Hartree-Fock-Bogoliubov Projected Spectra for Finite Nuclei

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Explicit expressions convenient for numerical calculations are derived for the number- and total-spin-projected spectrum of a nucleus from its Hartree-Fock-Bogoliubov solution. Suitable approximation to such a spectrum is found. The applications of these results are also carried out.

1. INTRODUCTION

IN detailed calculations of the properties of finite nuclei, the most useful method in the past has been configuration-mixing calculations within the framework of the shell model.[1] However, it is limited to near closed-shell nuclei because of certain difficulties underlying the many-particle configurations. The construction of good angular-momentum states and the computation of matrix elements between such states becomes very difficult. An alternative approach which avoids this difficulty has been suggested by Reflich, Kurath, and Picman.[2] This method makes use of the projection technique to calculate the low-lying states of a nucleus from its Hartree-Fock (HF) solution. This approach has been studied extensively.[3] While applying this approach to $2p$-$1f$ shell nuclei, we found two main features of these calculations[4]: (1) The projected first excited state from the HF solution was at about $\frac{1}{2}$ the experimental separation. (2) There were bands lying close to one another. One way to overcome this difficulty would be to carry out the band-mixing calculations. Another approach that is well known in the literature is the Hartree-Fock-Bogoliubov[5] (HFB) method. For the following reasons we prefer to investigate the projection of HFB wave functions for the $2p$-$1f$ shell nuclei. Since one knows that there exists a gap between the excited state of a nucleus and its HFB ground state, it is clear that the second difficulty will be removed. The first bad feature of the projected HF spectrum also will not be present, as the HFB correlations reduce the moment of inertia, thereby increasing the separation of the first excited state. Thus in the investigation of the equilibrium shapes of the nuclei in $2p$-$1f$ shell, one should include pairing effects together with the HF correlations.

Our aim in this paper is to carry out the essential algebra in order to obtain the low-lying nuclear spectra from such an intrinsic state. Since the HFB wave function is not an eigenfunction of the number operator N and the total angular momentum I, we have the same assumption as that made in the HF projection technique.[3] Namely, the low-lying nuclear-state wave functions now would be given by the N projection and I projection of the intrinsic HFB wave function. In Sec. 2 we discuss the relevant part of the HFB method. In Sec. 3 we derive the projected spectra and the expectation value with respect to this wave function of the one-body operator. In Secs. 4 and 5, we discuss the approximations for separate N and I projections. In Sec. 6 we estimate the effects of I and N projection on the HFB calculations based on the results of Secs. 4 and 5.

2. HFB SOLUTION

The HF theory gives deformed solutions for all nuclei except at closed shells. This is particularly true in the case of quadrupole-quadrupole force. It is well known that the pairing interaction tends to keep nuclei spherical. Therefore, in the investigation of the equilibrium shape of a nucelus one should include pairing effects together with the HF correlations. This also becomes essential when one gets large deformation from the HF solutions of nuclei in the middle of the shell. This problem has been treated in Ref. 5, where some advantages of this method over the HF calculations are pointed out.

Let the Hamiltonian H of the nucleus under consideration be as follows:

$$H=\sum\epsilon_j\, a_{jm}{}^{\dagger}a_{jm}+\tfrac{1}{2}\sum\langle j_1m_1j_2m_2|v|\,j_1'm_1'j_2'm_2'\rangle \times a_{j_2m_2}{}^{\dagger}a_{j_1m_1}{}^{\dagger}a_{j_1'm_1'}a_{j_2'm_2'}\,, \quad (1)$$

where v is the residual internucleon interaction and the ϵ_j are the single-particle energies; $a_{jm}{}^{\dagger}(a_{jm})$ are the creation (annihilation) operators for a state jm; and $\sum$ denotes the sum over all the indices occurring in Eq. (1).

