Possibility of the Speed of Sound Exceeding the Speed of Light in Ultradense Matter*

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We show that, in the classical physics of very dense matter, Lorentz invariance imposes no restriction on the speed of sound or on the ratio of pressure to energy density. Indeed, the simplest and most reasonable classical many-particle theory can manifest such apparently noncausal behavior whenever the calculated self-energy of a particle exceeds its observed (renormalized) rest energy. This comes about because ordinary mass renormalization subtracts out part of a particle's self-interaction energy without altering the interaction with other particles that contributes to pressure. Two types of models are exhibited which, at low densities, show normal behavior and, at high densities, become superluminal (speed of sound greater than speed of light in vacuum) and ultrabaric (pressure greater than energy density). One is a system of classical particles which, when stationary, repel each other by a short-range repulsive Yukawa interaction. Although the particles interact through ordinary retarded neutral vector fields, after mass renormalization there must always be a domain of sufficiently high densities where this matter becomes superluminal and ultrabaric. The second group of models is a class of classical Lorentz-invariant nonlinear field theories which, in the limit of low densities, reduces to a noninteracting Klein-Gordon field. If matter deep inside superdense stars could be ultrabaric, then the limiting gravitational red shift from the star's surface would be slightly under 2. This is perhaps suggestive of the observed clustering of quasar absorption-line red shifts at 1.95.

I. INTRODUCTION

WHILE the equation of state for ultradense matter is unknown, it has generally been assumed[1] that the pressure p cannot exceed the total energy density ϵ (including rest mass),

$$p<\epsilon. \tag{1.1}$$

In relativistic as in nonrelativistic mechanics, the speed of any compressional wave is[2]

$$c_s^2=\frac{dp}{d(\text{mass density})}=c^2\frac{dp}{d\epsilon}, \tag{1.2}$$

the only effect of relativity being to replace the local mass density by ϵ/c^2. Before matter can become *ultrabaric* ($p>\epsilon$) it must first become *superluminal* ($dp/d\epsilon>1$ or speed of sound exceeding speed of light in vacuum). Since this possibility seemingly contradicts the principle of causality, the possibility of ultrabaric matter has heretofore generally been rejected.

The existence of a fundamental velocity (c), the same in all inertial frames, is the basis of relativistic kinematics; the velocity with which signals propagate in a medium depends on the dynamics.[3] We will present several examples of classical (unquantized) theories which are Lorentz invariant but for which bulk matter can become superluminal ($dp/d\epsilon>1$) and ultrabaric ($p/\epsilon>1$). In fact, such matter is a conceivable end point of ordinary crushed nuclear matter if, because of the persistence of the known short-range repulsions, nuclear matter is incompressible enough. In our examples, matter is ordinary ($p\ll\epsilon$) at low densities and, as the density is increased, passes smoothly into the region $p>\epsilon$ without revealing any pecularity or discontinuity when $p=\epsilon$ or $p>\epsilon$. Sound waves correspond to massless particles, so that as their speed c_s passes through c the energy remains finite, unlike the case of massive particles for which $v=c$ is a barrier. Superluminal sound waves can exist in a dense medium for which the total energy density is positive for all observers[4] and in which the microscopic particle velocities can even be nonrelativistic.

In dilute gases, the pressure is principally due to thermal motion of particles and is small compared with the energy density. For an ideal gas of particles of

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[1] B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory & Gravitational Collapse* (The University of Chicago Press, Chicago, Ill., 1965); Ya. B. Zel'dovich and I. D. Novikov, Usp. Fiz. Nauk **84**, 377 (1964) [English transl.: Soviet Phys.—Usp. **1**, 763 (1965)]. See, however, W. Pauli, *Theory of Relativity* (Pergamon Press, Inc., New York, 1958).

[2] L. D. Landau and E. M. Lifshitz, *Fluid Dynamics* (Pergamon Press, Inc., New York, 1959). In the high-density or strong-coupling limit the distinction between collision-dominated and collision-free sound waves disappears. In particular, zero sound and ordinary sound become indistinguishable translations of the Fermi sphere and propagate with the same velocity. In the high-density and static limit, Eq. (1.2) remains applicable to any kind of sound waves.

[3] Some textbooks (e.g., Pauli, Ref. 1) distinguish carefully between the principles of relativity and constant light velocity (which lead to the kinematics of the Lorentz group and prevent any material body from moving faster than light) and another principle allowing no signal faster than light. For an excellent discussion of the paradox of advanced actions which become possible when superluminal signal velocities are admitted, see J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **21**, 425 (1949), particularly p. 427. For recent interpretations of the possibility of noncausality, see G. Feinberg, Ref. 4; R. G. Newton, Phys. Rev. **162**, 1274 (1967); and D. A. Kirzhnits and V. L. Polyachenko, Zh. Eksperim. i Teor. Fiz. **46**, 755 (1964) [English transl.: Soviet Phys.—JETP **19**, 514 (1964)] which also claims to present a theory in which sound propagation can be superluminal.

[4] Our classical superluminal sound waves are therefore unrelated to the quanta of imaginary mass proposed by G. Feinberg, Phys. Rev. **159**, 1089 (1967).

mass m and number density n,

$$\epsilon=nm\left\langle\frac{c^2}{(1-v^2/c^2)^{1/2}}\right\rangle,$$

$$p=\tfrac{1}{3}nm\left\langle\frac{v^2}{(1-v^2/c^2)^{1/2}}\right\rangle,$$

in terms of average particle velocities, so that even when these velocities are relativistic

$$p\leq\tfrac{1}{3}\epsilon. \tag{1.3}$$

The equality holds for a photon gas, as also follows directly from the tracelessness of the energy-momentum tensor of the electromagnetic field. If charged particles are minimally coupled to the electromagnetic field, their energy-momentum tensor has negative trace so that $p<\frac{1}{3}\epsilon$.[5]

Zel'dovich[6] considered the example of particles interacting through the exchange of neutral vector mesons of nonzero mass and found that in the limit of infinite density n, p approaches but does not exceed ϵ. This example was the first to show how the inequality (1.3) can be exceeded once one considers interactions of finite range which do indeed obtain in strong interactions. In a relativistic Fermi gas,[7] where the Fermi velocity $v_F\approx c$, the exclusion principle can cause pressure fluctuations in one dimension and the speed of zero sound

$$c_s\rightarrow v_F\approx c.$$

In this paper, we investigate Lorentz-invariant models in which, as the density increases, p can exceed ϵ and the speed of sound can exceed the speed of light in vacuum. This superluminal and ultrabaric behavior can exist when the self-energy of a particle exceeds its observed (renormalized) mass. The total energy of a system of particles consists in the energy of interaction between different particles together with the self-energy and "mechanical" mass of individual particles. The observed or renormalized particle mass is the sum of these latter contributions and is, both in classical and in quantum theory, conventionally specified arbitrarily and independently of what one might calculate for the self-energy. Indeed, for point particles the self-energy is infinite but the renormalized masses are given arbitrary, observed finite values.

