Second Series *June, 1921* *Vol. XVII, No. 6*

THE

PHYSICAL REVIEW.

THE DISTRIBUTED CAPACITY OF INDUCTANCE COILS.

By G. Breit.

Synopsis.

Effective Capacity of a Coil Defined.—Experiments show that if a coil is connected in series with a condenser of capacity C, the frequency $(\omega/2\pi)$ with which this combination is in resonance is given by $L(C + C_0) = 1/\omega^2$, where L and C_0 are constants.

The constant C_0 is called the *effective capacity* of the coil, sometimes simply "the capacity" of the coil. A general formula, equation (6), is derived for its calculation.

Single-layer Solenoid. The formula is applied to the short single-layer solenoid, used when grounded in an elliptical shield and when insulated from the shield, and to the short single-layer solenoid used when grounded and insulated in free space. An explanation is given of the remarkable *constancy of* C_0 as found by experiment in the case of short coils.

Experimental Verification.—An experimental verification is given by direct measurement of capacity and inductance. The current distribution in a coil has also been studied experimentally. The results have verified the theory.

Introduction.

THIS paper is intended to call the attention of physicists and mathematicians to some interesting aspects of the subject of distributed capacity of coils. The subject is of practical importance because inductance coils are used extensively in radio communication and because the distributed capacity, taken in connection with the value of inductance, determines the range of wave-lengths within which the inductance coil can be used to advantage. Furthermore, there is considerable mathematical interest connected with the calculation of the effective capacity caused by the capacity distributed along the wire of the coil.

The subject has been largely neglected by mathematical physicists. Lentz[1] and Drude[2] seem to be the only ones who have made a study of it.

[1] W. Lentz, Ann. d. Phys., *34*, p. 923–974, 1912. W. Lentz, Ann. d. Phys., *43*, p. 749–797, 1914.

[2] P. Drude, Ann. d. Phys., *9*, p. 293, 1902.

However, there are errors in Drude's mathematics, while Lentz's results are not adapted to numerical calculation and involve too many assumptions to be used generally. For example his treatment applies only to wires having circular section and further it combines the assumption of negligible curvature of a single turn with that of an infinitely long coil. This combination of assumptions may be justified for treatment of skin effect. Its applicability to capacity calculation is questionable.

In this paper an outline of the general method of calculating the effective capacity will be given and then illustrated by working out some special cases.

The Effective Capacity of Inductance Coils.

Definition of Effective Capacity.—It is an experimental fact that the resonance frequency $\omega/2\pi$ of an inductance coil across whose terminals a condenser of capacity C is connected is given by

$$L(C + C_0) = \frac{1}{\omega^2} \tag{1}$$

where L, C_0 are constants for the coil in question. The international electrical units are used in this paper, where not otherwise specified. Formula (1) means that if an electromotive force $E_0 \cos \omega t$ acts in some part of the circuit and the capacity of the condenser is varied then the current is a maximum when (1) is satisfied. The constant C_0 is called the *Effective Capacity* of the coil. It is due to the capacities which are distributed along the wire of the coil.

The assumptions made in the following calculation are as follows:

1. The field of the condenser does not affect the field of the coil appreciably.

2. The resistance of the coil is negligible compared with its reactance, so that the wire may be treated as a perfect conductor.

3. The dimensions of the coil are so small in comparison with the wavelength used that the retarded values of the scalar and vector potential may be equated to their contemporaneous values.

4. The value of the E.M.F. induced in the whole coil by the current in a small section of the coil at a given point of the coil is proportional to the length of the section.

5. In the special cases considered the diameter of the coil is very much larger than the depth of the coil's winding or the axial length of the coil.

These assumptions will now be discussed in detail. Consider the case when the field of the condenser does not affect appreciably the electric intensity at the wires of the coil. It is essential to understand that this may not always be the case as is seen from the following considerations.

The wire of the coil is connected to the condenser whose capacity is C. If there is a difference of potential between the two condenser plates the charges on the plates give rise to an electric field whose line integral along any curve between the two plates is equal to the difference of potential between the plates. If the curve is taken along the wire it follows that the electric intensity along all of the wire cannot be negligible since its line integral is equal to a large difference of potential.

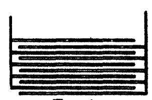

Fig. 1. Interleaving effect of condenser plates.

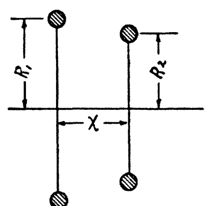

Fig. 2. Diametral section of two turns of wire.

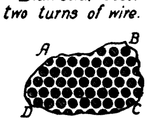

Fig. 3. Cross section of multi-layer coil by a plane through the axis.

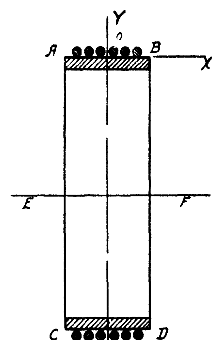

Fig. 4. Diametral section of short single-layer solenoid.

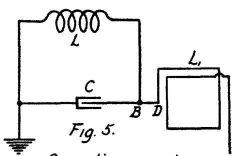

Fig. 5. Connections used in measuring the capacity of a coil to ground.

