\epsilon$, which is the only survivor when $J=0$.

The coupling constants a' and b' presumably depend on J and M in some way which is unknown at first. The bootstrap conditions will ultimately determine this dependence on trajectory values of J and M.

Using the vector-boson propagator

$$P_\delta{}^\epsilon = [(p+q)_\delta (p+q)^\epsilon / m^2 - \delta_\delta{}^\epsilon](s - m^2 + i\gamma)^{-1},$$

where γ is related to the width of the ρ, and the second vertex tensor $V_2{}^\delta = (q'-p')^\delta$, where the constant a' evaluated at $J=0$, $M=1$ has been absorbed into the a' and b' constants of the first vertex, we obtain

$$A_\lambda{}^{(s)} = (s-m^2+i\gamma)^{-1}\{2b' p_\eta{}' q_\mu \cdots q_\nu - [b' + a'(q'-p')\cdot(q-p)] q_\eta q_\mu \cdots q_\nu\} \times (\Phi_\lambda{}^J)^{\eta\mu\cdots\nu}. \quad (10)$$

The helicity quantization axis, or z axis, is in the direction $-\mathbf{q}$ in the center-of-mass (c.m.) system. In the rest frame of π^* obtained by a pure velocity transformation from the c.m. system, let the 3-momentum of q be $-Q\hat{z}$. We define the scattering angle θ in the s-channel c.m. system to be the angle between $\mathbf{p}$ and $\mathbf{p}'$. In the π^* rest frame, let this angle be ψ and the magnitude of $\mathbf{p}'$ be Q'. Then, by using the techniques of angular momentum tensors,[5] we find

$$q_\eta \cdots q_\nu (\Phi_\lambda{}^J)^{\eta\cdots\nu} = Q^J [J!/(2J-1)!!]^{1/2} \delta_{0,\lambda} \quad (11)$$

and

$$p_\eta{}' q_\mu \cdots q_\nu (\Phi_\lambda{}^J)^{\eta\mu\cdots\nu} = -Q' Q^{J-1} [(J+1)!/J(2J-1)!!]^{1/2} \times \{-\tfrac{1}{2}\sin\psi\,\delta_{1,\lambda} + [J/(J+1)]^{1/2}\cos\psi\,\delta_{0,\lambda} + \tfrac{1}{2}\sin\psi\,\delta_{-1,\lambda}\}. \quad (12)$$

The following kinematical definitions and relations are useful in carrying out further reduction of (10):

$$s=(p+q)^2, \quad t=(p-p')^2, \quad u=(p-q')^2,$$
$$S=[s^2-2s(M^2+1)+(M^2-1)^2]^{1/2},$$
$$T=[t^2-2t(M^2+1)+(M^2-1)^2]^{1/2},$$
$$2MQ=S, \quad 2MQ'=T,$$
$$ST\sin\psi = 2M[stu-(M^2-1)^2]^{1/2},$$
$$2ST\cos\psi = (t-u)(s+M^2-1)-S^2,$$
$$S(s-4)^{1/2}\cos\theta = s^{1/2}(2t+s-M^2-3),$$
$$S(s-4)^{1/2}\sin\theta = 2[stu-(M^2-1)^2]^{1/2}.$$

Inserting Eqs. (11) and (12) into (10) and simplifying, we obtain

$$A_0{}^{(s)} = (s-m^2+i\gamma)^{-1} f_0(s;J,M) d_{00}{}^1(\theta), \quad (13)$$

$$A_{\pm 1}{}^{(s)} = \pm(s-m^2+i\gamma)^{-1} f_1(s;J,M) d_{10}{}^1(\theta), \quad (14)$$

[5] C. Zemach, Phys. Rev. 140, B97 (1965).

where

$$f_0 = S^{J-1}\left(\frac{s-4}{s}\right)^{1/2}\left[aS^2 + b\left(\frac{J}{J+1}\right)^{1/2}\frac{s+M^2-1}{M\sqrt{2}}\right], \quad (15)$$

$$f_1 = bS^{J-1}(s-4)^{1/2}. \quad (16)$$

Non-energy-dependent factors have been absorbed into the coupling constants a' and b' by the replacements

$$a' = a(2M)^J[(2J-1)!!/J!]^{1/2}$$

and

$$b' = -b\sqrt{2}(2M)^{J-1}[J(2J-1)!!/(J+1)!]^{1/2}.$$

Note that

$$\lim_{J\to 0}[(2J-1)!!/J!]^{1/2} = 1,$$

and hence the vanishing of b' at $J=0$ implies the more moderate condition on b,

$$\lim_{J\to 0} b' = \lim_{J\to 0} b\sqrt{J} = 0. \quad (17)$$

C. Crossed Amplitudes

We define $A_\lambda{}^{(t)}$ and $A_\lambda{}^{(u)}$ to be the analytic continuations into the s-channel physical region of the t- and u-channel helicity amplitudes due to direct ρ poles in the t and u channels, respectively. Thus, to construct $A_\lambda{}^{(t)}$, we interchange s and t in Eqs. (13)–(16), obtaining t-channel helicity amplitudes with the ρ pole in the t variable, and then analytically continue the variables s and t into the s-channel physical region where $t<0$, $u<0$, and $s \geq (M^2-1)^2/tu$.