Renormalizing individual masses reduces the total mass-energy density without necessarily altering the pressure which depends upon interactions between *different* particles. In this way, the conventional process of mass renormalization (identifying a particle's mass with an arbitrary finite observable quantity) admits the possibility of pressure exceeding energy density.

The content of this paper is as follows. Section II considers the general conditions necessary for $p>\epsilon$ or $dp/d\epsilon>1$ and presents a simple many-particle theory in which these conditions are indeed realized whenever the observed individual particle's mass is less than its computed self-energy. The speed of sound is calculated from $dp/d\epsilon$ both directly and by solving dynamically for the motion of particles which interact among themselves via the usual retarded potentials. The different treatment afforded to mutual interactions and to self-interactions permits a macroscopic noncausality to appear in a theory based entirely on retarded interactions. An entirely microscopic version of this is known already[8,9] as the preacceleration phenomenon in the classical electrodynamics of point electrons.

Section III considers an entirely field description of matter which is Lorentz invariant, has positive-definite energy density, reduces to a free Klein-Gordon field at low densities, but becomes ultrabaric at high densities. In this theory too, the speed of sound is calculated from $dp/d\epsilon$ both directly and dynamically by studying the propagation of compressional waves through the continuum.

Thus we have separate classical particle theories and classical wave theories (field theories) in which matter becomes superluminal at high densities. A complete relativistic quantum-mechanical solution encompassing simultaneously both particle and wave properties is beyond us, but effects of quantization are discussed qualitatively in Sec. IV. If real matter becomes ultrabaric at all, this can happen only at densities some orders of magnitude greater than nuclear densities, conditions that may only be realized inside superdense stars. In Sec. IV, we show that light from the surface of such stars will exhibit a limiting gravitational red shift slightly below 2, which is suggestive of the observed clustering of quasar absorption-line red shifts at 1.95.

II. MANY-PARTICLE SYSTEM WITH REPULSIVE RETARDED INTERACTION

A. Conditions for Superluminal and for Ultrabaric Behavior

From the first law of thermodynamics or, otherwise, from the definition of pressure, we have

$$p=-(dE/dV)_S. \tag{2.1}$$

In terms of the number density $n=N/V$ and energy density $\epsilon=E/V$,

$$p=-\epsilon+nd\epsilon/dn, \tag{2.2}$$

$$dp/d\epsilon=(nd^2\epsilon/dn^2)/(d\epsilon/dn). \tag{2.3}$$

[5] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Pergamon Press, Inc., New York, 1962).
[6] Ya. B. Zel'dovich, Zh. Eksperim. i Teor. Fiz. **41**, 1609 (1961) [English transl.: Soviet Phys.—JETP **14**, 1143 (1962)].
[7] G. Kalman, Phys. Rev. **158**, 144 (1967).
[8] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938); see also F. Rohrlich, *Classical Charged Particles* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965).
[9] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).

(We consider only systems in their ground state; all processes are isoentropic or $T=0°$.) Thus if the total energy density (including rest mass) increases faster than the second power of the number density, then $p/\epsilon>1$ and $dp/d\epsilon>1$. In Secs. II and III, we consider Lorentz-invariant particle theories and field theories, respectively, in which this happens.

For a system of N particles at rest at points $\mathbf{r}_i$, with two-body interaction, we have

$$\phi_{ij}=\phi(\mathbf{r}_i-\mathbf{r}_j),$$
$$E=\sum_i m_ic^2+\tfrac{1}{2}\sum_{i\neq j}\phi_{ij}, \tag{2.4}$$
$$\epsilon=nmc^2+\frac{1}{2V}\sum_{i\neq j}\phi_{ij}. \tag{2.5}$$

For finite-range repulsive forces, the particles will, in the ground state, form a regular lattice with average spacing a. In the infinite density limit $a\rightarrow 0$, the lattice sum may be replaced by an integral

$$\sum_{i\neq j}\phi_{ij}=N\int\phi(r)d^3r\ (N/V) \tag{2.6}$$

and

$$\epsilon=nmc^2+\tfrac{1}{2}n^2v(0), \tag{2.7}$$
$$\frac{dp}{d\epsilon}=\frac{nv(0)}{mc^2+nv(0)}. \tag{2.8}$$

Here,

$$v(0)=\int\phi(r)d^3r \tag{2.9}$$

exists for finite-range interactions. In the continuum limit ($a\rightarrow 0$), each particle sits in a uniform potential due to all other particles homogeneously distributed, so that the potential energy density varies as n^2 in Eq. (2.7). Consequently,

$$p/\epsilon\leq 1,\quad dp/d\epsilon\leq 1, \tag{2.10}$$

the equality being approached for densities,

$$n\gg mc^2/v(0). \tag{2.11}$$

In the Zel'dovich model[6]

$$\phi=g^2e^{-\mu r}/r, \tag{2.12}$$
$$v(0)=4\pi g^2/\mu^2, \tag{2.13}$$

and for $m\approx 2\mu\approx$baryon mass, and strong coupling constant $g^2\approx 1$, the critical density

$$mc^2/v(0)=(1/4\pi)(mc^2/g^2)\mu^2 \tag{2.14}$$

is an order of magnitude greater than nuclear densities.

In Zel'dovich's treatment, p can never exceed ϵ nor $dp/d\epsilon$ exceed unity, because the correlations between particles are neglected. We shall now consider Zel'dovich's own model (which is a fairly realistic classical description of nuclear forces), and show that at densities high enough if the (negative) correlation energy exceeds nmc^2 in Eq. (2.5), matter must become superluminal and ultrabaric.