Nevertheless it is legitimate to neglect this electric intensity under conditions which are satisfied in most practical cases. Consider the field which arises from the charges on the plates of an ordinary condenser whose plates are interleaved as shown on Fig. 1. The electric field is most intense between the plates. It also has a value at all points outside. But this value becomes small at a large distance. The principal contribution to the line integral is, therefore, confined to a small length along the leads connecting the coil to the condenser, and even if it should happen that there is an intense field due to the condenser at some part of the coil, it usually happens that this field is either compensated in some other part of the coil or else is along the wires of the coil.

However, there are cases when this assumption cannot be justified. No general treatment of these cases can be given unless the dimensions and position of the condenser are known. For this reason the present treatment considers the ideal case when the condenser is connected to the coil by long leads and in practical applications one must constantly bear in mind that this assumption has been made.

Let, then, the electric intensity caused by the condenser be negligible and let the electromotive force be applied between the condenser and the coil in one of the long leads. Then the electric intensity at any point on the surface of the coil is due solely to two causes:

(1) the currents in the coil,

(2) the charges on the wire of the coil.

If further the material out of which the coil is made should be a perfect conductor then the electric intensity must be normal to the surface of the coil at every point.

The question arises immediately as to whether the current is the same through every cross section of the wire. It is apparent at once that it is not. In fact the current in the coil gives rise to an electric intensity which has a component tangential to the wire at least at some points because if this were not the case the coil would not have any self-inductance, because the self-inductance is the line integral along the wire of the electric field due to the current per unit rate of change of current. But the total electric intensity must be normal to the surface of the wire and therefore has no tangential component. Consequently there must be charges on some parts of the coil which give rise to an electric intensity whose tangential component is equal and opposite to the tangential component due to the current. In the case considered the tangential component of the current varies periodically. Therefore, the charges also vary periodically. If x is an arbitrary parameter along the wire such, *e.g.*, as a length measured from an arbitrary fixed point along the wire, if i is the current, and if Qdx is the net charge in an infinitesimal segment dx, *i.e.*, if Q is the charge per unit length, then it is easily shown that

$$\frac{\partial i}{\partial x} = -\frac{\partial Q}{\partial t} \tag{2}$$

by the conservation of charge. As shown

$$\frac{\partial Q}{\partial t} \neq 0.$$

Therefore, $\partial i/\partial x \neq 0$. Thus the current is different at different points on the wire.

Suppose that the frequency dealt with is so low that this non-uniformity of current is small. Then as an approximation the component of the electric intensity may be computed as if a uniform current i were flowing through the coil. Let (1) and (2) be arbitrarily selected points on the surface of the wire of the coil. Let the self-inductance of the portion of the coil included between (1) and (2) be L_{12} and denote the mutual inductance of this section to the rest of the coil by M_{12}. Then the line integral of the electric intensity due to i and taken along the wire from (1) to (2), i being reckoned positive when the current is flowing from (1) to (2), is

$$e_{12} = (L_{12} + M_{12}) \frac{\partial i}{\partial t}.$$

If the electrostatic potential is V_1 at (1) and V_2 at (2), the line integral of the electric intensity due to the charges and taken from (1) to (2) is

$$V_1 - V_2.$$

The total line integral is then

$$e_{12} + V_1 - V_2.$$

But the total electric intensity is normal to the wire and, therefore, the total line integral is zero. Hence

$$V_2 - V_1 = e_{12}, \tag{3}$$

i.e., the line integral of the field due to the current between any two points on the surface of the wire is equal to the difference of potential between the two points.

Attention must be called here to the fact that e_{12} is independent of the path on the surface of the wire along which the integration is effected because no matter how intricate this path may be the result of the integration must always give $V_2 - V_1$. This means that even though the electric field due to the currents cannot be derived from a single-valued potential at all points in space, the component tangential to the surface of a perfect conductor may be derived from a single-valued potential on the surface of that conductor.

The same may also be seen from the fact that the magnetic intensity has a constant normal component to the surface of a perfect conductor so that the flux of magnetic induction through any closed curve on the surface of the conductor is constant and the line integral of the electric intensity around any closed curve is zero.

Thus from a knowledge of i, $V_2 - V_1$ may be derived. If it is also known that some part of the coil is grounded then the point grounded is at zero potential and therefore V is known at all points of the coil. According to the assumption made as to the condenser the only charges which give rise to V are those on the wire of the coil and the objects in the neighborhood of the coil. Suppose that it is possible to solve the electrostatic problem of finding such a distribution of charges on the coil and its surroundings as to give rise to the specified value of V. This problem can be always solved and, moreover, the solution of it is unique because it is equivalent to finding V in a region of space within which V satisfies Laplace's equation and on the boundaries of which V is known. (Theoretically it is known that such a distribution and that only one such distribution exists.) Then Q in (2) is known and consequently i may be obtained from the formula

$$i = i_1 - \int_{x_1}^{x} \frac{\partial Q}{\partial t} dx,$$

where i_1 is the value of i at $x = x_1$.

