in $\mathfrak{H}$. Rotations, Galilei transformations, or Poincaré transformations then transform rays into rays which leads to a *ray representation* of the respective group[3] where each group element g is represented by a class $\tilde{U}(g)$ of unitary operators differing by an arbitrary phase factor $e^{i\alpha}$. For the rotation group $SO(3)$ and the connected part P of the Poincaré group one can, according to Bargmann,[4] select suitable members $U(g)$ of these equivalence classes which form a unitary vector representation of, respectively, $SO(3)$ and P (integer total spin) or of the corresponding covering groups $SU(2)$ and $\bar{P}$ (half-integer spin). For this so-called "reduction of phase" we refer to the clear exposition in the textbook of Ludwig.[5] As mentioned in the introduction, to one element of $SO(3)$ or P there correspond, in the latter case, still two physically equivalent operators in $SU(2)$ or $\bar{P}$, a reminder of the original starting point, the ray representation, and a warning not to take the term "rotation by 2π" too literally because it is in the same equivalence class $\tilde{U}(g)$ as the rotation by 0°. The reduction of phase is only a purely *mathematical* simplifying convention which leaves unaffected the fundamental arbitrariness of phase factors. And one should never expect physical consequences from a mathematical convention.

We would like to thank Professor E. P. Wigner for stimulating and helpful comments.

[3] See, for example, R. Hagedorn, Nuovo Cimento Suppl. **12**, 73 (1959).
[4] V. Bargmann, Ann. Math. **59**, 1 (1954).
[5] G. Ludwig, *Grundlagen der Quantenmechanik* (Springer-Verlag, Berlin, 1954).

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Testing Relativity with Laser Ranging to the Moon

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Precise laser ranging to the moon is shown to be capable of measuring the ratio (m_g/m_i) of gravitational mass to inertial mass of the earth to accuracy sufficient to detect gravitational self-energy contributions. If m_g/m_i of the earth differs from 1 by an amount of order (earth gravitational self-energy)/(earth total energy), then the lunar orbit will acquire a range oscillation of amplitude about 12 m.

In a recent paper, Baierlein studied possible tests of general relativity using laser ranging to the moon.[1] An accuracy of two parts in 10^{10} in laser round-trip time was assumed which corresponds to about 8-cm accuracy in the earth-moon distance.

Recently, the author has examined possible experiments to test the ratio of gravitational mass (m_g) to inertial mass (m_i) for massive bodies,[2] and in a separate paper has studied the aspects of gravitational theories which would be tested by a careful measurement of m_g/m_i for massive bodies.[3]

The purpose of this paper is to show that the laser ranging to the moon can be used to measure m_g/m_i for the earth—a possible experiment not considered by Baierlein.[1] The experimental effect discussed in this paper will be possibly two orders of magnitude larger than those discussed by Baierlein.

In the author's previous paper[2] the possibility was considered that the ratio of a massive body's gravitational mass to its inertial mass is given by

$$\frac{m_g}{m_i}=1+\eta\frac{G}{c^2}\int\rho(x)\rho(x')\frac{d^3xd^3x'}{|\mathbf{x}-\mathbf{x}'|}\Big/\int\rho(x)d^3x,\qquad(1)$$

where $\rho(x)$ is the body's mass density, η is a dimensionless constant of order 1, G is the gravitational constant, and c is the velocity of light. For a uniform mass of radius a, Eq. (1) yields

$$m_g/m_i=1+(6/5)\eta Gm/c^2a.\qquad(2)$$

The earth's mass is somewhat concentrated in the center. This should about double the sizes of the effects discussed in this paper. For the earth, the correction term (2) is much larger than for the moon, so the moon correction will be neglected in this paper.

As the earth and moon travel around the sun, (2) will lead to an excess acceleration of the earth toward the sun of

$$\delta a=(6/5)\eta(Gm_e/c^2a_e)GM_\odot/R^2,\qquad(3)$$

where R is the sun-earth distance. In the earth-centered coordinate system, the moon receives an acceleration which is the negative of (3). The effect of (3) then on the lunar orbit is now calculated.

Assume for simplicity a circular lunar orbit which satisfies the equations of motion

$$d^2r/dt^2=h^2/r^3-\mu/r^2+\delta a\cos\phi\qquad(4a)$$

and

$$dh/dt=-\delta a\, r\sin\phi.\qquad(4b)$$

[1] R. Baierlein, Phys. Rev. **162**, 1274 (1967).
[2] K. Nordtvedt, Phys. Rev. **169**, 1014 (1968).
[3] K. Nordtvedt, Phys. Rev. **169**, 1017 (1968).