B. One-Dimensional Chain of Particles

At zero temperature, particles with mutual finite-range repulsions form a regular lattice which, in one dimension, is a chain of equally spaced particles. We assume that the particles repel each other through a two-body repulsive interaction

$$\phi_{ij}=g^2e^{-\mu|r_i-r_j|}, \tag{2.15}$$

which is the case if the particles interact via a neutral vector-meson field. For a regular interparticle spacing a, the energy per lattice particle is

$$\epsilon/n=mc^2+\tfrac{1}{2}g^2\sum{}' e^{-|n|\mu a}$$
$$=mc^2+g^2[e^{-\mu a}/(1-e^{-\mu a})]. \tag{2.16}$$

The sum $\sum'$ is over all positive and negative integers, excluding zero, since the self-interaction is included in the renormalized mass mc^2. Then the energy per unit length is

$$\epsilon=\frac{1}{a}\left(mc^2+\frac{g^2}{e^{\mu a}-1}\right), \tag{2.17}$$

$$p=-\frac{\partial(\epsilon/n)}{\partial a}=\frac{g^2\mu e^{\mu a}}{(e^{\mu a}-1)^2}, \tag{2.18}$$

$$\frac{dp}{d\epsilon}=\eta(\mu a)^2\frac{e^{\mu a}+1}{e^{\mu a}-1}\frac{e^{\mu a}}{(e^{\mu a}-1)^2+\eta[e^{\mu a}(\mu a+1)-1]}, \tag{2.19}$$

where $\eta=g^2/mc^2$ is a dimensionless measure of the ratio of coupling constant to rest energy. For very dense matter, $\mu a\ll 1$, the speed of sound is

$$c_s^2=c^2\frac{dp}{d\epsilon}=c^2\left[\frac{2-\mu a}{2-\frac{3}{2}a\mu+(\mu a/\eta)}\right], \tag{2.20}$$

which contains Zel'dovich's infinite-density limit $c_s^2\rightarrow c^2$ when $a\rightarrow 0$. For μa small but finite, two possibilities occur: (i) For $\frac{1}{2}g^2<mc^2$, $c_s^2<c^2$; (ii) for $\frac{1}{2}g^2>mc^2$, $c_s^2>c^2$. Since $\frac{1}{2}g^2$ is just the self-energy of a particle (which is finite in one dimension), the condition for superluminal behavior is just that the calculated self-energy exceed the renormalized mass.

In the opposite regime, $\mu a\gg 1$, only nearest-neighbor interactions are important:

$$p/\epsilon=\frac{g^2(\mu a)e^{-\mu a}}{mc^2+g^2e^{-\mu a}}, \tag{2.21}$$

$$\frac{dp}{d\epsilon}=\frac{g^2(\mu a)^2e^{-\mu a}}{mc^2+g^2(\mu a)e^{-\mu a}}. \tag{2.22}$$

In this low-density regime, the coupling strength must satisfy

$$(g^2/mc^2)e^{-\mu a}>1/(\mu a)^2 \tag{2.23}$$

for the lattice to become superluminal and must satisfy

$$(g^2/mc^2)e^{-\mu a}>1/\mu a \qquad (2.24)$$

for the lattice to become ultrabaric.

C. Three-Dimensional Lattice

In three dimensions, the two-body repulsive interaction,

$$\phi_{ij}=g^2\frac{e^{-\mu|\mathbf{r}_i-\mathbf{r}_j|}}{|\mathbf{r}_i-\mathbf{r}_j|}, \qquad (2.25)$$

leads to a ground state whose particular lattice structure depends on the average interparticle spacing a.

For any system of N particles, the potential energy for a lattice,

$$V_L=\tfrac{1}{2}\sum_{i\neq j}\phi_{ij}, \qquad (2.26)$$

is less than the potential energy

$$V_C=N\int\frac{e^{-\mu r}}{r}d^3r\frac{N}{V}=\frac{N^2}{V}\frac{4\pi g^2}{\mu^2}=\frac{N^2}{V}v(0) \qquad (2.27)$$

that would obtain in the continuum limit $a\rightarrow 0$; (which of course is why the lattice structure forms the lowest-energy state). We will now compare V_L and V_C in the two limits $0<\mu a\ll 1$ and $\mu a\gg 1$.

1. High-Density Behavior, $\mu a\ll 1$

For small but finite lattice spacing, $0<a\ll 1/\mu$, the lattice energy V_L is related to the continuum energy V_C as follows. In each unit cell, the potential energy of a particle situated a distance ξ from the lattice point would be

$$V(\xi)=V_0+\xi\cdot\nabla V_0+\tfrac{1}{2}\xi_i\xi_j\partial_i\partial_j V_0+\cdots, \qquad (2.28)$$

where V_0 is the value at the lattice point. If in a single unit cell the discrete lattice charge were smeared uniformly over the unit cell of dimension a, the mean potential energy

$$\langle V\rangle=V_0+\langle\xi\rangle\nabla V_0+\tfrac{1}{2}\langle\xi_i\xi_j\rangle\partial_i\partial_j V_0+\cdots \qquad (2.29)$$

would be obtained. Now, the continuum results if this smearing out of charge is done in all unit cells. Then, since $\nabla^2 V_0=\mu^2 V_0$ and $\langle\xi_i{}^l\rangle=0$ for odd l,

$$V_C=V_L+\alpha(\mu a)^2V_L+\beta(\mu a)^4V_L+\cdots, \qquad (2.30)$$

where the $\alpha,\beta\cdots$ are dimensionless positive coefficients determined by the geometry of the individual unit cell. Therefore, for a very dense lattice

$$V_L=[1-\alpha(\mu a)^2]V_C, \quad \mu a\ll 1.$$

The general results are independent of the particular lattice structure assumed. For simplicity of calculation, we assume a simple cubic lattice in which each particle is surrounded by six nearest neighbors. Then $\tfrac{1}{2}\langle\xi_i\xi_j\rangle\times\partial_i\partial_jV_0=\tfrac{1}{2}\times\tfrac{1}{3}(\tfrac{1}{2}\alpha)^2(\mu^2V_0)=(1/24)(\mu a)^2V_0$ and $\alpha=2/24=1/12$.