Further, since the relation between the charge density $Q(x)$ and the "difference of potential" between the coil terminals is necessarily linear one can conveniently write

$$Q(x) = + \alpha(x) \frac{di_1}{dt}, \tag{4}$$

where $\alpha(x)$ is a function of x whose form depends on the shape and dimensions of the coil and the nature of the medium around the coil. The expression di/dt is written only as an approximation, which, however, is legitimate if $i - i_1$ is small because in the computation carried out only the terms depending on the first power of the frequency will be taken into account and $Q(x)$ itself depends on the first power and powers higher than the first. Since $i - i_1$ is zero for direct current, it is small for a sufficiently low frequency. Thus

$$i(x) = i_1 - \left(\frac{d^2 i}{dt^2}\right) \int_{x_1}^{x} \alpha(x) dx, \tag{5}$$

where x_1 is the value of x at one of the terminals of the coil.

The coil may be divided mentally into a number of small sections. Thus if the coil has 100 turns each of the turns may be looked at as one such small section. Within one of the sections the current is uniform. If the parameter assigned to one of them is x, the current in the section is $i(x)$. Then $i(x)$ is given by the preceding equation (5). If the section is small enough it is possible to find such a function $M(x)$ that $M(x)\Delta x$ is the sum of the self-inductance and the mutual inductance to the rest of the soil for a section included between x and $x + \Delta x$. The electromotive force induced between the coil terminals due to the current in the section is then

$$M(x) \frac{\partial i(x)}{\partial t} \Delta x.$$

The electromotive force due to all sections is

$$e_{12} = - \Sigma M(x) \frac{\partial i(x)}{\partial t} \Delta x,$$

where the summation is extended over all the sections. If Δx becomes infinitesimal this summation is

$$e_{12} = - \int_{x_1}^{x_2} M(x) \frac{\partial i(x)}{\partial t} dx$$

where x_1, x_2 are the values of x at the ends of the coil. Substituting the value of $i(x)$ given by (5) the electromotive force is:

$$e_{12} = - \frac{di_1}{dt} L + \frac{d^3 i_1}{dt^3} \int_{x_1}^{x_2} M(x) \left\{ \int_{x_1}^{x} \alpha(x) dx \right\} dx,$$

where

$$L = \int_{x_1}^{x_2} M(x) dx$$

and is therefore the self-inductance of the coil. If the oscillations in the current i are simple harmonic, *i.e.*, if they are represented by

$$i = I_0 \cos (\omega t - \theta),$$

then

$$\frac{d^3 i_1}{dt^3} = - \omega^2 \frac{di_1}{dt},$$

so that the electromotive force induced in the whole coil is

$$e_{12} = - L \frac{di_1}{dt} \left[1 + \omega^2 \int_{x_1}^{x_2} \frac{M(x)}{L} \left\{ \int_{x_1}^{x} \alpha(x) dx \right\} dx \right].$$

The current i_1 is the current which flows through the condenser connected to the coil to constitute an oscillatory circuit. Let C be the capacity of the condenser and $- Q_c$ the charge on the plate connected to the ungrounded point of the coil given by $x = x_1$. Then

$$\frac{dQ_c}{dt} = i_1$$

and

$$\frac{di_1}{dt} = - \omega^2 Q_c.$$

Therefore, in terms of Q_c the electromotive force is

$$e_{12} = L \omega^2 Q_c \left[1 + \omega^2 \int_{x_1}^{x_2} \frac{M(x)}{L} \left\{ \int_{x_1}^{x} \alpha(x) dx \right\} dx \right].$$

But the difference between the potential of the plate connected to $x = x_2$

and the plate connected to $x = x_1$ is Q_c/C. Therefore,

$$\frac{Q_c}{C} = Q_c L\omega^2\left[1 + \omega^2\int_{x_1}^{x_2}\frac{M(x)}{L}\left\{\int_{x_1}^{x}\alpha(x)dx\right\}dx\right].$$

If

$$\omega^2\int_{x_1}^{x_2}\frac{M(x)}{L}\left\{\int_{x_1}^{x}\alpha(x)dx\right\}dx$$

is small compared to 1, that is, if ω is sufficiently small, the above formula gives

$$\omega^2 = \frac{1}{LC}.$$

Using this value of ω^2 and substituting in the integral term of the above expression a second approximation is obtained in the form

$$\frac{1}{\omega^2} = L\left[C + \int_{x_1}^{x_2}\frac{M(x)}{L}\left\{\int_{x_1}^{x}\frac{\alpha(x)}{L}dx\right\}dx\right].$$

If this is to be identical with

$$\frac{1}{\omega^2} = L(C + C_0)$$

which is the experimental relation then

$$C_0 = \int_{x_1}^{x_2}\frac{M(x)}{L}\left\{\int_{x_1}^{x}\frac{\alpha(x)}{L}dx\right\}dx. \qquad (6)$$

Although approximations were made in the formula it is general and correct as long as C_0 is a constant because the formula is exact at low frequencies. For convenience of reference the meaning of the symbols in this formula is restated.

If the charge in an element dx is $Q(x)dx$, then

$$\alpha(x) = \frac{Q(x)}{\left(\frac{di}{dt}\right)},$$

$M(x)dx$ is the mutual inductance between the section dx and the whole coil. L is the self-inductance. x is an arbitrary parameter. x_1, x_2 are the terminal values of x, x_1 being the value at the ungrounded condenser terminal.