Consider the analytic continuation of a function defined in the t-channel physical region to that of the s channel. If branch points are encountered, we follow the usual rule of starting in the t channel from a point $t_1 + i(\epsilon'+\epsilon'')$, $s_1 - i\epsilon'$, $u_1 - i\epsilon''$, with $\epsilon', \epsilon'' > 0$, maintaining these infinitesimal imaginary parts to avoid branch points, and analytically continuing to the neighborhood of the central point $s=t=u=1+\frac{1}{3}M^2$. Near the central point the path of continuation crosses the real s and t axes and then continues to the final point $s_2 + i(\epsilon'+\epsilon'')$, $t_2 - i\epsilon'$, $u_2 - i\epsilon''$ in the s channel.

We find that for $0<M<3$, the functions S, T, and $[stu-(M^2-1)^2]^{1/2}$ are positive real in the three physical regions, while $t^{1/2}$ and $(t-4)^{-1/2}$ are negative imaginary in the s- and u-channel physical regions if the positive real roots are taken in the t channel.

The diagram obtained from Fig. 3 by crossing lines p' and q contains the ρ pole in the t channel. The corresponding s-channel helicity amplitudes may be found by using helicity crossing relations.[6] These amplitudes are

$$B_\lambda{}'(s,\cos\theta) = (-)^\lambda \sum_{\lambda'} A_{\lambda'}{}^{(t)} d_{\lambda'\lambda}{}^J(\psi),$$

[6] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) 26, 322 (1964).

while the s-channel amplitudes corresponding to the ρ pole in the u channel are

$$B_\lambda''(s,\cos\theta)=\sum_{\lambda'}(-)^{\lambda'}A_{\lambda'}{}^{(u)}d_{\lambda'\lambda}{}^J(\chi),$$

where $\psi\leftrightarrow\chi$ when $u\leftrightarrow t$ or $\cos\theta\leftrightarrow-\cos\theta$. It follows that $B_\lambda''(s,\cos\theta)=(-)^{\lambda+1}B_\lambda'(s,-\cos\theta)$, and so it is sufficient to determine B_λ'. Interchanging s and t in the amplitudes f_0 and f_1 and in the rotation matrix elements $d_{00}{}^1(\theta)=\cos\theta$ and $d_{10}{}^1(\theta)=-\sin\theta/\sqrt{2}$ and carrying out the t- to s-channel continuations, we find

$$B_\lambda'=(-)^\lambda(m^2-t)^{-1}\Bigg\{b\sqrt{2}[stu-(M^2-1)^2]^{1/2}\times T^{J-2}[d_{1\lambda}{}^J(\psi)-d_{-1\lambda}{}^J(\psi)]-T^{J-2}\left[aT^2+b\left(\frac{J}{J+1}\right)^{1/2}\times\left(\frac{t+M^2-1}{\sqrt{2}M}\right)\right](2s+t-M^2-3)d_{0\lambda}{}^J(\psi)\Bigg\},\quad(18)$$

where we have dropped the $i\gamma$, since the denominator (m^2-t) cannot vanish in the s-channel physical region. The sum of the two crossed amplitudes then gives the complete Born amplitude

$$B_\lambda=\tfrac{1}{2}(B_\lambda'+B_\lambda''),\quad(19)$$

where the factor $\frac{1}{2}$ and relative plus sign is appropriate for the isotopic spin$=1$ projection.

D. Partial-Wave Projection

The expansion of amplitude (19) into amplitudes of definite total angular momentum L is[7]

$$B_\lambda(s,\cos\theta)=\sum_L\bar{B}_\lambda{}^L(s)d_{\lambda0}{}^L(\theta).\quad(20)$$

Inverting this and picking out $L=1$, we obtain

$$\bar{B}_\lambda(s)=\frac{3}{2}\int_0^\pi d_{\lambda0}{}^1(\theta)B_\lambda(s,\cos\theta)\sin\theta d\theta,\quad(21)$$

the helicity partial-wave Born amplitude.

Rotation matrix elements $d_{\lambda'\lambda}{}^J(\psi)$ are simply related to Jacobi polynomials,[8] which in turn are conveniently written as combinations of Legendre polynomials[9] when J is arbitrary, as long as $|\lambda'\pm\lambda|$ is a small integer.[10]

[7] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

[8] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957), p. 58.

[9] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. II, p. 173.

[10] For the general result, see A. H. Mueller and T. L. Trueman, Phys. Rev. **160**, 1296 (1967).