$$E=Nmc^2+[1-\alpha(\mu a)^2](N^2/V)v(0), \qquad (2.31)$$

$$\epsilon=n\{mc^2+\tfrac{1}{2}nv(0)[1-\alpha(\mu a)^2]\}, \qquad (2.32)$$

$$p=\tfrac{1}{2}n^2v(0)[1-\tfrac{1}{3}\alpha(\mu a)^2], \qquad (2.33)$$

$$\frac{dp}{d\epsilon}=\frac{nv(0)[1-(2/9)\alpha(\mu a)^2]}{mc^2+nv(0)[1-\tfrac{2}{3}\alpha(\mu a)^2]}=\frac{1-(2/9)\alpha(\mu a)^2}{1-\tfrac{2}{3}\alpha(\mu a)^2+(mc^2/4\pi g^2)a(\mu a)^2}, \quad \mu a\ll 1. \qquad (2.34)$$

Comparison with Eqs. (2.7) and (2.8) shows that the negative correlation energy in Eq. (2.31) reduces the energy density more than the pressure is reduced and makes the speed of sound in (2.34) greater than what it would be in the continuum limit $a=0$. Whenever the density is high enough, so that the interparticle spacing satisfies

$$a<(g^2/mc^2)(8\pi/9)\alpha, \qquad (2.35)$$

then

$$c_s>c.$$

For $a\ll(g^2/mc^2)(8\pi/9)\alpha$,

$$(c_s/c)^2=1+(4/9)\alpha(\mu a)^2, \qquad (2.36)$$

which reduces to Zel'dovich's result when $a\rightarrow 0$ or when correlation α is ignored. For point particles interacting by finite-range repulsive forces in three dimensions, there is *always*, for any nonvanishing coupling strength, a regime of high densities in which the speed of sound exceeds the speed of light in vacuum. According to Eq. (2.35), this regime obtains whenever g^2/a, the particle's field energy outside of a, exceeds its renormalized mass. This was also the criterion for superluminal behavior in one dimension.

2. Low-Density Behavior, $\mu a\gg 1$

In the other limiting regime where $\mu a\gg 1$, only nearest-neighbor interactions are significant. In a simple cubic lattice, there are six nearest neighbors and the lattice energy is

$$E=N[mc^2+\tfrac{1}{2}\times 6g^2(e^{-\mu a}/a)], \qquad (2.37)$$

$$p=(\mu/a^2)(e^{-\mu a}/a)+O(1/\mu a), \qquad (2.38)$$

$$\frac{p}{\epsilon}=\frac{(\mu a)(g^2e^{-\mu a}/a)}{mc^2+3g^2(e^{-\mu a}/a)}, \quad \mu a\gg 1. \qquad (2.39)$$

When a compressional wave propagates along a principal axis of a cubic crystal, the lattice spacing changes only along this one axis. The pressure variation is anisotropic and the change in energy density involves only $2/6=\tfrac{1}{3}$ of the nearest-neighbor interactions. The resultant speed of sound is

$$\left(\frac{c_s}{c}\right)^2=\frac{dp}{d\epsilon}=\frac{(\mu a)^2(g^2e^{-\mu a}/a)}{mc^2+(\mu a)(g^2e^{-\mu a}/a)},\quad \mu a\gg 1. \tag{2.40}$$

When

$$g^2(e^{-\mu a}/a)\gtrsim mc^2/(\mu a),\quad (c_s/c)^2\approx(\mu a)\gg 1. \tag{2.41}$$

At still higher densities, when

$$g^2(e^{-\mu a}/a)\gtrsim mc^2, \tag{2.42}$$

$$p/\epsilon\approx\tfrac{1}{3}\mu a>1. \tag{2.43}$$

The condition (2.42) for matter to be ultrabaric again is that the field energy in the asymptotic region has to exceed the particle's observed (renormalized) rest mass; this will happen at low densities only if the coupling is very strong. Note that the conditions (2.41) and (2.42) depend only on the asymptotic two-particle interaction at large distances.

D. Dynamical Calculation of Sound-Wave Propagation

A dynamical calculation must in the limit of infinite wavelengths reproduce the static result

$$c_s^2=c^2dp/d\epsilon, \tag{2.44}$$

obtained in Eqs. (2.34) and (2.40). We present a dynamical calculation for didactic purposes and to extend the earlier results to shorter wavelengths and higher frequencies. To avoid analytic complications, we consider only the simple cubic lattice with nearest-neighbor interactions ($\mu a\gg 1$).

The usual retarded Green's function,

$$G_{\text{ret}}(r,t)=\int_{-\infty}^{\infty}e^{-[\mu^2-(\omega/c)^2]^{1/2}r}e^{i\omega t}\left(\frac{d\omega}{2\pi}\right), \tag{2.45}$$

is the solution of

$$\left[\frac{\partial^2}{(c\partial t)^2}-\nabla^2+\mu^2\right]G_{\text{ret}}=4\pi\delta^3(\mathbf{r})\delta(t) \tag{2.46}$$

with the branches of the square root chosen so that

$$G_{\text{ret}}(r,t)=0,\quad r>ct. \tag{2.47}$$

It is important to recall that the usual phase retardation occurs only for the frequency components $\omega>c\mu$; for $\omega<c\mu$, an amplitude reduction appears in place of phase retardation.

The charge current density of all the other lattice particles produces at the nth lattice site the retarded external fields

$$\phi_n(\mathbf{r},t)=\int\sum_{l\neq n}G_{\text{ret}}(\mathbf{r}-\mathbf{r}',t-t')j_0(\mathbf{r}',t)d^3r'dt', \tag{2.48}$$

$$\mathbf{A}_n(\mathbf{r},t)=\int\sum_{l\neq n}G_{\text{ret}}(\mathbf{r}-\mathbf{r}',t-t')\mathbf{j}(\mathbf{r}',t')d^3r'dt', \tag{2.49}$$

where

$$j_0(\mathbf{r}',t')=g\delta(\mathbf{r}'-\mathbf{a}_l-\mathbf{x}_l(t')), \tag{2.50}$$

$$\mathbf{j}(\mathbf{r}',t')=g\dot{x}_l(t')\delta(\mathbf{r}'-\mathbf{a}_l-\mathbf{x}_l(t')), \tag{2.51}$$

$\mathbf{a}_l$ being the equilibrium position of the lth particle and $\mathbf{x}_l(t')$ its displacement from equilibrium at time t'. Self-interaction of the nth particle with itself produces mass renormalization and other effects to be discussed later. The external force on the nth particle,

$$\mathbf{F}_n(r,t)=g(\nabla\phi_n-\dot{\mathbf{A}}_n)=\mathbf{F}_{n\phi}=\mathbf{F}_{n\mathbf{A}}, \tag{2.52}$$

arises from its own motion and that of its neighbors.