The main steps in the derivation of this formula are summarized below:

(*a*) Since the wire was assumed to be a perfect conductor the E.M.F. induced between any two points is equal to the difference of potential between the points on the surface.

(*b*) This difference of potential calls for charges on the surface of the wire.

(*c*) The charges on the wire cause a non-uniform distribution of current in the wire.

(*d*) When the frequency is low the E.M.F. induced between any two points may be calculated as if the current were uniform. Hence the potential at all points of the wire is known and, therefore, the charge in any section is known.

(*e*) Knowing the distribution of charge the distribution of current is derived.

(*f*) From a knowledge of the connection between the current at the ungrounded condenser terminal and the current at any other point the self-induced E.M.F. is expressed in terms of the current at the condenser terminal.

(*g*) This E.M.F. is equated to the potential difference between the condenser plates expressed in terms of the charge and capacity. By comparison with the formula

$$L(C + C_0) = \frac{1}{\omega^2}$$

C_0 is derived.

(*h*) The theory given so far applies strictly only to very low frequencies. However, it will be shown later that C_0 is independent of the frequency for many cases. This is true experimentally in all important cases. Therefore, in the cases when C_0 does not vary with the frequency the formula (6) is general.

Applications to Problems Solved by Two-Dimensional Methods.

In this paper only problems solved by two-dimensional methods will be considered. A large class of problems reduce themselves to two-dimensional problems. This is the case with any circular coil whose diameter is large compared to the maximum distance between two of its turns.

Let R_1 be the radius of one of the turns. The charge density per unit length of wire is sensibly constant within one turn, if the number of turns is large. Let this charge density be ρ_1. The potential due to the turn considered at any point in space is now obtained by direct integration. By symmetry it is the same at all points of any circle coaxial with the first. Let this circle be of radius R_2, and let x denote the distance between the planes of the two circles. (See Fig. 2.) Then the potential in volts is

$$V = \frac{c^2}{10^9 K} \int_0^{2\pi} \frac{\rho_1 R_1 d\varphi}{\sqrt{(R_1 \cos \varphi - R_2)^2 + R_1^2 \sin^2 \varphi + x^2}},$$

where $c = 2.9982 \times 10^{10}$ and is the ratio of the electromagnetic to the electrostatic unit of charge. K is the specific inductive capacity of the medium around the coil. Now

$$\int_0^{2\pi} \frac{d\varphi}{\sqrt{(R_1 \cos\varphi - R_2)^2 + R_1^2 \sin^2\varphi + x^2}}$$
$$= 4\int_0^{\pi/2} \frac{d\varphi}{\sqrt{x^2 + (R_1 + R_2)^2 - 4R_1R_2\cos^2\varphi}}$$
$$= \frac{4}{\sqrt{x^2 + (R_1 + R_2)^2}} F\left(\frac{2\sqrt{R_1R_2}}{\sqrt{x^2 + (R_1+R_2)^2}}\right),$$

where $F(k)$ denotes complete elliptic integral of the first kind with modulus k.

If x is small compared to both R_1 and R_2

$$\frac{2\sqrt{R_1R_2}}{\sqrt{(R_1 + R_2)^2 + x^2}}$$

is nearly 1. Consequently the elliptic integral

$$F\left(\frac{2\sqrt{R_1R_2}}{\sqrt{(R_1 + R_2)^2 + x^2}}\right)$$

is approximately[1]

[1] See Appendix

$$\log \frac{4}{\sqrt{1 - \frac{4R_1R_2}{(R_1 + R_2)^2 + x^2}}} = \log\left(\frac{8R}{d}\right),$$

where

$$d = \sqrt{(R_1 - R_2)^2 + x^2}$$

and where

$$\sqrt{(R_1 + R_2)^2 + x^2}$$

is replaced by $2R$, R now being used for R_1 as well as R_2. Thus

$$V = \frac{2 \times 10^{-9}c^2}{K} \rho_1 \log\left(\frac{8R}{d}\right) = \frac{17.978 \times 10^{11}}{K} \rho_1 \log\left(\frac{8R}{d}\right). \quad (7)$$

This is the approximate expression for the potential caused by a circular ring at a small distance from it. It is essential to note here that the potential caused by the same ring at infinity vanishes to the first order.

Let then $ABCD$ (see Fig. 3) represent a section of the coil by a plane through its axis, the small circles being the sections of the wires of the coil. Each one of the wires is charged, the charge being distributed with a certain surface density, say σ. Let ds be an element of length along one of the circles. Then $\sigma\, ds$ is the corresponding linear density of charge for a ring of width ds and on the surface of the wire. If r is the distance from any point in the cross section to ds, then the potential at that point is obtained as

$$V = \Sigma \frac{10^{-9}c^2}{K} \int 2\sigma \log\left(\frac{8R}{r}\right) ds = \Sigma \frac{8.989 \times 10^{11}}{K} \int 2\sigma \log\left(\frac{8R}{r}\right) ds,$$

where the integration extends over the whole circle and the summation covers all the circles. V can be broken up into two parts by writing

$$V = -\Sigma \frac{8.989 \times 10^{11}}{K} \int 2\sigma \log(r) ds + (\log 8R) \Sigma \frac{8.989 \times 10^{11}}{K} 2\sigma ds.$$