The relations needed in Eq. (18) are

$$d_{00}{}^J(\psi)=P_J(\cos\psi),$$

$$d_{01}{}^J(\psi)=[J(J+1)]^{-1/2}\sin\psi\,P_J'(\cos\psi)=\left(\frac{J}{J+1}\right)^{1/2}\frac{1}{\sin\psi}[P_{J-1}(\cos\psi)-\cos\psi\,P_J(\cos\psi)],\quad(22)$$

$$d_{10}{}^J(\psi)-d_{-10}{}^J(\psi)=-2d_{01}{}^J(\psi),$$

$$d_{11}{}^J(\psi)-d_{-11}{}^J(\psi)=2[P_J(\cos\psi)-\cos\psi\,P_J'(\cos\psi)/J(J+1)].$$

Equation (21) becomes, with (18), (19), the above relations, and some kinematical manipulations,

$$\bar{B}_0(s)=\frac{3}{2}\int_0^\pi\Bigg\{-\frac{b\sqrt{2}}{M}\left(\frac{J}{J+1}\right)^{1/2}\times ST^{J-1}[P_{J-1}(\cos\psi)-\cos\psi P_J(\cos\psi)]-T^{J-2}\left[aT^2+b\left(\frac{J}{J+1}\right)^{1/2}\left(\frac{t+M^2-1}{\sqrt{2}M}\right)\right]\times(2s+t-M^2-3)P_J(\cos\psi)\Bigg\}\sin\theta\cos\theta d\theta/(m^2-t)\quad(23)$$

and

$$\bar{B}_1(s)=\frac{3}{2\sqrt{2}}\int_0^\pi\Bigg\{b\sqrt{2}S(s-4)^{1/2}T^{J-2}\times[P_J(\cos\psi)-[\cos\psi/J(J+1)]P_J'(\cos\psi)]\sin\theta-T^{J-2}\left[aT^2+b\left(\frac{J}{J+1}\right)^{1/2}\left(\frac{t+M^2-1}{M\sqrt{2}}\right)\right]\times(2s+t-M^2-3)[J(J+1)]^{-1/2}\sin\psi\,P_J'(\cos\psi)\Bigg\}\times\sin^2\theta d\theta/(m^2-t).\quad(24)$$

E. Kinematic Singularities

The expression (16) shows that f_1 has factorizable kinematic singularities carried by the factor

$$\rho_1(s;J,M)=S^{J-1}(s-4)^{1/2},\quad(25)$$

while (15) shows that the singularities are not uniformly factorizable for arbitrary values of J in the pure $\lambda=0$ amplitude. On the other hand, the linear combination of amplitudes f_0-cf_1, where

$$c=c(s;J,M)=\frac{s+M^2-1}{M}\left[\frac{J}{2s(J+1)}\right]^{1/2},\quad(26)$$

does contain the factorizable kinematic term

$$\rho_{01}(s;J,M)=S^{J+1}[(s-4)/s]^{1/2}.\quad(27)$$

The same linear combination is found from an arbitrary vertex tensor constructed from the four-momenta p and q, and is not merely peculiar to the particular tensor defined in Eq. (8). Thus we expect that the same combination of helicity partial-wave Born amplitudes will have factorizable singularities, and this factorizability should be preserved in the unitarization procedure. The *type* of singularity or zero, however, does depend on the choice of tensor, since the tensor can always be multiplied by an arbitrary function of kinematic invariants. Our choice happens to be the simplest one giving the *standard* threshold behaviors.

It is of some interest to discuss these explicit s singularities contained in the expressions (15) and (16). The factor $(s-4)^{1/2}$ common to both f_0 and f_1 is the normal kinematic threshold behavior for the p wave of the two final pions, while the factors of S carry the normal threshold and pseudothreshold singularities at $s=(M+1)^2$ and $s=(M-1)^2$ for the initial state of $\pi\pi^*$. The initial-state kinematical singularities or zeros should be of the *standard* form S^l, where l is the smallest angular momentum consistent with $(-)^l=-1=$ product of the intrinsic parities of π, π^*, and ρ that can be formed from the spins.[11] Thus we must have $l=1$ for $J=0$ and $l=J-1$ for $J\geq 2$. We note that with Eq. (17), f_0 does indeed have this behavior, while f_1 has the correct behavior for $J\geq 2$, which is the physical domain of J for this amplitude. We also note that when $J=0$ and $M=1$, the singularity at $s=0$ in f_0 cancels the pseudothreshold singularity, leaving only a zero at $s=4$, which is appropriate to the p-wave Born amplitude of equal-mass spin-zero particles.

F. Unitarity

Assuming two-particle or elastic unitarity, we choose intermediate states containing only pions, as discussed in the paragraph containing Eq. (4), in which case the statement of unitarity above the two-pion threshold is simply the phase condition[12]

$$\mathrm{Im}\beta_\lambda(s;J,M)=(48\pi)^{-1}[(s-4)/s]^{1/2}\times\beta_\lambda(s;J,M)\beta_0{}^*(s;0,1),$$

where β_0 and β_1 are the unitarized versions of $\bar{B}_0$ and $\bar{B}_1$, respectively. It follows that the phase of $\beta_\lambda(s+i\epsilon;J,M)$ is equal to that of $\beta_0(s+i\epsilon;0,1)$ modπ for $4<s<s_1$, where s_1 is the first inelastic threshold. However, the square-root branch point in $\beta_\lambda(s;J,M)$ at $s=(M+1)^2$ when $M\neq 1$ produces a discontinuous change in phase of $\frac{1}{2}\pi$, which means that for real values of M such that $1<M<(\sqrt{s_1})-1$, unitarity cannot be satisfied over the entire interval $4<s<s_1$. Now, if there existed a stable recurrence of mass M_r, we would have $s_1=(M_r+1)^2$, and physically realizable processes would be attained as M approached either endpoint in the interval $1<M<(\sqrt{s_1})-1$. Unitarity, therefore, cannot be simultaneously satisfied in these two real processes, and hence the existence of a stable recurrence *necessarily* violates two-pion unitarity in this model. Furthermore, if M is real, two-pion unitarity can be satisfied only if $M\leq 1$, and therefore our dynamical recipe is consistently defined only for $0<M\leq 1$.