We consider a compressional mode in which all displacements $\mathbf{x}_n$ are parallel to one principal crystal axis and have the same frequency ω. To lowest order in x, the six nearest neighbors give forces along the principal axis:

$$F_{n\mathbf{A}}=\frac{g^2}{a}e^{[\mu^2-(\omega/c)^2]^{1/2}a}(i\omega)\times[\dot{x}_{n-1}(t)+\dot{x}_{n+1}(t)+4\dot{x}_n(t)], \tag{2.53}$$

$$F_{n\phi}=-g^2\left[\frac{\partial^2}{\partial a^2}\left(\frac{e^{-\mu a}}{a}\right)2x_n-\frac{\partial^2}{\partial a^2}\left(\frac{e^{-[\mu^2-(\omega/c)^2]^{1/2}a}}{a}\right)\times(x_{n-1}+x_{n+1})\right]-g^2\left[4\frac{\partial}{\partial a}\left(\frac{e^{-\mu a}}{a}\right)\left(\frac{x_n}{a}\right)-\frac{4\partial}{\partial a}\left(\frac{e^{-[\mu^2-(\omega/c)^2]^{1/2}a}}{a}\right)\left(\frac{x_n}{a}\right)\right]. \tag{2.54}$$

In $F_{n\mathbf{A}}$, the term proportional to $4\dot{x}_n$ arises from the in-phase motion of the four neighbors above, below, and to the sides of particle n; the first two terms are due to the motion of the neighbors behind and before the nth particle. In $F_{n\phi}$, the first bracket contains the force on the nth particle by these fore and aft neighbors due to their motion and due to the nth particle's motion; the second bracket is due to the four neighbors above, below, and to the sides, which move in phase with the nth particle.

At low frequencies the external force $\mathbf{F}_n$ is equal to $m\ddot{\mathbf{x}}_n$, where m is the measured (renormalized) mass. At higher frequencies, other effects of self-interaction should be included, such as the radiation damping force (for $\omega/c>\mu$). At low frequencies,

$$\mathbf{F}_n=-m\omega^2\mathbf{x}_n+O(\omega^4). \tag{2.55}$$

We seek normal modes for which

$$x_n\sim e^{ikan}e^{-i\omega t},\quad \text{all } n, \tag{2.56}$$

and obtain from Eqs. (2.53)–(2.55) the dispersion relation

$$-g^2\left[2\mu^2\frac{e^{-\mu a}}{a}-2\mu^2\frac{e^{-[\mu^2-(\omega/c)^2]^{1/2}a}}{a}\cos ka\right]=-m\omega^2, \tag{2.57}$$

valid for $\mu a\ll 1$ and $(\omega/c)\ll\mu$. On the left-hand side, only the terms in $F_{n\phi}$ due to the two front and back neighbors have survived. The same result would be obtained from a scalar-meson theory, if scalar interaction led to repulsion between particles.

In the dispersion relation (2.57), for $k\to 0$, $\omega\to 0$, and

$$\omega^2=\frac{(\mu a)^2(g^2e^{-\mu a}/a)}{mc^2+(\mu a)(g^2e^{-\mu a}/a)}(kc)^2, \tag{2.58}$$

which gives a phase velocity = group velocity, agreeing with Eq. (2.40). We emphasize that this result, that the velocity of sound can exceed the velocity of light, was obtained by the use of retarded interactions and depended only on the long-range part of the interparticle interaction and the use of the renormalized mass in Eq. (2.55).

Our lattice model can be extended to higher frequencies by specifying the sources more completely. We now assume point sources in order to determine the higher-frequency terms on the right-hand side of Eq. (2.55). Following Bhabha,[10] the retarded self-interaction of a point particle in neutral vector-meson theory contains a formally infinite self-energy which is discarded (renormalized), a structure-independent radiation damping term, and a term due to the possibility of the particle catching up with the meson field which it has earlier emitted. This last term is missing in electrodynamics ($\mu=0$) where the emitted field always propagates with velocity c, faster than the particle velocity. Bhabha's renormalized equation of motion for a point particle in an external field is, for particle speeds $\dot{x}\ll c$,

$$\mathbf{F}=m'\ddot{\mathbf{x}}-\frac{2g^2}{3c^3}\frac{d^3\mathbf{x}}{dt^3}-g^2\mu^2\int_0^\infty\frac{dy}{y^2}J_2(\mu y)\times\left[\mathbf{x}(t)-\mathbf{x}(t-y)-y\frac{\dot{\mathbf{x}}}{c}(t-y)\right]. \tag{2.59}$$

Since the last term effectively contains terms proportional to $\ddot{\mathbf{x}}$, m' is only part of the renormalized mass. For oscillatory motion with frequency $(\omega/c)\ll\mu$, Eq. (2.59) gives

$$F=-m\omega^2x-(g^2\omega^4x/4\mu c^4), \tag{2.60}$$

where the renormalized mass,

$$m=m'-\tfrac{1}{2}g^2\mu, \tag{2.61}$$

is m' only in electrodynamics, where $\mu=0$. The ω^4 term in Eq. (2.60) represents the oscillating particle's increase in self-energy with increasing frequency that occurs because in Eq. (2.45), the range of the meson field increases from $1/\mu$ at $\omega=0$ to ∞ at $\omega=\mu$.