After making a canonical transformation from a_{jm} to quasifermions α_{im}, one obtains the HFB equations by equating to zero the off-diagonal bilinear quasiparticle part of the transformed Hamiltonian.[5] The required transformation is

$$a_{jm}=\sum_i\,(U_{ji}{}^m\alpha_{im}+V_{ji}{}^m\alpha_{i\,-m}{}^{\dagger})$$

and

$$a_{j\,-m}{}^{\dagger}=\sum_i\,(U_{ji}{}^m\alpha_{i\,-m}{}^{\dagger}+V_{ji}{}^m\alpha_{im})\,. \quad (2)$$

[1] J. P. Elliot and B. M. Flowers, Proc. Roy. Soc. (London) **A229**, 536 (1955).
[2] M. Redlich, Phys. Rev. **110**, 468 (1958); D. Kurath and L. Picman, Nucl. Phys. **10**, 313 (1959).
[3] W. H. Bassichis and G. Ripka, Phys. Letters **15**, 320 (1965); W. H. Bassichis, B. Giraud, and G. Ripka, Phys. Rev. Letters **13**, 52 (1965); I. Kelson, Nucl. Phys. **89**, 387 (1966); M. Bouten, P. Van. Leuven, H. Depudyt, and L. Schotsmans, Nucl. Phys. **A100**, 90 (1967); M. R. Gunye and C. S. Warke, Phys. Rev. **156**, 1087 (1967).
[4] S. B. Khadkikar and M. R. Gunye, Nucl. Phys. **A110**, 472 (1968).
[5] S. T. Belyaev, Kgl. Danske Videnskab Selskab, Mat. Fys. Medd. **31**, No. 11 (1959); M. Baranger, Phys. Rev. **122**, 992 (1961), and other references quoted here.

In Eq. (2), U and V satisfy the proper orthonormality relations. The problem then reduces to finding the solution of the following HFB equations:

$$E_{im}U_{ji}{}^{m}=(\epsilon_j-\lambda)U_{ji}{}^{m}+\sum_p[\Gamma_{jp}{}^{m}U_{pi}{}^{m}+\Delta_{jp}{}^{m}V_{pi}{}^{m}],$$

$$E_{im}V_{ji}{}^{m}=-(\epsilon_j-\lambda)V_{ji}{}^{m}-\sum_p[\Gamma_{jp}{}^{m*}V_{pi}{}^{m}-\Delta_{jp}{}^{m*}U_{pi}{}^{m}], \quad (3)$$

where

$$\Gamma_{jp}{}^{m}=\Gamma_{pj}{}^{m*}=\sum_{m_2,q,r}\langle jmqm_2|\tilde{v}|pmrm_2\rangle\rho_{rq}{}^{m_2},$$

$$\Delta_{jp}{}^{m}=-\Delta_{pj}{}^{m}=\sum_{m_2,q,r}\tfrac{1}{2}\langle jmp\bar{m}|\tilde{v}|q\bar{m}_2rm_2\rangle\sigma_{rq}{}^{m_2},$$

$$\rho_{qr}{}^{m}=\sum_i V_{qi}{}^{m}V_{ri}{}^{m},$$

and

$$\sigma_{rq}{}^{m}=\sum_i V_{qi}{}^{m}U_{ri}{}^{m},$$

and the chemical potential λ is to be determined from the number-conservation conditions.

Using Bloch and Messiah's theorem,[6] we write

$$U_{ji}{}^{m}=u_i{}^{m}C_{ji}{}^{m} \text{ and } V_{ji}{}^{m}=v_i{}^{m}C_{ji}{}^{m}. \quad (4)$$

Then from Eqs. (3) and (4) one obtains

$$E_{im}=(\eta_{im}+\delta_{im})^{1/2}, \quad (5)$$

$$\delta_{im}=\sum_{i'm'}\langle imi-m|\tilde{v}|i'm'i'-m'\rangle\delta_{i'm'}/E_{i'm'}, \quad (6)$$

and

$$\Gamma_{jp}{}^{m}=\tfrac{1}{2}\sum_{i_1m_1}\langle jmi_1m_1|\tilde{v}|pmi_1m_1\rangle[1-\eta_{i_1m_1}/E_{i_1m_1}]. \quad (7)$$

In the above equations, $\tilde{v}$ deontes the antisymmetric matrix element of v. The eigenvalues η_{im} and the eigenvectors $C_{ji}{}^{m}$ are to be determined from

$$\eta_{im}C_{ji}{}^{m}=(\epsilon_j-\lambda)C_{ji}{}^{m}+\sum_p\Gamma_{jp}{}^{m}C_{pi}{}^{m}. \quad (8)$$