We define $\bar{B}_{01}=\bar{B}_0-c\bar{B}_1$, which is expected to have the kinematic singularities of ρ_{01}, while $\bar{B}_1$ alone has the singularities of ρ_1. Since β_0 and β_1 have the same phase under elastic unitarity, the combination $\beta_{01}=\beta_0-c\beta_1$ also has the same phase.

In the following statements, the subscript λ is understood to have the values 01 or 1. Removing the kinematic factors ρ_λ, we write the unitarized amplitudes as

$$\beta_\lambda(s;J,M)=\rho_\lambda(s;J,M)N_\lambda(s;J,M)/D(s),$$

where N_λ carries the left-hand cut from $s=0$ to $-\infty$, D has a right-hand cut, starting at $s=4$, such that D^{-1} has a phase equal to $\delta_1(s)$, the $I=1$ p-wave elastic $\pi\pi$ phase shift, and ρ_λ carries square-root branch cuts from $s=4$ to $(M+1)^2$ and to the left of $s=(M-1)^2$. The D function is given by

$$D(s)=\exp\left[-\frac{s}{\pi}\int_4^\infty\frac{\delta_1(s')ds'}{(s'-s)s'}\right].$$

We proceed with a simple-minded approximation scheme which can be solved exactly. Let

$$N_\lambda/D=\bar{B}_\lambda/\rho_\lambda+C_\lambda/D,$$

where $\bar{B}_\lambda/\rho_\lambda$ has only the left-hand cut from 0 to $-\infty$ and C_λ has only a right-hand cut starting at $s=4$. Thus we assume that the discontinuity on the left-hand cut of the Born term is a true representation of the left-hand cut. We find the solution

$$\beta_\lambda(s;J,M)=\bar{B}_\lambda(s;J,M)-\frac{\rho_\lambda(s;J,M)}{\pi D(s)}\times\int_4^\infty\frac{\bar{B}_\lambda(s';J,M)\,\mathrm{Im}D(s')ds'}{(s'-s)\rho_\lambda(s';J,M)}. \quad (28)$$

4. BOOTSTRAP EQUATIONS

A. Matching Conditions

The input terms Eq. (28) are to be matched in the bootstrap sense to the appropriate linear combinations $\alpha_\lambda(s;J,M)$ of the direct amplitudes $A_0{}^{(s)}/d_{00}{}^1(\theta)$ and $A_1{}^{(s)}/d_{10}{}^1(\theta)$. These combinations are, from Eqs. (13)–(16) and (26),

$$\alpha_{01}=a\rho_{01}\mu \quad\text{and}\quad \alpha_1=b\rho_1\mu, \quad (29)$$

where

$$\mu=\mu(s)=(s-m^2+i\gamma)^{-1}. \quad (30)$$

In the conventional bootstrap procedure, we would equate the position and residue of the ρ pole in Eq. (28) to those of (29), which would yield two equations for

[11] T. W. B. Kibble, Phys. Rev. **131**, 2282 (1963).

[12] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959); S. Mandelstam, *ibid.* **4**, 84 (1960); R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).

each value of λ. However, since the statement of unitarity in our model, where just one external particle is continued, is just a *phase* condition, it is the D function which carries this phase, and therefore also contains the ρ pole if analytically continued onto the unphysical sheet. The input term $\bar{B}_\lambda$ has no effect on the pole position, which is always where $\tan\delta_1(s)=i$ when s is continued from the neighborhood of the resonance on the physical sheet, and thus Eq. (28) identically satisfies the position aspect of the bootstrap.

We are permitted, therefore, to go a step beyond the conventional bootstrap in the direction toward implementing full crossing symmetry and require that more of the behavior of the amplitude near the pole be matched in the bootstrap conditions. If we could analytically continue to the neighborhood of the pole, we could then match the pole residue and the next-higher-order term. It is far easier, however, to match the amplitudes and first derivatives of the amplitudes at the point where the phase is $\frac{1}{2}\pi$ on the physical cut. This procedure is nearly equivalent to the pole-matching procedure when the pole is very near the cut.