Equations (4a) and (4b) include the perturbing force generated by (3). Linearizing (4a) and (4b) about a circular orbit ($r \to r_0+\delta r$, $h \to h_0+\delta h$), we obtain

$$\delta h = (\delta a\, r_0/\Omega)\cos\Omega t\,, \qquad (5)$$

where $\Omega=\omega_0-\omega_s$, ω_0 being the lunar angular frequency, ω_s being the sun's angular frequency. Using (5) in the linearized version of (4a) we get

$$\delta\ddot{r} = -(3h_0^2/r_0^4-2\mu/r_0^3)\delta r+(2h_0/r_0^3)\delta h+\delta a\cos\Omega t\,, \qquad (6)$$

which yields

$$\delta\ddot{r}+\omega_0^2\delta r=(1+2\omega_0/\Omega)\delta a\cos\Omega t\,, \qquad (7)$$

where we have used $\mu/r_0^3=\omega_0^2$ and $h_0=r_0^2\omega_0$. Equation (7) has the solution

$$\delta r=[3\delta a/(\omega_0^2-\Omega^2)]\cos\Omega t\,, \qquad (8)$$

where we have used $\omega_0 \cong \Omega$ as $\omega_s \cong \omega_0/13$. The resonance denominator in (8) which enhances our effect is due to the fact that circular orbits in the $1/r$ potential problem have a divergent polarizability for static external fields. (The quantum-mechanical analog of this is the linear Stark effect of the hydrogen atom.)

Setting $\omega_0^2-\Omega^2 \simeq 2\omega_0\omega_s \simeq (2/13)\omega_0^2$ and using (3), we finally get

$$\delta r=-(117/5)(GM_\odot/c^2)(r_e^3/a_eR^2)\eta\cos\Omega t \simeq -1200\eta\cos\Omega t\ \mathrm{cm}\,, \qquad (9)$$

where r_e is the radius of the lunar orbit. For η of order 1, (9) represents a change of three parts in 10^{-8} in the lunar orbital radius. This is about an order of magnitude smaller than present observational accuracy. However, if laser ranging as discussed by Baierlein[1] is able to improve our range-measurement accuracy several orders of magnitude, then (9) should offer a possibility of measuring η, the m_g/m_i ratio of earth, to good accuracy.

As we have shown in another paper,[3] Einstein's gravitational theory predicts $\eta=0$, although this null result is due to the exact cancelation of effects from many metric terms—metric terms which have not been measured to date in gravitational experiments. Other gravitational theories, in particular the scalar-tensor theory of Brans and Dicke,[4] are expected to yield $\eta\neq 0$.[3]

[4] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

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Spinning Shell as a Source of the Kerr Metric

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A method is described which develops the interior field and the physical properties of a slowly spinning, nearly spherical thin mass shell as a power series in the angular velocity ω, on the presumption that the exterior field is the Kerr metric. Results are worked out explicitly, correct to the third order in ω. The ellipticity of the shell is arbitrarily assignable to within quantities of order ω^2. If it is suitably prescribed, the interior field tends to flatness, in the limit where the radius of the shell approaches the gravitational radius, and the extended inertial frames thus defined in the interior are dragged around rigidly by the shell in "Machian" fashion. This amplifies a previous first-order result due to Brill and Cohen.

1. INTRODUCTION

THE axially symmetric stationary vacuum metric

$$\begin{aligned} ds^2 &= (r^2+a^2\cos^2\theta)[dr^2/(r^2-2mr+a^2)+d\theta^2] \\ &+\left(r^2+a^2+\frac{2mra^2\sin^2\theta}{r^2+a^2\cos^2\theta}\right)\sin^2\theta d\varphi^2 \\ &+\frac{4amr}{r^2+a^2\cos^2\theta}\sin^2\theta d\varphi dt-\left(1-\frac{2mr}{r^2+a^2\cos^2\theta}\right)dt^2 \end{aligned} \qquad (1)$$

discovered by Kerr[1] is generally believed to represent the exterior gravitational field of a spinning spherical (or nearly spherical) body of mass m and angular velocity proportional to $-a$. Until quite recently, the sole basis for this belief was the agreement between the weak gravitational field of such a body, computed from the linearized Einstein equations, and the asymptotic

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[1] R. P. Kerr, Phys. Rev. Letters **11**, 237 (1963). The above form of the metric is due to R. H. Boyer and R. W. Lindquist, J. Math. Phys. **8**, 265 (1967). For alternative derivations of Kerr's solution, see, R. P. Kerr and A. Schild, in *Proceedings of the Galileo Galilei Centenary Meeting on General Relativity*, edited by G. Barbera (Firenze, 1965), p. 222; F. J. Ernst, Phys. Rev. **167**, 1175 (1968).