If the force from Eqs. (2.53)–(2.54) is now inserted in Eq. (2.60), the dispersion relation for longitudinal phonons becomes

$$(\omega/c)^2(mc^2+g^2\mu e^{-\mu a}\cos ka)+\frac{g^2(\omega/c)^4}{4\mu}=4\mu^2\frac{g^2e^{-\mu a}}{a}\sin^2(\tfrac{1}{2}ka). \tag{2.62}$$

As $\omega\to 0$, this reduces to Eq. (2.58), but for all k the solutions of this equation for phonons of frequency ω are real and periodic in ka. (The other, high-frequency, root corresponds to the spurious runaway solutions, well known in electrodynamics, which are suppressed by the asymptotic conditions.) The lattice of point particles is stable against longitudinal low-frequency density fluctuations at all wavelengths and admits a maximum frequency at $ka=\pi$ which is

$$(\omega/c)^2\sim 4\mu^2(e^{-\mu a}/\mu a)^{1/2}\ll\mu^2.$$

The group velocity, which was superluminal at $k=0$, reduces to zero at $k=\pi/a$.

Sound propagation faster than light has appeared because the ordinary mass renormalization subtracts out part of a particle's self-interaction without altering the interaction with other particles that gives rise to pressure. Such a noncausality in the face of the usual retarded interactions occurs also in the classical electrodynamics of point charges when the infinite self-energy is replaced by a finite mass.[8] However, in electrodynamics the resultant "pre-acceleration" occurs only microscopically without leading to any macroscopic effects. In a system of many charges interacting electromagnetically, the pre-acceleration time vanishes rapidly with increasing number of particles.[9]

III. CLASSICAL FIELD THEORY

A complete quantum-mechanical description of matter would exhibit both particle and wave (field) properties. As an alternative description of classical matter to the particle model of Sec. II, we now consider the other extreme of a pure field. The classical field theory to be discussed below is Lorentz invariant, has positive-definite energy, and reduces to a free field in the low-density limit. It does, however, propagate low-frequency sound waves which, although slow at low densities, become superluminal at high densities until finally the matter itself becomes ultrabaric.

We consider a complex scalar field described by the first-order Lagrangian,

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I, \tag{3.1}$$

$$\mathcal{L}_0=\psi_\mu^\dagger\partial^\mu\phi+\partial^\mu\phi^\dagger\psi_\mu-|\psi_\mu|^2-\mu^2|\phi|^2, \tag{3.2}$$

$$\mathcal{L}_I=-gf(j_\mu^2),\quad g>0, \tag{3.3}$$

in which the current

$$j_\mu=\tfrac{1}{2}i(\phi^\dagger\psi_\mu-\psi_\mu^\dagger\phi) \tag{3.4}$$

is self-coupled. Because of the gauge invariance of this Lagrangian, this current is conserved; we will be

[10] H. J. Bhabha, Proc. Roy. Soc. (London) **A172**, 384 (1939); see also P. Havas, Phys. Rev. **87**, 309 (1952).

interested in a sector in which

$$N=\int nd^3x,\quad n=j_0 \tag{3.5}$$

has a fixed nonvanishing value. The interaction (3.3), with arbitrary $f(x)$, will give an energy density depending on $j_\mu^2=n^2-\mathbf{j}^2$ in a way to be chosen. For $f\equiv 0$, the above Lagrangian describes a conventional free Klein-Gordon field. For $f(x)=x$, we expect to recover the Zel'dovich example in which the pressure approaches but does not exceed the energy density. If $f(x)$, while vanishing when $x=0$, increases more rapidly than x, we expect to describe a continuum which is normal at low densities and ultrabaric at high densities.

The equations of motion obtained from the Lagrangian (3.1) are

$$D_\mu\phi-\psi_\mu=0, \tag{3.6}$$

$$D_\mu\psi^\mu+\mu^2\phi=0, \tag{3.7}$$

where

$$D_\mu\equiv\partial_\mu+igf'j_\mu, \tag{3.8}$$

$$f'\equiv f'(j_\mu^2)=\frac{d}{dj_\mu^2}f(j_\mu^2). \tag{3.9}$$

Thus

$$j_\mu=\tfrac{1}{2}i(\phi^\dagger\overleftrightarrow{\partial}_\mu\phi)-2gf'j_\mu\phi^\dagger\phi \tag{3.10}$$

$$=\frac{s_\mu}{1+2gf'S}, \tag{3.11}$$

where

$$s_\mu\equiv\tfrac{1}{2}i(\phi^\dagger\overleftrightarrow{\partial}_\mu\phi) \tag{3.12}$$

would be the conserved current in the absence of interaction and

$$S\equiv\phi^\dagger\phi. \tag{3.13}$$

With $\pi=\psi_0{}^\dagger$ canonically conjugate to ϕ, the Hamiltonian density is

$$\mathcal{H}=|\pi|^2+|\nabla\phi|^2+\mu^2|\phi|^2+gf-4g^2(f')^2|\phi|^2\mathbf{j}^2. \tag{3.14}$$

The noncovariant term in $\mathbf{j}^2$ appears because the interaction is effectively of derivative form. Because

$$s_\mu+\tfrac{1}{2}i\partial_\mu S=i\phi^\dagger(\partial_\mu\phi), \tag{3.15}$$

$$s_\mu-\tfrac{1}{2}i\partial_\mu S=-i(\partial_\mu\phi^\dagger)\phi, \tag{3.16}$$

$$|\nabla\phi|^2=(|\mathbf{s}|^2+\tfrac{1}{4}|\nabla S|^2)/S, \tag{3.17}$$

so that

$$|\nabla\phi|^2-4g^2(f')^2|\phi|^2\mathbf{j}^2 =(1/S)[\tfrac{1}{4}(\nabla S)^2+\mathbf{j}^2(1+4gf'S)]\geqslant 0. \tag{3.18}$$

The Hamiltonian density is thus positive definite.

The ground state of the system is the translationally invariant state

$$\nabla\phi=0=\mathbf{j},\quad \phi=\phi(t) \tag{3.19}$$

$$n=\tfrac{1}{2}i(\phi^\dagger\pi^\dagger-\pi\phi)=\text{constant}. \tag{3.20}$$

For given n,

$$\pi^\dagger=n/i\phi^\dagger,\quad \pi=-n/i\phi, \tag{3.21}$$

$$\mathcal{H}(n,S)=n^2/S+\mu^2S+gf(n^2), \tag{3.22}$$

whose minimum is at

$$S_{\min}=|n|/\mu, \tag{3.23}$$

$$\mathcal{H}_{\min}=2|n|\mu+gf(n^2)=\epsilon. \tag{3.24}$$

With this energy density, we obtain

$$p=-\epsilon+nd\epsilon/dn =-gf(n^2)+2gn^2f'(n^2)\geqslant 0, \tag{3.25}$$

so that

$$\frac{p}{\epsilon}=\frac{-gf(n^2)+2gn^2f'(n^2)}{2|n|\mu+gf(n^2)}, \tag{3.26}$$

$$\frac{dp}{d\epsilon}=\frac{2gnf'+4gn^3f''}{2\mu+2gnf'}=\left(\frac{c_s}{c}\right)^2. \tag{3.27}$$

If $f(n^2)=n^2$, $f'(n^2)=1$, and $f''=0$, we recover Eqs. (2.7)–(2.8); but if $f(n^2)$ varies as a higher power of n^2, the matter becomes superluminal and ultrabaric for n large enough.

