Critical Remark on the Observability of the Sign Change of Spinors under 2π Rotations

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In a recent paper, Aharonov and Susskind claimed rotations by 2π to have observable effects on spinors. Their reasoning is shown to be inconclusive.

IN the standard quantum-mechanical description of spin-$\frac{1}{2}$ particles the wave functions transform under space rotations according to a unitary representation of $SU(2)$, the covering group of the rotation group $SO(3)$. As is well known, the correspondence between $SO(3)$ and $SU(2)$ is not one-to-one; to every rotation there belong two elements of $SU(2)$ which differ by a factor of -1. With rotations around the x axis by an angle θ, for example, one can associate the matrix

$$U(\theta)=\begin{pmatrix}\cos\frac{1}{2}\theta & i\sin\frac{1}{2}\theta\\ i\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta\end{pmatrix} \quad (1)$$

as well as the matrix $-U(\theta)$. The same matrices describe the transformation of spinor indices of a spin-$\frac{1}{2}$ wave function under the corresponding rotation. Choosing the sign positive, one finds that for $\theta=2\pi$, which in $SO(3)$ of course gives the identity, relation (1) yields $U(2\pi)=-1$. Therefore the element -1 in $SU(2)$ is sometimes called a rotation by 2π. Since an over-all change of the sign of a wave function does not affect expectation values, the minus sign connected with a rotation by 2π of spinors should have no observable effect.

In a recent paper by Aharonov and Susskind (AS),[1] this last statement was claimed invalid. They propose a gedanken experiment which can be described in a somewhat (but unessentially) simplified form as follows: Divide, by means of suitable boxes I and II, the wave function ψ of a spin-$\frac{1}{2}$ particle into two parts, ψ_1 in box I and ψ_2 in box II, so that $\psi=\psi_1+\psi_2$. Separate box I and box II spatially. Rotate box I by $2\pi n$ around a fixed axis, thereby transforming ψ_1 into $(-1)^n\psi_1$. Then again join both boxes, i.e., both parts of the wave function. The resulting wave function $\psi'=(-1)^n\psi_1+\psi_2$ then clearly depends on the number n of rotations by 2π of box I. The peculiar minus sign connected with rotations by 2π of spinors should thus, by interference, lead to observable effects, at least in principle.

It follows immediately that either quantum mechanics is not self-consistent or that this reasoning is inconclusive because of the fact that *by exactly the same argument one can show the rotation by 0°, i.e., no rotation at all, to have an observable effect on spinors.* Indeed, quantum mechanics tells us that instead of $U(\theta)$ in (1) we may equally well choose $-U(\theta)$ to represent a rotation of spinors. So let us do this. Then we have $U(2\pi)=1$ and $U(0)=-1$. Performing the same experiment as before, we find that a rotation by 0° changes $\psi=\psi_1+\psi_2$ into $-\psi_1+\psi_2$.

It is easy to exhibit a flaw in the argument of Aharonov and Susskind. The separate rotation of box I considered above is not at all a rotation of a quantum state; it is instead a time development of the total wave function ψ which has to be described by $\psi\rightarrow e^{iHt}\psi$ with an appropriate Hamiltonian H, but not by formally applying to ψ_1 the spinor transformation law[2] with the minus sign as required in AS.

The usual definition of "rotation of a quantum state" is the following: If, by some suitable apparatus, a quantum system is prepared in a certain state, then the rotated state is *the state prepared by the rotated apparatus.* Rotations in this sense of the quantum state in box I, i.e., rotations of the whole experimental setup preparing this state, will evidently rotate the whole wave function ψ instead of only a part ψ_1 of it. It is not possible to consider box I alone as preparing apparatus, since then the experiment would be a measuring process, with box II as a measuring apparatus. By standard theory one then would have to couple the subsystems by employing the *direct product* of their states. This is clearly inappropriate for *interference* experiments of the type considered here.

It seems to us that in the reasoning of Ref. 1 there is an underlying misunderstanding of the origin of the spinor transformation law. The very existence of spinor particles is closely related to one of the most fundamental properties of quantum theory, the representation of physical states not by unit vectors $|\psi\rangle$ of a Hilbert space $\mathcal{H}$, but by unit *rays* $\{e^{i\alpha}|\psi\rangle$, α arbitrary$\}$

[1] Y. Aharonov and L. Susskind, Phys. Rev. 158, 1237 (1967).