The second of these two vanishes if

$$\Sigma \int \sigma ds = 0,$$

i.e., if the total charge on the wires of the coil remains constant. Such is the cause if the coil is not grounded. In this case of the ungrounded coil the first term is the only term and consequently the potential is the same as that which would exist if the wires of the coil were straight because a straight wire may be considered as the limiting case of a circular wire of infinite radius. *Thus the case of the ungrounded short coil is always reducible to a two-dimensional problem.*

If the short coil is grounded the only modification is that introduced by the second term. This contributes the same potential at all points of the cross section. Thus only a slight modification in the first solution is introduced by grounding. It must be carefully remembered here that in the two-dimensional case the logarithmic potential is infinite both at the filament causing it and at infinity. This makes it impossible to apply the two-dimensional treatment in general. If, however, the total charge is zero, the potential vanishes at infinity in the two-dimensional case and as was shown above is approximately the same as in the case of circular turns. It is only in this case, therefore, that the two-dimensional treatment applies.

There is an additional simplification in the case considered. This is introduced by the fact that *the E.M.F. induced in any one turn is practically the same as that in any other.*

In fact the expression for the mutual inductance between two coaxial circles whose radius R is large compared to their distance apart d is

$$M = 4\pi R\left[\log\left(\frac{8R}{d}\right) - 2\right].$$

Draw a system of rectangular axes (OX, OY) in the plane perpendicular to the wires of the coil. To each circle coaxial with the coil there corresponds a point (x, y), the point of intersection of the circle with the plane.

Suppose the current distribution is given by the fact that a current

$$\frac{1}{\mu} f(x, y)dxdy$$

crosses an area $dx\, dy$ at (x, y), μ being here the permeability of the medium. Then the flux through a circle passing through the point (x_0, y_0) is

$$N = 4\pi R \int\int \left\{ \log\left(\frac{8R}{\epsilon^2 \sqrt{(x - x_0)^2 + (y - y_0)^2}} \right) \right\} f(x, y)dxdy,$$

where the double integral is taken over the entire area through which a current may pass, and ϵ is the natural base. Let a be the maximum possible value of

$$\sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Then

$$\log \frac{a}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

is always either zero or positive.
But

$$N = 4\pi R \int\int \left\{ \log\left(\frac{8R}{a\epsilon^2}\right) \right\} f(x, y)dxdy + 4\pi R \int\int \left\{ \log \frac{a}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right\} f(x, y)dxdy.$$

Therefore,

$$N > 4\pi R \left\{ \log\left(\frac{8R}{a\epsilon^2}\right) \right\} \int\int f(x, y)dxdy. \tag{8}$$

Further,

$$\frac{\partial N}{\partial x_0} = 4\pi R \int\int \frac{x_0 - x}{(x_0 - x)^2 + (y_0 - y)^2} dxdy,$$
$$\frac{\partial N}{\partial y_0} = 4\pi R \int\int \frac{y_0 - y}{(x_0 - x)^2 + (y_0 - y)^2} dxdy, \tag{9}$$

and by virtue of (8)

$$\frac{\partial(\log N)}{\partial x_0} < \frac{1}{\log\left(\frac{8R}{a\epsilon^2}\right)} \frac{\int\int \frac{(x_0 - x)dxdy}{(x_0 - x)^2 + (y_0 - y)^2}}{\int\int f(x, y)dxdy},$$

$$\frac{\partial(\log N)}{\partial y_0} < \frac{1}{\log\left(\frac{8R}{a\epsilon^2}\right)} \frac{\int\int \frac{(y_0 - y)dxdy}{(x_0 - x)^2 + (y_0 - y)^2}}{\int\int f(x, y)dxdy}.$$

Thus by making $\log (8R/a)$ large the maximum variation in $\log N$ can be made negligible. This fact is not by any means self-evident because the magnetic field at an infinitesimal distance from a current is infinite.

Therefore, in coils with sufficiently large diameter the magnetic flux through any one turn is the same as that through any other.

These results will now be applied to the case of a short closely wound solenoid.

(*a*) *Short Closely Wound Solenoid.*—Let $A\ B\ C\ D$ (Fig. 4) be a diametral section of a circular cylinder, the axis of the cylinder being $E\ F$. The coil is imagined to be wound on the surface of the cylinder, each turn being circular, the turns being wound close together and the number of turns being large. The distance $A\ B$ is taken to be negligible in comparison with the distance $A\ C$.

In the diametral plane points will be named by means of a cartesian system of axes OX, OY. The origin O of the system is placed at the middle of the line AB. OX is along $A\ B$, and OY is perpendicular to $A\ B$. The number of turns being large the current is uniform within a few turns. Thus the x coördinate of a turn may be chosen as the parameter previously denoted by x. The width of the coil is denoted by $2a$.