Since the strength and position of the ρ pole is not an adjustable parameter in the matching procedure, very special conditions would have to be met in order that the phase representation reduces to the elastic unitarity condition in the case $M=1$, $J=0$. In fact, as is well known, the ρ bootstrap does not seem to be perfectly self-supporting,[13] and we should therefore make some allowance for this imperfection by including an adjustable additional term in the partial-wave Born amplitudes. We propose adding a pole term of the form $\rho_{01}R_0/(s+s_0)$ to $\bar{B}_{01}$ corresponding to a highly simplified treatment of other exchanged particles in the amplitude. Such a term lumps these exchanges together with an effective mass related to s_0, which, after the initial adjustment in the case $M=1$, $J=0$, will be regarded as a fixed parameter. This lumped exchange would also contribute to the pure $\lambda=1$ amplitude, and hence we add a corresponding term $\rho_1R_1/(s+s_0)$ to $\bar{B}_1$. The residues $R_0(J,M)$ and $R_1(J,M)$ play roles similar to the coupling constants a and b, and therefore have some *a priori* unknown dependence on J and M.

We now write down the matching equations which are to determine the π^* trajectory. Adding the suggested adjustable pole terms to the partial-wave Born amplitudes (23) and (24), we can write (for arbitrary J and M)

$$\begin{aligned}\bar{B}_0&=ax_0+by_0+(R_0\rho_{01}+cR_1\rho_1)/(s+s_0)\,,\\ \bar{B}_1&=ax_1+by_1+R_1\rho_1/(s+s_0)\,,\end{aligned}\tag{31}$$

where

$$x_0(s;J,M)=-\frac{3}{2}\int_0^\pi T^J(2s+t-M^2-3)P_J(\cos\psi)\frac{\cos\theta\,\sin\theta d\theta}{m^2-t},$$

$$y_0(s;J,M)=-\frac{3}{2}\left(\frac{J}{J+1}\right)^{1/2}\int_0^\pi\left\{\frac{\sqrt{2}}{M}ST^{J-1}[P_{J-1}(\cos\psi)-\cos\psi P_J(\cos\psi)]\right.$$
$$\left.+T^{J-2}\left(\frac{t+M^2-1}{M\sqrt{2}}\right)(2s+t-M^2-3)P_J(\cos\psi)\right\}\frac{\cos\theta\,\sin\theta d\theta}{m^2-t},$$

$$x_1(s;J,M)=-\frac{3}{2\sqrt{2}}\int_0^\pi T^J(2s+t-M^2-3)[J(J+1)]^{-1/2}\sin\psi\,P_J'(\cos\psi)\frac{\sin^2\theta d\theta}{m^2-t},$$

$$y_1(s;J,M)=\frac{3}{2\sqrt{2}}\int_0^\pi\left\{\sqrt{2}S(s-4)^{1/2}\left[P_J(\cos\psi)-\frac{\cos\psi}{J(J+1)}P_J'(\cos\psi)\right]\sin\theta\right.$$
$$\left.-\left(\frac{t+M^2-1}{M\sqrt{2}}\right)(2s+t-M^2-3)\frac{\sin\psi\,P_J'(\cos\psi)}{J+1}\right\}T^{J-2}\frac{\sin^2\theta d\theta}{m^2-t}.$$

Inserting Eqs. (31) into (28), we find

$$\beta_{01}=\rho_{01}(aw_0+bz_0+R_0u)\,,\qquad \beta_1=\rho_1(aw_1+bz_1+R_1u)\,,\tag{32}$$

where (suppressing the J, M dependence on the right-hand side)

$$w_0(s;J,M)=\frac{x_0(s)-c(s)x_1(s)}{\rho_{01}(s)}-\frac{1}{\pi D(s)}\int_4^\infty\frac{x_0(s')-c(s')x_1(s')}{(s'-s-i\epsilon)\rho_{01}(s')}\,\mathrm{Im}D(s')ds'\,,\tag{33}$$

$$z_0(s;J,M)=\frac{y_0(s)-c(s)y_1(s)}{\rho_{01}(s)}-\frac{1}{\pi D(s)}\int_4^\infty\frac{y_0(s')-c(s')y_1(s')}{(s'-s-i\epsilon)\rho_{01}(s')}\,\mathrm{Im}D(s')ds'\,,\tag{34}$$

[13] J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. 137, B1242 (1965); L. A. P. Balázs and S. M. Vaidya, *ibid.* 140, B1025 (1965).

$$u(s)=\frac{1}{s+s_0}-\frac{1}{\pi D(s)}\int_4^\infty \frac{\mathrm{Im}D(s')ds'}{(s'+s_0)(s'-s-i\epsilon)}, \tag{35}$$

$$w_1(s;J,M)=\frac{x_1(s)}{\rho_1(s)}-\frac{1}{\pi D(s)}\int_4^\infty \frac{x_1(s')\,\mathrm{Im}D(s')ds'}{(s'-s-i\epsilon)\rho_1(s')}, \tag{36}$$

$$z_1(s;J,M)=\frac{y_1(s)}{\rho_1(s)}-\frac{1}{\pi D(s)}\int_4^\infty \frac{y_1(s')\,\mathrm{Im}D(s')ds'}{(s'-s-i\epsilon)\rho_1(s')}. \tag{37}$$

The question of convergence of these integrals is easily settled. We find the following asymptotic behaviors for large s: $c\sim\sqrt{s}$, ρ_1, x_0, $x_1\sim s^J$, y_0, $y_1\sim s^{J-1}$, and $\rho_{01}\sim s^{J+1}$. Hence, if $\mathrm{Im}D(s)\to 0$ at least as fast as $(\ln s)^{-k}$, with $k>1$, then the integral in w_1, which is the least rapidly convergent of the above five integrals, will converge.