[2] The effect of e^{iHt} on ψ depends, of course, strongly on the details of the experimental arrangement, and therefore cannot be calculated by means of $U(\theta)$ which is universal. For instance, if box I is axially symmetric around the rotation axis, the electron spin and a possible homogeneous magnetic field being oriented along this same axis, then $e^{iHt}\psi$ clearly does not depend at all on the angle of rotation θ. This counterexample shows that formal application of $U(\theta)$ to ψ_1 is not a correct recipe for solving the Schrödinger equation. Interference experiments of the type considered here can therefore show only dynamical properties of quantum systems and have nothing to do with kinematics, e.g., spinor transformation law and fermion superselection rule. E. P. Wigner (private communication) has suggested the above counterexample to us and has also stressed the nonkinematical nature of the AS experiment.

in $\mathcal{H}$. Rotations, Galilei transformations, or Poincaré transformations then transform rays into rays which leads to a *ray representation* of the respective group[3] where each group element g is represented by a class $\tilde{U}(g)$ of unitary operators differing by an arbitrary phase factor $e^{i\alpha}$. For the rotation group $SO(3)$ and the connected part P of the Poincaré group one can, according to Bargmann,[4] select suitable members $U(g)$ of these equivalence classes which form a unitary vector representation of, respectively, $SO(3)$ and P (integer total spin) or of the corresponding covering groups $SU(2)$ and $\bar{P}$ (half-integer spin). For this so-called "reduction of phase" we refer to the clear exposition in the textbook of Ludwig.[5] As mentioned in the introduction, to one element of $SO(3)$ or P there correspond, in the latter case, still two physically equivalent operators in $SU(2)$ or $\bar{P}$, a reminder of the original starting point, the ray representation, and a warning not to take the term "rotation by 2π" too literally because it is in the same equivalence class $\tilde{U}(g)$ as the rotation by 0°. The reduction of phase is only a purely *mathematical* simplifying convention which leaves unaffected the fundamental arbitrariness of phase factors. And one should never expect physical consequences from a mathematical convention.

We would like to thank Professor E. P. Wigner for stimulating and helpful comments.

[3] See, for example, R. Hagedorn, Nuovo Cimento Suppl. **12**, 73 (1959).
[4] V. Bargmann, Ann. Math. **59**, 1 (1954).
[5] G. Ludwig, *Grundlagen der Quantenmechanik* (Springer-Verlag, Berlin, 1954).

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Testing Relativity with Laser Ranging to the Moon

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Precise laser ranging to the moon is shown to be capable of measuring the ratio (m_g/m_i) of gravitational mass to inertial mass of the earth to accuracy sufficient to detect gravitational self-energy contributions. If m_g/m_i of the earth differs from 1 by an amount of order (earth gravitational self-energy)/(earth total energy), then the lunar orbit will acquire a range oscillation of amplitude about 12 m.

In a recent paper, Baierlein studied possible tests of general relativity using laser ranging to the moon.[1] An accuracy of two parts in 10^{10} in laser round-trip time was assumed which corresponds to about 8-cm accuracy in the earth-moon distance.

Recently, the author has examined possible experiments to test the ratio of gravitational mass (m_g) to inertial mass (m_i) for massive bodies,[2] and in a separate paper has studied the aspects of gravitational theories which would be tested by a careful measurement of m_g/m_i for massive bodies.[3]

The purpose of this paper is to show that the laser ranging to the moon can be used to measure m_g/m_i for the earth—a possible experiment not considered by Baierlein.[1] The experimental effect discussed in this paper will be possibly two orders of magnitude larger than those discussed by Baierlein.

In the author's previous paper[2] the possibility was considered that the ratio of a massive body's gravitational mass to its inertial mass is given by

$$\frac{m_g}{m_i}=1+\eta\frac{G}{c^2}\int\rho(x)\rho(x')\frac{d^3xd^3x'}{|\mathbf{x}-\mathbf{x}'|}\bigg/\int\rho(x)d^3x, \quad (1)$$

where $\rho(x)$ is the body's mass density, η is a dimensionless constant of order 1, G is the gravitational constant, and c is the velocity of light. For a uniform mass of radius a, Eq. (1) yields

$$m_g/m_i=1+(6/5)\eta Gm/c^2a. \quad (2)$$

The earth's mass is somewhat concentrated in the center. This should about double the sizes of the effects discussed in this paper. For the earth, the correction term (2) is much larger than for the moon, so the moon correction will be neglected in this paper.

As the earth and moon travel around the sun, (2) will lead to an excess acceleration of the earth toward the sun of

$$\delta a=(6/5)\eta(Gm_e/c^2a_e)GM_\odot/R^2, \quad (3)$$

where R is the sun-earth distance. In the earth-centered coordinate system, the moon receives an acceleration which is the negative of (3). The effect of (3) then on the lunar orbit is now calculated.

Assume for simplicity a circular lunar orbit which satisfies the equations of motion

$$d^2r/dt^2=h^2/r^3-\mu/r^2+\delta a\cos\phi \quad (4a)$$

and

$$dh/dt=-\delta a\, r\sin\phi. \quad (4b)$$

[1] R. Baierlein, Phys. Rev. **162**, 1274 (1967).
[2] K. Nordtvedt, Phys. Rev. **169**, 1014 (1968).
[3] K. Nordtvedt, Phys. Rev. **169**, 1017 (1968).