It was shown that the flux through one turn of the coil is the same as that through any other. *Therefore in* (6) *the quantity* $M(x)$ *is a constant.* Since

$$\int_{-a}^{+a} M(x)dx = L,$$

$$\frac{M(x)}{L} = \frac{1}{2a}.$$

Hence

$$C_0 = \frac{1}{2a}\int_{-a}^{+a}\left\{\int_{-a}^{x}\frac{\alpha(x)}{L}dx\right\}dx. \tag{10}$$

It now remains to compute the quantity $\alpha(x)$. This depends on the objects surrounding the coil. A simple condition is that in which all these objects are removed to an indefinite distance. However, practically this is not always realized. For this reason in the design of a standard inductance with a definite distributed capacity it seems advisable to surround the coil with a metallic shield. The mathematics in the special instance considered will be simplest if the shield is in the form of a solid of revolution obtained by constructing an ellipse whose foci are at A and B and whose plane is that of $A\ B\ C\ D$, and revolving the ellipse about the axis $E\ F$. The potential of the shield will be arbitrarily taken as zero. The middle of the coil (*i.e.*, the point O) is considered as connected to the shield by a short wire so that the potential of O is also zero.

Then by virtue of (3) the potential at x on $A\ B$ is

$$V = -\frac{di}{dt}L\frac{x}{2a}, \tag{11}$$

the positive direction of i being from negative to positive values of x.

In order to find $\alpha(x)$ it is necessary to find $Q(x)$. This is attained by finding the distribution of potential between $A\ B$ and the shield.

The problem is solved without difficulty by the use of elliptical coordinates defined by the transformation

$$x + jy = a \cosh (u + jv), \tag{12}$$

where

$$j = \sqrt{-1}$$

or its equivalent:

$$\begin{aligned} x &= a \cosh u \cos v, \\ y &= a \sinh u \sin v. \end{aligned} \tag{13}$$

It follows from (12) that the real or the imaginary part of any monogenic function of $u + jv$ is a solution of Laplace's equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Again if u is kept constant then in virtue of (13)

$$\frac{x^2}{a^2 \cosh^2 u} + \frac{y^2}{a^2 \sinh^2 u} = 1,$$

so that the point (x, y) lies on an ellipse whose foci are $(\pm a, o)$. Let then u_0 be the value of u which is assigned to the ellipse of the shield. Then

$$V = -\frac{L}{2}\frac{di_1}{dt}\frac{(\epsilon^{-u} - \epsilon^{u-2u_0}) \cos v}{1 - \epsilon^{-2u_0}}, \tag{14}$$

satisfied Laplace's equation because $(\epsilon^{-u} - \epsilon^{u-2u_0}) \cos v$ is the real part of $\epsilon^{-(u+jv)} - \epsilon^{-2u_0}\epsilon^{u+jv}$. Also if $u = u_0$, then $V = 0$, and if $u = 0$ $V = -L(di_1/dt)(x/2a)$ so that (11) is satisfied. Finally the above expression for V, as well as its first derivatives are finite and continuous in the space between $u = 0$ and $u = u_0$ except when the segment $A\ B$ is crossed. Consequently this is the potential which exists in the space under the conditions of the problem.

The segment $A\ B$ is given, of course, by $u = 0$. If u is slightly greater than 0 the segment expands into an ellipse. The half of the ellipse corresponding to positive values of y is given by the values of v between 0 and π. The negative values of y are similarly given by values of v between π and 2π. Thus the line AB is the limiting state of such an ellipse for which $u = \delta$, δ being infinitesimal.

Differentiating (12) one obtains:

$$\frac{d(x+jy)}{d(u+jv)} = a \sinh (u+jv) = a[\sinh u \cos v + j \cosh u \sin v];$$

$$\therefore \left| \frac{d(x+jv)}{d(u+jv)} \right| = a \sqrt{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v}. \tag{15}$$

The surface density of charge on the ellipse $u = \delta$ is

$$-\frac{10^{-11}K}{35.956\pi}\left(\frac{\partial V}{\partial n}\right)_{u=\delta},$$

where $\partial/\partial n$ is the directional derivative as to the normal drawn out from $u \mp \delta$. But by (15)

$$\frac{\partial V}{\partial n} = \frac{1}{a\sqrt{\sinh^2 u \cos^2 v + \cosh^2 u \sin v}}\frac{\partial V}{\partial u}.$$

Therefore the surface density

$$\sigma_1 = -\frac{10^{-11}K\left(\frac{\partial V}{\partial n}\right)_{u=\delta}}{35.956\pi a\sqrt{\sinh^2 \delta \cos^2 v + \cosh^2 \delta \sin^2 v}}. \tag{16}$$

Substituting $2\pi - v$ for v in (16) it is seen that the surface density at the reflection of the point given by v in $A\,B$ is also σ_1. The sum of the two surface densities is then

$$\sigma = 2\sigma_1 = -\frac{10^{-11}K}{17.978\pi a}\frac{\left(\frac{\partial V}{\partial n}\right)_{u=\delta}}{\sqrt{\sinh^2 \delta \cos^2 v + \cosh^2 \delta \sin^2 v}}.$$

In the limit when $\delta = 0$ it is found from (14) that

$$\sigma = -\frac{10^{-11}KLl \coth u_0 \cos v}{35.956\pi a\,|\sin v|}\frac{di_1}{dt}.$$

Let l be the length of one turn of the wire of the coil. Then the charge in the infinitesimal interval $(x, x+dx)$ is

$$-\frac{10^{-11}KLl}{35.956\pi a}\coth u_0 \frac{\cos v}{|\sin v|}\frac{di_1}{dt}dx.$$