The chemical potential is related to the particle number N by

$$\tfrac{1}{2}\sum_{im}[1-\eta_{im}/E_{im}]=N. \quad (9)$$

In writing the matrix elements in Eqs. (6) and (7), we used the new single-particle states

$$\varphi_{im}=\sum_j C_{ji}{}^{m}\varphi_{jm}. \quad (10)$$

Thus the nonlinear coupled Eqs. (6)–(9) can be solved by the usual iterative method as is done in the case of HF solution. The HFB wave function and its energy are given by

$$|\text{HFB}\rangle=\prod_{im}[u_i{}^{m}+v_i{}^{m}b_{im}{}^{\dagger}b_{i-m}{}^{\dagger}]|0\rangle$$

and

$$E_{\text{HFB}}=\tfrac{1}{4}\sum_{im}(T_{im}-\lambda-E_{im})[1-\eta_{im}/E_{im}], \quad (11)$$

where

$$T_{im}=\langle im|T|im\rangle=\sum_j\epsilon_j(C_{ji}{}^{m})^2,$$

$$b_{im}{}^{\dagger}=\sum_j C_{ji}{}^{m}a_{jm}{}^{\dagger},$$

and

$$b_{i-m}{}^{\dagger}=\sum_j C_{ji}{}^{m}(-1)^{j+m}a_{j-m}{}^{\dagger},$$

and the vacuum state $|0\rangle$ is defined by $a_{jm}|0\rangle=0$. Since φ_{im} forms a complete set, we can rewrite Eq. (1) in the form

$$H=\sum\langle im|T|pm\rangle b_{im}{}^{\dagger}b_{pm}+\tfrac{1}{2}\sum\langle impm_1|v|i'm'p'm_1'\rangle\times b_{pm_1}{}^{\dagger}b_{im}{}^{\dagger}b_{i'm'}b_{p'm_1'}. \quad (12)$$

3. HFB PROJECTED SPECTRA AND TRANSITION AMPLITUDES

The operators which project good angular-momentum states IM and the total number of particles, n, from the $|\text{HFB}\rangle$ wave functions are

$$P_M{}^{I}=\frac{2I+1}{8\pi^2}\int\mathfrak{D}_{MK}{}^{I}(\alpha,\beta,\gamma)\mathfrak{R}(\alpha,\beta,\gamma)d\Omega$$

and

$$P_n=\frac{1}{2\pi}\int e^{-in\theta}e^{i\theta N}d\theta, \quad (13)$$

where N is the number operator and $\mathfrak{R}(\alpha,\beta,\gamma)$, $\mathfrak{D}_{MK}{}^{I}(\alpha,\beta,\gamma)$ are the rotation operator and the matrices as defined in Ref. 7. Knowing the transformation properties of $a_{jm}{}^{\dagger}$ and a_{jm} under rotation $\mathfrak{R}(\alpha,\beta,\gamma)$, it is not difficult to verify that

$$\mathfrak{R}(\alpha,\beta,\gamma)b_{im}{}^{\dagger}\mathfrak{R}^{-1}(\alpha,\beta,\gamma)=\sum_{i_1m_1}\langle i_1m_1|\mathfrak{R}|im\rangle b_{i_1m_1}{}^{\dagger}$$

and

$$\mathfrak{R}(\alpha,\beta,\gamma)b_{im}\mathfrak{R}^{-1}(\alpha,\beta,\gamma)=\sum_{i_1m_1}\langle i_1m_1|\mathfrak{R}|im\rangle^{*}b_{i_1m_1}, \quad (14)$$

where

$$\langle i_1m_1|\mathfrak{R}(\alpha,\beta,\gamma)|im\rangle=\sum C_{ji}{}^{m}\mathfrak{D}_{m_1m}{}^{j}(\alpha,\beta,\gamma)C_{ji_1}{}^{m_1}. \quad (15)$$

Using the facts that H commutes with $\mathbf{J}$ and N, and that the projection operator P satisfies $P^2=P$, we obtain

[6] C. Bloch and A. Messiah, Nucl. Phys. 39, 95 (1962).