In order to confirm that Eq. (3.27) actually represents the propagation of some kind of density wave, we now consider small density fluctuations about the ground state:

$$\phi^{(0)}=\left(\frac{|n|}{\mu}\right)^{1/2}e^{-i\omega_0 t},\quad \omega_0\equiv\mu+gnf'(n^2). \tag{3.28}$$

We return to Eqs. (3.6)–(3.7), which can be written

$$[(\partial_\mu+igf'j_\mu)^2+\mu^2]\phi=0, \tag{3.29}$$

and consider perturbed solutions of the form

$$\phi=\phi^{(0)}(1+C\cos k\cdot x+iD\sin k\cdot x)\equiv\phi^{(0)}+\phi^{(1)},\quad k\cdot x=\omega t-\mathbf{k}\cdot\mathbf{x}. \tag{3.30}$$

Inserting in Eq. (3.29) and using current conservation, we obtain the dispersion relation

$$\left(\frac{\omega}{ck}\right)^2=\left(\frac{c_s}{c}\right)^2+\frac{\mu}{\omega_0}\frac{\omega^2-c^2k^2}{\omega^2-c^2k^2-4\mu^2}, \tag{3.31}$$

where

$$\left(\frac{c_s}{c}\right)^2\equiv\frac{2gnf'+4gn^3f''}{2\mu+2gnf'} \tag{3.32}$$

is the same quantity which was computed statically in Eq. (3.27). The dispersion relation (3.31) is a quadratic:

$$\omega^4-\omega^2(Dk^2+c^2k^2+4\mu^2)+(ck)^2[Dk^2+4\mu^2(c_s/c)^2]=0,$$

$$D\equiv c_s{}^2+c^2\mu/\omega_0,$$

whose solution is

$$\omega^2=\tfrac{1}{2}[4\mu^2+(D+c^2)k^2]\pm\tfrac{1}{2}\{[4\mu^2+(c^2-D)k^2]^2+16c^2k^2\mu^3/\omega_0\}^{1/2}.$$

The discriminant is positive and there are two stable modes that do not cross:

(1) a low-frequency (acoustic) branch

$$\omega^2 \approx c_s^2 k^2 + O(k^4),$$

(2) a high-frequency (optic) branch

$$\omega^2 \approx 4\mu^2 + O(k^2).$$

The acoustic modes thus do propagate at long wavelengths with the speed of sound (3.27) calculated. At long wavelengths, $\omega = c_s k$ and the group and phase velocities are equal. At all wavelengths, the phase velocity and the group velocity of the acoustic branch is either subluminal or superluminal depending on whether in the zero-frequency limit the density makes $c_s < c$ or $c_s > c$.

IV. CONCLUDING DISCUSSION

A. Quantization

We have shown that in a conventional classical particle theory and in a simple classical field theory matter can become superluminal and ultrabaric when the density is high enough. Because canonical mass renormalization, in quantum theory just as in classical theory, reduces the energy density of a many-particle system without reducing the pressure proportionally, superluminal and ultrabaric behavior may possibly persist in quantum theories. Unfortunately, exactly solvable models of suggestive relativistic quantum many-particle systems or field theories are unavailable.[11]

One need not look far to see how different a nonlinear quantum field theory can be from a classical field theory with the same Lagrangian. A boson field with quadrilinear repulsive self-coupling,

$$g\phi^\dagger(x)\phi^\dagger(x)\phi(x)\phi(x),$$

gives a density-dependent energy in classical theory. In quantum theory, however, the δ-function repulsion between different particles in three dimensions leads to no interparticle interaction whatsoever.

The quantized version of noncausal classical particle theories presumably still describes particles whose bare models even have sensible vacuum states, i.e., whether there is a lowest energy state (especially for large coupling constants). Quantum zero-point motion opposes lattice formation particularly for small $g^2/\hbar c$ sufficiently large that the lattice would otherwise be maintained. We can examine acausality for various standard approximations to such systems although they are not necessarily relevant to the exact results. In a very dense Fermi system, zero-sound speed already approaches the speed of light, when calculated in the Hartree-Vlasov approximation which ignores exchange.[7] If the exchange energy is included, a sufficiently dense Fermi system, strongly coupled to a neutral vector-meson field, becomes ultrabaric if ϵ is calculated to lowest order.[12] In a dense Bose system, the positional anticorrelation needed for $p > \epsilon$ can come about only through the repulsive forces. In S-matrix theories, noncausal ghost poles may appear in solvable models or approximate solutions but are deliberately rejected. Indeed, practically all models satisfying unitarity do show such ghosts in elastic scattering whenever the coupling constants exceed certain bounds.[13]

There are thus some indications that, in quantum theory as in classical theory, causality is logically distinct from Lorentz invariance, and noncausal sound propagation is not impossible at high densities.

TABLE I. Maximum mass-radius ratio and surface red shift z_s assuming constant density ϵ. The slightly higher values in parentheses refer to Bondi's optimal models in which $\epsilon(r)$ increases centrally but always is as large as $p(r)$ or as $3p(r)$.

Limiting central pressure-density ratio $(p/\epsilon)_c$	M/R		z_s	
∞	0.444		2	
1	0.375	(0.390)	1	(1.14)
$\frac{1}{3}$	0.278	(0.319)	0.5	(0.65)

B. An Observational Consequence

Assuming provisionally that real matter may become ultrabaric at high densities, we now discuss an observable consequence. Only in superdense (neutron or hyperon) stars might the requisite densities obtain.