Thus by the definition of $\alpha(x)$ given in (4)

$$\alpha(x) = -\frac{10^{-11}KLl}{35.956\pi a}\cosh u_0 \frac{\cos v}{|\sin v|}.$$

Substituting in (6) and remembering that A corresponds now to $x = -a$,

$$C_0 = \frac{10^{-11}K}{8.989}\int_{-a}^{+a}\frac{1}{2a}\left\{\int_{-a}^{x}\frac{-Ll}{(4\pi aL)}\coth u_0 \frac{\cos v}{|\sin v|}\,dx\right\}dx.$$

The integration is effected without difficulty by the substitution

$$x = a \cos v.$$

The result is

$$C_0 = \frac{10^{-11} Kl}{16 \times 8.989} \coth u_0 = 0.06952 \times 10^{-12} Kl \coth u_0. \qquad (17)$$

Since the major semi-axis of the ellipse $u = u_0$ is $a \cosh u_0$, and the minor is $a \sinh u_0$, the result could be also written as

$$C_0 = 0.06952 \times 10^{-12} Kl \frac{\alpha}{\beta} \text{ farads}$$

$$= 0.06952 Kl \frac{\alpha}{\beta} \mu\mu f,$$

where α = major semi-axis,
β = minor semi-axis.

In particular if the major and minor semi-axes are both made infinite the ratio α/β becomes unity. Thus the *effective capacity of a short single layer solenoid used with its middle grounded and undisturbed by surrounding objects is*

$$C_0 = 0.06952 Kl \mu\mu f = \frac{Kl}{16} \text{ cgs electrostatic units of capacity.} \qquad (19)$$

It is remarkable that according to the formula (18) the only modification introduced by the shield is given by the factor α/β which even for comparatively narrow shields becomes approximately 1.

(*b*) *Short, Closely Wound Solenoid Grounded at Terminal When in Shield.*—Consider now the case when the terminal B is connected to the shield. The method of solution is almost identical with that of the preceding case. The only difference is that the potential at any point, x, is

$$-\frac{L(x-a)}{2a}\frac{di_1}{dt}$$

rather than

$$-\frac{Lx}{2a}\frac{di_1}{dt}.$$

It is clear that if on the solution given by (14) a solution for Laplace's equation which is 0 at $u = u_0$ and $(L/2)(di/dt)$ at $u = 0$ be superposed, the result will satisfy all the conditions of the problem. The expression

$$\frac{L}{2}\frac{di}{dt}\left(\frac{u_0 - u}{u_0}\right)$$

actually vanishes when $u = u_0$ and becomes $(L/2)(di/dt)$ when $u = 0$.

Therefore,

$$V = \frac{L}{2}\left[\frac{u_0 - u}{u_0} - \frac{\epsilon^{-u} - \epsilon^{u-2u_0}}{1 - \epsilon^{-2u_0}}\cos v\right]\frac{di}{dt} \tag{20}$$

is the required solution for the potential.

From (20)

$$\left(\frac{\partial V}{\partial u}\right)_{u=0} = \frac{L}{2}\left[-\frac{1}{u_0} + \coth u_0 \cos v\right]\frac{di}{dt},$$

so that

$$\alpha(x) = -\frac{10^{-11}KLl}{35.956\pi a}\left[-\frac{1}{u_0} + \coth u_0 \cos v\right]\frac{1}{|\sin v|} \tag{20a}$$

and

$$C_0 = \frac{10^{-11}L}{17.978a}\int_{-a}^{+a}\left\{\int_{-a}^{+x}\frac{l}{4\pi a}\left[\frac{1}{u_0} - \coth u_0 \cos v\right]\frac{dx}{|\sin v|}\right\}dx$$

$$= 0.06952 \times 10^{-12}Kl\left[\coth u_0 + \frac{2}{u_0}\right]\text{farads}$$

or if it is preferrable to express the result in terms of α, and β, as before, then

$$C_0 = 0.06952 \times 10^{-12}Kl\left[\frac{\alpha}{\beta} + \frac{2}{\log\sqrt{\dfrac{\alpha+\beta}{\alpha-\beta}}}\right]$$

$$= 0.06952Kl\left[\frac{\alpha}{\beta} + \frac{2}{\log\sqrt{\dfrac{\alpha+\beta}{\alpha-\beta}}}\right]\mu\mu f. \tag{22}$$

In the derivation of this formula the reader probably noticed that the current at A was chosen as i_1 while the terminal B was grounded. The reason for this choice is that when the capacity of a condenser is measured one set of plates is connected to ground permanently and the other is connected to and disconnected from the measuring apparatus. Thus the charge measured (either directly or used indirectly in theoretical derivations of formulas) is the charge supplied to the *ungrounded plate*. This is, therefore, the charge Q_c used in the derivation of formula (6).

Care should be taken not to use the formula (22) in cases when $\alpha/\beta - 1$ becomes very small because then the dimensions of the ellipse becomes large and the equation

$$\frac{\partial^2 V}{\partial z^2} = 0$$

is no longer true.

Only if α is negligible in comparison with R is this legitimate.