[7] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957).

for the projected energies E_n^I as follows:

$$E_n^I=\langle \mathrm{HFB}|HP_M{}^IP_n|\mathrm{HFB}\rangle/\langle \mathrm{HFB}|P_M{}^IP_n|\mathrm{HFB}\rangle,$$
$$\equiv h_n{}^I/p_n{}^I,$$
$$=\frac{\int_0^\pi \sin\beta d\beta\, d_{KK}{}^I(\beta)h_n(\beta)}{\int_0^\pi \sin\beta d\beta\, d_{KK}{}^I(\beta)p_n(\beta)}. \quad (16)$$

From Eqs. (12)–(14) it is not difficult to show for a system of N neutrons and P protons that

$$p_n(\beta)=p_n(N,\beta)p_n(P,\beta),$$
$$h_n(\beta)=h_n(N,\beta)p_n(P,\beta)+h_n(P,\beta)p_n(N,\beta)+h_n(N,P,\beta), \quad (17)$$

with

$$p_n(N,\beta)=\sum_{\Gamma_N\Gamma_{N'}} F(\Gamma_N,\Gamma_N'),$$
$$F(\Gamma_N,\Gamma_N')=\mathrm{Det}^{2N}[\langle\Gamma|\mathfrak{R}|\Gamma'\rangle]\times V(\Gamma_N)V(\Gamma_N')U(\bar\Gamma_N)U(\bar\Gamma_N').$$

$\Gamma_N\equiv\{i_1m_1, i_1-m_1,\cdots, i_Nm_N, i_N-m_N\}$ is a configuration within our functional space $\varphi_{i_k}m_k$, and $\bar\Gamma_N$ is a total space excluding Γ_N. $\mathrm{Det}^{2N}[\langle\Gamma|\mathfrak{R}|\Gamma'\rangle]$ denotes a $2N\times 2N$ determinant obtained from the matrix $\langle\Gamma_N|\mathfrak{R}|\Gamma_N'\rangle$ defined in Eq. (15) for the configurations Γ_N and Γ_N'. Further, in Eq. (17)

$$V(\Gamma_N)=\prod_{\alpha\in\Gamma_N}|v_\alpha|^{1/2}$$

and

$$U(\bar\Gamma_N)=\prod_{\alpha\in\bar\Gamma_N}|u_\alpha|^{1/2},$$

$$h_n(N,\beta)=\sum_{\Gamma_N\Gamma_{N'}}F(\Gamma_N,\Gamma_N')\times\Big\{\sum_{\substack{\alpha\in\Gamma_N,\\ \gamma\in\Gamma_N'}}(\alpha|T|\mathfrak{R}\gamma)(\alpha|\mathfrak{R}^{-1}|\gamma)+\tfrac{1}{2}\sum_{\substack{\alpha\beta\in\Gamma_N,\\ \alpha'\beta'\in\Gamma_N'}}(\alpha\beta|\bar v|\mathfrak{R}\alpha'\beta')\times(\alpha|\mathfrak{R}^{-1}|\alpha')(\beta|\mathfrak{R}^{-1}|\beta')\Big\} \quad (18)$$

and

$$h_n(N,P,\beta)=\sum_{\Gamma_N\Gamma_{N'}\Gamma_P\Gamma_{P'}}F(\Gamma_N,\Gamma_N')F(\Gamma_P,\Gamma_P')\times\sum_{\substack{\alpha\in\Gamma_N,\ \beta\in\Gamma_N',\\ \gamma\in\Gamma_P,\ \delta\in\Gamma_P'}}(\alpha\gamma|\bar v|\mathfrak{R}\beta\delta)\times(\alpha|\mathfrak{R}^{-1}|\beta)(\gamma|\mathfrak{R}^{-1}|\delta). \quad (19)$$

We have denoted the inverse of the matrix $(\Gamma_N|\mathfrak{R}|\Gamma_N')$ by $(\Gamma_N|\mathfrak{R}^{-1}|\Gamma_N')$. $\mathfrak{R}$ and $\bar v$ in the matrix element are the usual rotation operator and the interaction with the antisymmetrization sign, respectively. Corresponding expressions for protons in Eq. (17) can similarly be written. Equations (17)–(19) have a very similar structure to that found in the HF projected spectrum.[3] For the sake of completeness we quote the expectation value of the one-body tensor operator with respect to the projected state.