A quasistatic star must be in hydrostatic equilibrium and also stable against radial oscillations. These requirements, together with the equation of state of the stellar matter, determine a maximum mass-radius ratio M/R and hence a maximum gravitational red shift z_s from the star's surface.

The hydrostatic equilibrium is governed by the Tolman-Oppenheimer-Volkoff condition

$$-\frac{dp/dr}{p+\epsilon} = \frac{m+4\pi r^3 p}{r^2(1-2m/r)}, \tag{4.1}$$

where

$$m(r) = \frac{G}{c^2}\int_0^r \epsilon(r) 4\pi r^2 dr, \tag{4.2}$$

the local effects of general relativity being only to alter the Newtonian gravitational force by the addition of pressure terms and space-time dilatation factors. Any local distribution of ultrabaric matter, in which of

[11] Approximate models for many-particle systems showing ultrabaric behavior are not hard to find. Electrons interacting through neutrino pair exchange exhibit G^2/r^5 repulsion in lowest order in the four-fermion coupling constant G. This repulsion is strong enough at short distances to make such a system superluminal if the observable electron mass is finite. The unrenormalizability of the theory makes corrections to the Born-approximation result not well defined.

[12] While a given Feynman diagram, of course, gives causal propagation, this need not be so for the Green's function corresponding to an infinite set of Feynman diagrams. When the speed of sound is calculated from $dp/d\epsilon$, an infinite set of diagrams is summed making superluminal behavior possible.

[13] M. Ruderman, Phys. Rev. 127, 312 (1962).

course the energy-momentum tensor density is locally conserved, $\partial^\nu T_{\mu\nu}=0$, and is of positive trace, $T^\mu{}_\mu=3p+\epsilon>0$, and for which the energy density ϵ is also positive, will be compatible with general relativity. For a given radius R, the most massive star will be that which has as much mass density ϵ/c^2 on the outside as possible and, to support it, the greatest central pressure. Stability against radial oscillations demands, however, that for a fluid $\epsilon(r)$ not increase outwards.

We consider first a star of constant density ϵ (Schwarzschild interior solution). Then $m=\frac{4}{3}(G/c^2)\pi r^3\epsilon$,

$$\frac{-dp}{(p+\epsilon)(p+\frac{1}{3}\epsilon)}=\frac{4\pi r dr}{1-(8/3)(G/c^2)\pi\epsilon r^2} \tag{4.3}$$

and

$$\frac{p(r)+\epsilon}{p(r)+\frac{1}{3}\epsilon}=3\left(\frac{1-2u_s}{1-2u(r)}\right)^{1/2}, \tag{4.4}$$

where

$$u(r)=m/r,$$

$$u_s=u(R)=M/R. \tag{4.5}$$

The pressure,

$$p(r)=\epsilon\,[(1-2u)^{1/2}-(1-2u_s)^{1/2}]/[3(1-2u_s)^{1/2}-(1-2u)^{1/2}], \tag{4.6}$$

increases inwards, and at the center

$$(p/\epsilon)_c=z_s/(2-z_s), \tag{4.7}$$

where

$$1+z_s=(1-2u_s)^{-1/2}=(1-2M/R)^{-1/2} \tag{4.8}$$

is the red shift of light from the surface of the star. M/R and z_s are tabulated as a function of $(p/\epsilon)_c$ in Table I, for this constant-density model. The maximum red shift $z_s=2$ is realized only if infinite central pressure and p/ϵ are permitted. This possibility can be a limiting state of superdense matter only if superluminal speeds of sound are admitted.

If the central pressure p_c cannot be infinite, then the constant-density star is not quite optimal from the point of view of maximizing the star's total mass or surface red shift. As Bondi has shown,[14] if the pressure is density-limited, a modest increase of density towards the star's center permits a slightly larger red shift than would obtain in a constant-density model. The values of M/R and z_s obtained by Bondi in this optimal case by numerical integration are listed in parentheses in Table I.

What is significant is that a large gravitational red shift, approaching but not exceeding 2, obtains only if matter is allowed to be ultrabaric in the star's core. For $z_s=2$, $p(r)>\epsilon$ only where $(r/R)^2<5/8$; only the central third of the star's volume is ultrabaric, this core being surrounded by normal matter in which the pressure decreases from $p=\epsilon$ to $p=0$ at the surface. Although the mass of a star whose center is infinitely ultrabaric is only $(4/9)/(3/8)=1.2$ times greater than otherwise possible; this 20% increase in mass changes the surface red shift from 1.14 to 2.0. Since infinitely ultrabaric matter is only a limiting case and the star's density must actually decrease smoothly through its envelope, the value $z_s=2$ could not be quite realized in any actual ultrabaric star.

It is interesting to observe that in those quasars that show several absorption lines, unique absorption-line red shifts near 1.95 are reported.[15] A unique red shift is difficult to understand on any cosmological model, so these data have lent some support to the view[16] that quasar red shifts may be gravitational in origin. We see that the gravitational red shift from superdense neutron stars with ultrabaric cores would give almost precisely the absorption-line red shift reported.

The possibility of ultrabaric or superluminal behavior deep within certain stable stars would be revealed most clearly by a surface gravitational red shift which exceeds 1.14. Other tests appear extremely difficult. The possibility of superluminal or ultrabaric matter would not seem to change cosomological models in any qualitative sense. As long as energy density and pressure remain positive, for example, bouncing universes or the avoidance of singularities in the past are still impossible in classical general relativity.[17]

Our principal point is one of principle: Lorentz invariance permits (in classical theory at least) speed of sound greater than speed of light and pressure greater than energy density. Indeed, the possibility of superluminal and ultrabaric behavior in ultradense systems is inevitable (at least classically) once finite-range repulsive interaction energy exceeds the renormalized rests mass.

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[14] H. Bondi, Proc. Roy. Soc. (London) A282, 303 (1964).

[15] G. R. Burbidge and E. M. Burbidge, Astrophys. J. 148, L107 (1967).

[16] See, for example, F. Hoyle and W. Fowler, Nature 213, 373 (1967); and Ref. 15.

[17] R. Penrose, Phys. Rev. Letters 14, 57 (1965); S. W. Hawking and G. F. R. Ellis, Phys. Letters 17, 246 (1965); and S. W. Hawking, Phys. Rev. Letters 17, 444 (1966).