We let primes denote derivatives with respect to s and the tilde notation denote evaluation at $s=m^2$. Then, matching Eqs. (29) to Eq. (32), we have

$$a\tilde{\rho}_{01}\tilde{\mu}=\tilde{\rho}_{01}(a\tilde{w}_0+b\tilde{z}_0+R_0\tilde{u}),$$
$$b\tilde{\rho}_1\tilde{u}=\tilde{\rho}_1(a\tilde{w}_1+b\tilde{z}_1+R_1\tilde{u}),$$
$$[a\tilde{\rho}_{01}\tilde{\mu}]'=[\tilde{\rho}_{01}(a\tilde{w}_0+b\tilde{z}_0+R_0\tilde{u})]',$$
$$[b\tilde{\rho}_1\tilde{\mu}]'=[\tilde{\rho}_1(a\tilde{w}_1+b\tilde{z}_1+R_1\tilde{u})]'.$$

B. Solution of Equations: Determination of Trajectory

Since these equations are homogeneous in the unknowns a, b, R_0, and R_1, we can at most determine the unknown ratios

$$r=b/a,\quad r_0=R_0/a,\quad \text{and}\quad r_1=R_1/a. \tag{38}$$

Removing the kinematic factors $\tilde{\rho}_\lambda$ and reinserting the dependence on J and M in the quantities defined in Eqs. (33)–(37), where in fact the J, M dependence is explicitly known, we have

$$\tilde{\mu}=\tilde{w}_0(J,M)+r\tilde{z}_0(J,M)+r_0\tilde{u}, \tag{39}$$

$$r\tilde{\mu}=\tilde{w}_1(J,M)+r\tilde{z}_1(J,M)+r_1\tilde{u}, \tag{40}$$

$$\tilde{\mu}'=\tilde{w}_0'(J,M)+r\tilde{z}_0'(J,M)+r_0\tilde{u}', \tag{41}$$

$$r\tilde{\mu}'=\tilde{w}_1'(J,M)+r\tilde{z}_1'(J,M)+r_1\tilde{u}'. \tag{42}$$

Setting $\tilde{v}=\tilde{u}'/\tilde{u}$ and eliminating r_0 between Eqs. (39) and (41), we find

$$r=\frac{[\tilde{\mu}-\tilde{w}_0(J,M)]\tilde{v}-[\tilde{\mu}'-\tilde{w}_0'(J,M)]}{\tilde{v}\tilde{z}_0(J,M)-\tilde{z}_0'(J,M)}, \tag{43}$$

while from Eqs. (40) and (42) we find

$$r=\frac{\tilde{v}\tilde{w}_1(J,M)-\tilde{w}_1'(J,M)}{\tilde{v}[\tilde{\mu}-\tilde{z}_1(J,M)]-[\tilde{\mu}'-\tilde{z}_1'(J,M)]}. \tag{44}$$

Upon setting the right-hand sides of (43) and (44) equal, we tentatively obtain the equation satisfied by the hypothetical π^* trajectory. Before carrying this step out, we must consider the behavior of these results as $J\to 0$, where we must have $M=1$. We find that the following quantities tend to constants in the limit $J\to 0$[14]: x_0, $y_0J^{-1/2}$, $x_1J^{-1/2}$, y_1, and $cJ^{-1/2}$. Then in the same limit, from Eqs. (34) and (36), we have that $z_0J^{-1/2}$ and $w_0J^{-1/2}$ tend to constants, and hence the denominator of Eq. (43) and the numerator of Eq. (44) vanish like $J^{1/2}$. Now, from Eq. (17) we have $b=o(J^{-1/2})$ as $J\to 0$, or, equivalently, $r=o(J^{-1/2})$, since a cannot vanish in this limit, or else the $\pi\pi$ scattering amplitude would be identically zero. Thus the numerator of Eq. (43) must vanish when $J\to 0$, while the denominator of (44), if it should vanish, must not vanish faster than J.