$$\frac{\langle P_M{}^IP_n(\mathrm{HFB})|T_\mu{}^\lambda|P_M{}^IP_n(\mathrm{HFB})\rangle}{p_n{}^I}=(I\lambda M\mu|IM)\sum_\nu(I\lambda K-\nu\nu|IK)\times\int_0^\pi\frac{d_{K-\nu\nu}{}^I(\beta)T_{\nu,n}{}^\lambda(\beta)\sin\beta d\beta}{p_n{}^I}, \quad (20)$$

where

$$T_{\nu,n}{}^\lambda(\beta)=T_{\nu,n}{}^\lambda(N,\beta)p_n(P,\beta)+T_{\nu,n}{}^\lambda(P,\beta)p_n(N,\beta),$$

with

$$T_{\nu,n}{}^\lambda(N,\beta)=\sum_{\Gamma_N\Gamma_{N'}}F(\Gamma_N,\Gamma_N')\times\sum_{\substack{\alpha\in\Gamma_N,\\ \beta\in\Gamma_N'}}(\alpha|T_\nu{}^\lambda|\mathfrak{R}\beta)(\alpha|\mathfrak{R}^{-1}|\beta).$$

A corresponding expression for protons can similarly be written. In the future, we plan to carry out the calculations based on Secs. 2 and 3. However, here we approximately estimate the effects of the I and N projection on the HFB calculations based on the results of the next two sections.

4. APPROXIMATION TO N PROJECTION

It is known that the Bardeen-Cooper-Schrieffer (BCS) theory[8] does not conserve the number of particles. However, it has been proved by Bogoliubov[9] that the effect of this nonconservation on the physical properties of a system consisting of large N is very small. It is pointed out that the discrepancy arising from this is quite large for a system of finite number of particles as is the case in nuclear physics.[10] In such a situation it is desirable to improve the BCS calculations and several attempts have been made in the literature to achieve this.[11] The main starting point of these methods is the method of N projection. We shall adopt a different point of view and use the method of moments to approximate the N projection. Essentially, it amounts to the expansion in powers of the number fluctuation $\sigma=\langle(N-\bar N)^2\rangle^{1/2}$ where $\bar N=\langle N\rangle$ (the expectation values in this section refer to the BCS state). Let us assume that the energy

[8] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).
[9] N. N. Bogoliubov, Zh. Eksperim. i Teor. Fiz. 34, 58 (1958) [English transl.: Soviet Phys.—JETP 7, 41 (1958)].
[10] A. K. Kerman, R. D. Lawson, and M. H. MacFarlane, Phys. Rev. 124, 162 (1961).
[11] B. F. Bayman, Nucl. Phys. 15, 33 (1960); F. Iwamoto and H. Onishi, Progr. Theoret. Phys. (Kyoto) 37, 682 (1967), and other references quoted here.

TABLE. I. Ground-state energies in MeV obtained by various methods.

Nucleus	Ni^{58}	Ni^{60}	Ni^{62}	Ni^{64}	Ni^{66}
E_{KLM}[a](exact)	−1.49	−2.11	−1.75	−0.15	1.70
E_{KS}[b](BCS)	−1.13	−1.51	−1.09	0.22	2.48
Present	−1.61	−2.22	−1.90	−0.55	1.91
PR Model[c] $N=6$ G	0.50	0.80	1.00	1.25	
E_{exact}[c]	9.998	8.213	6.828	4.953	
E_{PBCS}[d]	10.095	8.249	6.850	4.966	
E_{BCS}[d]	10.562	9.215	8.082	6.510	
Present	10.040	8.019	6.480	4.419	

[a] Reference 10.
[b] L. S. Kisslinger and R. A. Soresonn, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 32, No. 9 (1960).
[c] A. Pawlikowski and W. Rybarska, Zh. Eksperim. i Teor. Fiz. 43, 543 (1963) [English transl.: Soviet Phys.—JETP 16, 388 (1963)].
[d] Reference 11.

E_n of the projected state of n particles is $E_n=E_1n+E_2n^2$. It is easy to see that $E_{\bar{N}}$, the projected energy of $\bar{N}$ particles, is

$$E_{\bar{N}}=\langle N\rangle+\langle(N-\bar{N})^2\rangle^2\langle H\rangle/[\langle N^3\rangle\langle N\rangle-\langle N^2\rangle\langle N^2\rangle]. \quad (21)$$

First one observes that the numerator of the second term in Eq. (21) is a negative definite quantity (since $\langle H\rangle$ is negative). Let us consider now

$$\langle N^2\rangle^2=[\sum_n n^2p_n]^2,$$

where $p_n=\langle P_n\rangle\geqslant 0$. We can also write

$$[\sum_n n^2p_n]^2=\{\sum_n [n^{3/2}p_n{}^{1/2}][n^{1/2}p_n{}^{1/2}]\}^2,$$

which from Schwartz's inequality is

$$[\sum_n n^2p_n]^2\leqslant[\sum n^3p_n][\sum np_n]=\langle N^3\rangle\langle N\rangle.$$

Thus we have proved that

$$\langle N^3\rangle\langle N\rangle-\langle N^2\rangle^2\geqslant 0. \quad (22)$$

From Eqs. (21) and (22) it is clear that $E_{\bar{N}}<\langle H\rangle$. In order to check how good the approximation $E_{\bar{N}}$ is to the projected BCS energy, we carry out the numerical calculations for various known BCS solutions. The results of these calculations are exhibited in Table I. It is interesting to notice that in the degenerate case our approximation gives the exact energy if we include the contribution of the pairing force to the single-particle energies while solving the BCS equations for a chemical potential λ and the gap parameter Δ. From Table I it is also clear that Eq. (21) is a good approximation to the N projection. Since the above effect is neglected in some of the calculations while evaluating Δ and λ, we get energies lower than the exact energies.

5. APPROXIMATION TO I PROJECTION

In the strong-coupling limit of the unified model of Bohr and Mottelson,[12] the nuclei possess a permanent deformation, and their energy spectrum is similar to that of an axially symmetric rotator. However, it is also known that nuclear rotational spectra deviate from such a simple picture because of the coupling between various collective motions[12,13]: Rotational spectra then have a modified form

$$E(I)=AI(I+1)-BI^2(I+1)^2+CI^3(I+1)^3+\cdots. \quad (23)$$

In this section we show that in the case of maximally deformed HF solution the projection method also gives a similar spectrum. Let the projection operator that projects out the total spin I_1 from the HF state be

$$P_{I_1}=\prod_{i\neq 1}\frac{J^2-I_i(I_i+1)}{I_1(I_1+1)-I_i(I_i+1)}.$$

The difference between the projected energies $E_{I_1}-E_{I_2}$ can then be shown to be

$$E_{I_1}-E_{I_2}=\frac{A[I_2(I_2+1)-I_1(I_1+1)]}{[1-BI_1(I_1+1)][1-BI_2(I_2+1)]}, \quad (24)$$

where

$$A=[\langle\alpha\rangle\langle HJ^2\alpha\rangle-\langle J^2\alpha\rangle\langle H\alpha\rangle]/\langle J^2\alpha\rangle^2,$$
$$B=\langle\alpha\rangle/\langle J^2\alpha\rangle,$$

and

$$\alpha=\prod_{i\neq 1,2}[J^2-I_i(I_i+1)]. \quad (25)$$

At this stage a comment on what we mean by a maximally deformed HF state is necessary. From Eq. (25) one observes that α depends implicitly on I_1 and I_2. When this dependence on I_1 and I_2 is very weak, we call that state a maximally deformed HF state. This essentially would be valid if each I_i were equally probable in the HF state. With this assumption Eq. (24) can be rewritten as

$$E_{I_1}-E_{I_2}=A\{I_2(I_2+1)-I_1(I_1+1)+B[I_2^2(I_2+1)^2-I_1^2(I_1+1)^2]+B^2[I_2^3(I_2+1)^3-I_1^3(I_1+1)^3]\cdots\}. \quad (26)$$

This expression is similar to that in Eq. (23). Further we want to show that one can also obtain a spectrum like that assumed by Sood[13] for even-even nuclei. From Eq. (24), the excited-state energy of an even-even nucleus measured from its ground state ($I=0$) is

$$\epsilon_{I_2}=\frac{[A]_{I_1=0}I_2(I_2+1)}{[1-(B)_{I_1=0}I_2(I_2+1)]}. \quad (27)$$

In general $[B]_{I_1=0}$ will depend on I_2, but we still assume that this dependence would be insensitive, so that $[B]_{I_1=0}$ is almost a constant independent of I_2. In order