When $J\to 0$, the matching equations (39)–(42) then reduce to

$$\tilde{\mu}=\tilde{w}_0(0,M)+r_0\tilde{u}, \tag{45}$$

$$r\tilde{\mu}=r\tilde{z}_1(0,M)+r_1\tilde{u}, \tag{46}$$

$$\tilde{\mu}'=\tilde{w}_0'(0,M)+r_0\tilde{u}', \tag{47}$$

$$r\tilde{\mu}'=r\tilde{z}_1'(0,M)+r_1\tilde{u}'. \tag{48}$$

Note that the pair of equations (45) and (47) implicitly determine the unknown s_0 by explicitly determining the quantity $\tilde{v}$; s_0 is the solution of

$$\tilde{v}=\frac{\tilde{\mu}'-\tilde{w}_0'(0,1)}{\tilde{\mu}-\tilde{w}_0(0,1)}. \tag{49}$$

The other pair of equations, if taken seriously, could independently determine s_0, which would be the solution of

$$\tilde{v}=\frac{\tilde{\mu}'-\tilde{z}_1'(0,1)}{\tilde{\mu}-\tilde{z}_1(0,1)}, \tag{50}$$

and there is no reason to believe that the solutions of Eqs. (49) and (50) are the same. However, the second pair of equations (46) and (48) has no physical basis, since $J=0$ is a "nonsense" value for the helicity$=1$ amplitude. Furthermore, the requirement that the numerator of Eq. (43) must vanish when $J\to 0$ and $M=1$ demands that Eq. (49) be satisfied, while

[14] In the standard interpolation (see Sec. 6), we have $d_{00}{}^J(\theta)=1$, $d_{11}{}^J(\theta)-d_{-11}{}^J(\theta)=2(1+\cos\theta)^{-1}$, and $d_{01}{}^J(\theta)\sim J^{1/2}\sin\theta(1+\cos\theta)^{-1}$ when $J\to 0$.

Eq. (50) must be satisfied only in the exceptional case that the denominator of Eq. (44) happens to vanish in the same limit. Assuming that this unlikely coincidence does not occur, Eq. (49) is clearly the preferred route to s_0, and this choice selects one of the two possible ways in which the equation of the trajectory must be satisfied when $M=1$ and $J=0$. An immediate consequence of the nonvanishing of the denominator of (44) is that r and b in fact must vanish like $J^{1/2}$. Also, the invalidity of Eq. (50) is not in contradiction to Eqs. (46) and (48), since these latter equations reduce to $0=0$ when $J=0$.

We set the right-hand sides of Eqs. (43) and (44) equal in order to obtain the trajectory equation

$$\frac{\tilde{v}[\tilde{\mu}-\tilde{w}_0(J,M)]-[\tilde{\mu}'-\tilde{w}_0'(J,M)]}{\tilde{v}\tilde{z}_0(J,M)-\tilde{z}_0'(J,M)} =\frac{\tilde{v}\tilde{w}_1(J,M)-\tilde{w}_1'(J,M)}{\tilde{v}[\tilde{\mu}-\tilde{z}_1(J,M)]-[\tilde{\mu}'-\tilde{z}_1'(J,M)]}, \quad (51)$$

with $\tilde{v}$ given by Eq. (49).

5. UNIQUENESS

The problem of determining a unique interpolation between integral values of J can now be discussed. Roughly speaking, the problem is resolved if the asymptotic behavior of the amplitude can somehow be specified as $|J|\to\infty$. Considering the J dependence only, while ignoring a purely energy-dependent factorizable term, the amplitude $f_\lambda(s;J,M)$ is proportional to the square root of the energy-plane residue of the Jth partial-wave forward-scattering amplitude $g_{\lambda\lambda}{}^J$ for $\pi\rho$ elastic scattering when the energy is evaluated at the hypothetical π^* pole of mass M. This follows from the observation that the J dependence of f_λ is entirely contained in the vertex V_1, and this same vertex would occur twice in $\pi\rho$ elastic scattering where an intermediate state of total angular momentum J and mass M is formed. The helicities of initial and final ρ, necessarily equal in the case of forward scattering, are denoted by the subscripts $\lambda\lambda$.

Although a bound has been set on the asymptotic behavior of the J-plane residue of a Regge pole in nonrelativistic potential theory for potentials that go to zero rapidly enough at infinity,[15] the form of this bound is actually of little help to us, since our procedure generates the *ratio* of amplitudes rather than the amplitudes themselves. A simple assumption, in the context of potential theory, leading heuristically to a determination of the asymptotic behavior of this ratio, is that the two residue functions of $g_{00}{}^J$ and $g_{11}{}^J$ belong to Regge trajectories of two different potentials.

It would be most desirable to investigate superpositions of Yukawa potentials, or at least two pure Yukawa potentials of the same range but of different strengths. In any event, such arguments are persuasive at best, and it is expedient here for illustrative purposes to take up a more tractable potential. In the case of two potentials of the same range, if we let the range become infinite, we obtain Coulomb potentials. Although the infinite range destroys some of the plausibility, the Coulomb potential is explicitly soluble. For example, the energy-plane residue is proportional to[16]

$$\frac{1}{(J+n)^3}\frac{1}{\Gamma(2J+n+1)},$$

where n labels a particular one of the infinitely many poles of the amplitude. The essential singularity of this residue at $|J|=\infty$ is more singular than can be tolerated by Carlson's theorem for a unique interpolation between integral J values. However, the *ratio* of the residues of the trajectories of two different potentials is proportional to

$$\left(\frac{J+n''}{J+n'}\right)^3\frac{\Gamma(2J+n''+1)}{\Gamma(2J+n'+1)},$$

and the square root of this ratio behaves asymptotically like $J^{1/2(n''-n')}$. Such an asymptotic power law permits a unique interpolation, since uniqueness is ensured if the ratio is bounded by $O(e^{k|J|})$, where $k<\frac{1}{2}\pi$ for a function defined on the non-negative even integers. It is certainly not clear that an asymptotic power law would be obtained for a potential of finite range or in relativistic models of Regge poles. This illustration merely suggests that residues possessing an intolerable essential singularity at infinity may, in the form of their ratio, satisfy one of the conditions for a unique interpolation.