[12] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **26**, No. 14 (1952); A. Bohr and B. R. Mottelson, *ibid.* **27**, No. 16 (1953).
[13] P. C. Sood, Phys. Rev. **161**, 1063 (1967).

to keep the I_2 dependence in $[A]_{I_1=0}$, we write

$$\left[\frac{\langle HJ^2\alpha\rangle}{\langle H\alpha\rangle}\right]_{I_1=0}=I_2(I_2+1)+\left[\frac{\langle H[J^2-I_2(I_2+1)]\alpha\rangle}{\langle H\alpha\rangle}\right]_{I_1=0}, \quad (28)$$

where the numerator of the second term is independent of I_i. From Eqs. (27) and (28) we rewrite the excited-state energy as

$$\epsilon_I=\frac{a[1-yI(I+1)]I(I+1)}{[1-xI(I+1)]}, \quad (29)$$

where

$$a=\frac{-\{\langle H\alpha\rangle_{I_1=0}-x\langle H[J^2I_2(I_2+1)]\alpha\rangle_{I_1=0}\}}{\langle J^2\alpha\rangle_{I_1=0}}$$

and

$$y=-\langle H\alpha\rangle_{I_1=0}/a\langle J^2\alpha\rangle_{I_1=0}; \quad x=[B]_{I_1=0}$$

in Eq. (29) is of the same form as that used in the empirical analysis[13] of the rotational spectra of even-even nuclei. Since the success of this form over the ordinary one in Eq. (23) is extensively discussed with many applications in Ref. 13, we do not carry out any applications of it.

The other approximation to the spectrum of a deformed nucleus is the method of moments due to Peierls and Yoccoz.[14] In this approach the projected energy is assumed to have a form

$$\epsilon_I=A+BI(I+1). \quad (30)$$

The parameters A and B are then determined by taking moments of the angular-momentum operator J with respect to the HF state $|K\rangle$ and H as a weighting factor. The expressions for A and B thus obtained are[15]

$$A=E_{\rm HF}-BK(K+1)-B\langle K|J_-J_+|K\rangle \quad (31)$$

and

$$\langle K|HJ_-J_+|K\rangle=A\langle K|J_-J_+|K\rangle+B\langle K|J_-J^2J_+|K\rangle. \quad (32)$$

Our aim here is to prove that this approach fails when the lowest-state energy $E_{I=K}$ turns out to be higher than $E_{\rm HF}$. In this case the moment of inertia parameter B turns out to be negative. From Eqs. (30) and (31) we have

$$\epsilon_{I=K}=E_{\rm HF}-B\langle J_+K|J_+K\rangle. \quad (33)$$

In Eq. (33), $\langle J_+K|J_+K\rangle$ is a positive definite quantity, from which we obtain the required result that $B<0$ if $\epsilon_{I=K}>E_{\rm HF}$ and $B>0$ if $\epsilon_{I=K}<E_{\rm HF}$.

[14] R. E. Peierls and J. Yoccoz, Proc. Phys. Soc. (London) **A70**, 381 (1957).

[15] S. Das Gupta and M. Harvey, Nucl. Phys. **A44**, 602 (1967).

TABLE II. ^{28}Si moment of inertia ϑ obtained by various methods.

Deformation δ	$\hbar^2/2\vartheta$ keV Expt.	Present	Sup. fluid[a]	Rig. bod.[a]	Irrot.[a]	Crank.[a]
0.25	322	292.5	290	94	1492	404
−0.50		247.6	242			

[a] See Ref. 17.

6. APPLICATION TO ^{28}Si

In order to estimate the effects of the N and I projection on the HFB calculations, we now make use of the results of the Secs. 4 and 5. Let us assume that the projected energy ϵ_{In} has the form

$$\epsilon_{In}=A_n+B_nI(I+1).$$

Further, we put the conditions that $\epsilon_{I=K,n}$ should be of the same form as that in Eq. (21), which is valid only for N-projected ground-state energy. This suggests that A_n and B_n should be such that

$$\epsilon_{In}=[A+BI(I+1)]\left\{1+\frac{\langle(N-\bar{N})^2\rangle^2}{[\langle N^3\rangle\langle N\rangle-\langle N^2\rangle^2]}\right\}, \quad (34)$$

where A and B in Eq. (34) are given by Peierls-Yoccoz expressions in Eqs. (32). This essentially amounts to assuming that the scale factor that occurs in the N projection for the ground state $(I=K, n)$ energy is approximately the same for all the low-lying projected state (I,n) energies. From Eq. (34), we observe that the multiplying factor in the curly brackets is positive definite and is greater than unity. Thus our approximation will spread out the spectra obtained from HF projection only. In other words, the moment of inertia will be decreased as compared to that obtained from the Cranking model[16] without N and I projection. In carrying out the numerical calculations we used the Nilsson BCS solution of ^{28}Si from Ref. 17. The results are exhibited in Table II. The small change observed in our values compared to the superfluidity nuclear model[5] is not surprising, since the BCS solution of ^{28}Si is carried out for all the nucleons, and naturally the number fluctuation in such a case would be small. As seen from Eq. (34) our correction factor is proportional to it. We will get larger corrections when the number fluctuation in the HFB state is large. The trend of the correction to the moment of inertia for ^{28}Si is in the right direction.

7. CONCLUSION

The HFB equations are put in a form convenient for numerical solution. The explicit expressions for the

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N- and I-projected spectra and the expectation value of a one-body tensor operator are derived from the projected HFB wave function. A suitable approximation is found for the N-projected BCS energy. From its application it is observed that this approximation is better than those known in the literature. With suitable approximations for the I projection it is shown that the I-projected spectrum has the form

$$\epsilon_I = \frac{a[1-yI(I+1)]I(I+1)}{[1-xI(I+1)]},$$

where a, x, and y are the constants independent of I. With the above two approximations for N and I projection, the ^{28}Si moment of inertia is calculated from the projected Nilson-BCS wave function. Though in this case the calculated value is not very much different from that of the Cranking superfluidity nuclear model, the trend is in the right direction.

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Lifetimes of the First Four Excited States in Si^{29}†

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The attenuated-Doppler-shift method was used to measure the lifetimes of the first four excited states in Si^{29}. The states were populated by the $Si^{28}(d,p)$ reaction and the direction of a recoil was defined by the direction of the outgoing proton with which it was in coincidence. The shifts were measured both for the stopping of recoils in a heavy material (gold) and for stopping in a light material (silicon or carbon). The lifetimes of these excited states were found to be $3.1_{-0.8}^{+1.1}\times10^{-13}$ sec for the first, $3.5_{-0.8}^{+0.9}\times10^{-13}$ sec for the second, $(2.0\pm0.7)\times10^{-14}$ sec for the third, and $(2.3\pm1.1)\times10^{-14}$ sec for the fourth. It is concluded that no simple picture adequately describes the low-lying states in Si^{29}, but that a mixture of two rotational bands shows promise.

INTRODUCTION

A VARIETY of nuclear-reaction studies[1] have established the energies, spins, and parities of the low-lying states in Si^{29}. Information about the shell-model configurations of these states has been obtained from analysis of $Si^{28}(d,p)$, $Si^{30}(d,t)$, and $Si^{30}(He^3,\alpha)$ data,[2-4] from which spectroscopic factors have been extracted by use of the techniques of direct-interaction spectroscopy. γ-ray branching ratios as well as some $E2/M1$ mixing ratios have been determined from particle-γ correlation studies[5]; these studies also removed any remaining uncertainties about the spin assignments for the first five excited states (Fig. 1). In the present work we report measurements of the lifetimes of the first four excited states.

The lifetimes were measured by the attenuated-Doppler-shift method. The development of the lithium-drifted germanium detector, whose resolution is 1–2 orders of magnitude better than that of sodium iodide, has rendered it feasible to use this method to measure

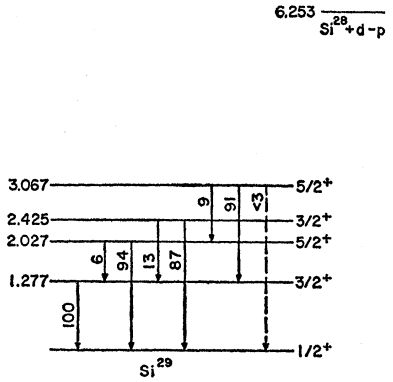

FIG. 1. Energy-level diagram of Si^{29}. The γ-ray branching ratios are from Ref. 5.

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