Let us then make the explicit assumption that, on the bootstrap-generated trajectory, the interpolated ratio must satisfy

$$\left|\frac{f_0(m^2;J,M)}{f_1(m^2;J,M)}\right|=O(e^{k|J|}) \quad (52)$$

as $|J|\to\infty$, where $k<\frac{1}{2}\pi$ and $\mathrm{Re}J\geq 0$.

If the bootstrap procedure is to generate a realistic trajectory, the amplitudes (or residues) must have a physical cut on the real axis in the J plane starting at J_t, the value of J corresponding to the fixed threshold $M=3$ and running to $+\infty$. This is true because the threshold behaviors of the trajectory and residue as functions of energy imply the presence of a branch-point in the J plane even when the energy is eliminated and the residue is considered as an analytic function of J. If, in taking the ratio of the two helicity amplitudes, the cut does not cancel, then Carlson's theorem is not immediately applicable without due consideration to cut. The necessary extension is as follows: If the

[15] R. G. Newton, *The Complex j-Plane* (W. A. Benjamin, Inc., New York, 1964), pp. 78, 79.

[16] V. Singh, Phys. Rev. **127**, 632 (1962).

analytic continuation of the ratio from above the cut onto the second sheet also satisfies Eq. (52), but only for $\mathrm{Re}J \geq n$, where n is some even integer greater than J_t, and if the ratio is meromorphic with a finite number of poles in this domain of the J plane, then an application of Carlson's theorem tells us that the ratio has a unique interpolation between integral $J \geq n$, and hence a unique analytic continuation for $\mathrm{Re}J \geq n$. This domain of uniqueness can then be increased by analytic continuation to $\mathrm{Re}J \geq J_t$ and also continued counterclockwise around the branchpoint to $\mathrm{Re}J \geq 0$.

On the trajectory $M_0(J)$ the relation between the ratios $r(J) = b/a$ and f_0/f_1 is

$$\frac{f_0(m^2; J, M_0(J))}{f_1(m^2; J, J_0(J))} = [mr(J)]^{-1} \times \{m^2 - [M_0(J) - 1]^2\}\{m^2 - [M_0(J) + 1]^2\} + \left(\frac{J}{J+1}\right)^{1/2} \left[\frac{m^2 + M_0^2(J) - 1}{M_0(J) m\sqrt{2}}\right]. \quad (53)$$

If, for example, $M_0(J)$ is bounded as $|J| \to \infty$, the behavior of Eq. (52) applies to r as well. In any event, Eq. (52) with relation (53) and the question of meromorphy of f_0/f_1 to the right of J_t constitute an *a posteriori* test of uniqueness of the interpolation.

There are several explicit sources of dependence on J in the system of Eqs. (39)–(42), and each has a "natural" analytic extension to continuous values of J. The rotation matrix elements $d_{\lambda'\lambda}{}^J$ have a standard interpolation based on the hypergeometric differential equation.[17] Note that the $d_{\lambda'\lambda}{}^J$ functions to which we refer occur in the crossing relations of Sec. 6 and are not used to carry out angular-momentum projections. The factor $[J/(J+1)]^{1/2}$ coming from a Clebsch-Gordan coefficient and the threshold factors ρ_{01} and ρ_1 with the S^J dependence both have obvious extensions onto the complex J plane. A reasonable procedure, then, of establishing the unique interpolation would be to assume that the "natural" one is correct, determine the trajectory and its asymptotic behavior as $|J| \to \infty$, and then test this behavior against Eq. (52).

[17] J. M. Charap and E. J. Squires, Ann. Phys. (N. Y.) 21, 8 (1963).

6. CONCLUDING REMARKS

The principal result of this paper is Eq. (51), which implicitly determines a trajectory carrying the internal quantum numbers of the pion, and, more generally, shows that a bootstrap hypothesis may generate the Regge trajectory of an external particle in a two-body scattering process. The explicit determination of the trajectory function $J_0(M^2)$, or at least some of its features, depends on the possibility of establishing uniqueness, which in turn depends on a more rigorous validation of the assumption (52). Exactly what form such a validation should take is the subject of future work.

Given that the natural analytic continuations of the sources of J dependence should pass the tests of uniqueness mentioned in Sec. 5, the approximation scheme of linearizing Eqs. (39)–(42) in J and M near $J=0$ and $M=1$ would vastly simplify the numerical determination of the trajectory and would be especially interesting in that the slope of the trajectory at the position of the pion could be accurately obtained.

Another area of future work is the more proper treatment of the ρ as a Regge particle. The entire ρ trajectory, regarded as given input data, could be fed into the model, and the residue function together with a sufficient number of its derivatives could be matched at the position of the physical ρ.