For this reason it is essential to work out the distributed capacity of a short solenoid for the case when it is grounded at one terminal and is kept at a considerable distance from surrounding objects.

Short Single Layer Solenoid Grounded in Free Space.—Consider now the same coil removed from surrounding objects and grounded at some point by a fine wire. Let B be grounded. It is evident that

$$V_1 = \frac{-L}{2}\frac{di}{dt}\epsilon^{-u}\cos v \tag{23}$$

is an expression for the potential which gives correct differences of potential for the case considered. However, at B, $V_1 = 0$. In fact its value at B is

$$-\frac{L}{2}\frac{di}{dt}.$$

It is thus necessary to find a solution of Laplace's equation which becomes

$$+\frac{L}{2}\frac{di_1}{dt}$$

at the coil and which vanishes at infinity and then to add this solution to V_1. Or if it is hard to find this solution it is sufficient to know what distribution of charges on the coil will give rise to such a solution. The answer is given by the expression (7). This expression is interpreted most easily by writing

$$d' = \frac{d}{8R},$$

for then

$$V = -10^{-9}c^2\frac{2\rho_1}{K}\log d'.$$

This shows that if $8R$ were the unit of length V would be the same as is taken in the two-dimensional case when the wires are straight. Consequently, the distribution of charges is the same as in the two-dimensional case. Now in this case the charge density varies as $1/|\sin v|$, *i.e.*, as $1/\sqrt{a^2 - x^2}$, as is seen from (20*a*). Consequently such a distribution gives a uniform potential over $A\,B$. The exact value of this potential may be ascertained by letting the charge density

$$\sigma = \frac{\rho}{\sqrt{a^2 - x^2}}. \tag{24}$$

The potential of the coil due to this charge is obtained most easily at the center of the winding of the coil and as has been shown above is the same at the center as at any other point of the coil. It is

$$V_2 = \frac{8.989 \times 10^{11}}{K}\int_{x=-a}^{x=+a}\frac{2\rho\log\left(\frac{8R}{|x|}\right)}{\sqrt{a^2 - x^2}}dx = \frac{8.989 \times 10^{11}}{K}2\pi\rho\log\left(\frac{16R}{a}\right).$$

But this must be $+ (L/2)(di_1/dt)$. Therefore,

$$\rho = + \frac{10^{-11} K L \dfrac{di_1}{dt}}{35.956\pi \log\left(\dfrac{16R}{a}\right)}$$

and the charge density corresponding to this value of ρ is

$$\therefore \sigma_2 = + \frac{10^{-11} K \dfrac{di_1}{dt} L}{35.956\pi \log\left(\dfrac{16R}{a}\right)\sqrt{a^2 - x^2}}$$

$$= + \frac{10^{-11} K L \dfrac{di_1}{dt}}{35.956\pi a \left\{\log\left(\dfrac{16R}{a}\right)\right\}|\sin v|}. \quad (24a)$$

The charge density called for by V_1 (see (23)) is

$$\sigma_1 = - \frac{10^{-11} K L}{35.956\pi a}\frac{di_1}{dt}\frac{\cos v}{|\sin v|}.$$

Hence the total charge density

$$\sigma = \sigma_1 + \sigma_2 = \frac{10^{-11} K L}{35.956\pi a}\frac{di_1}{dt}\left[\frac{1}{|\sin v| \log\left(\dfrac{16R}{a}\right)} - \frac{\cos v}{|\sin v|}\right]. \quad (25)$$

If as before l is the length of one turn of wire (*i.e.*, $l = 2\pi R$) then the charge in the infinitesimal interval $(x, x + dx)$ is

$$\sigma l x d = \frac{10^{-11} K L l}{35.956\pi a}\frac{di_1}{dt}\left[\frac{1}{|\sin v| \log\left(\dfrac{16R}{a}\right)} - \frac{\cos v}{|\sin v|}\right]$$

and

$$\alpha(x) = \frac{10^{-11} L l K}{35.956\pi a}\left[\frac{1}{|\sin v| \log\left(\dfrac{16R}{a}\right)} - \frac{\cos v}{|\sin v|}\right]. \quad (26)$$

Therefore (see (6))

$$C_0 = \frac{10^{-11} K}{35.956}\int_{-a}^{+a}\frac{1}{2a}\left\{\int_{-a}^{x}\frac{Ll}{\pi a L}\left[\frac{1}{|\sin v| \log\left(\dfrac{16R}{a}\right)} - \frac{\cos v}{|\sin v|}\right] dx\right\} dx.$$

Performing the integrations:

$$C_0 = \frac{10^{-11}Kl}{8.989}\left[\frac{1}{16} + \frac{1}{8\log\left(\frac{16R}{a}\right)}\right] \text{ farads}$$
$$= 1.1124\left[\frac{1}{16} + \frac{1}{8\log\left(\frac{16R}{a}\right)}\right] Kl \text{ micromicrofarads.} \tag{27}$$

The formulas (22) and (27) are both seen to be made up of two parts. The first part is the effective capacity of the coil when ungrounded. The second may, therefore, be called the effective capacity due to grounding.

In the following paragraph, it will be shown that *for the case of a coil used in free space the effective capacity due to grounding is* $\frac{1}{4}$ *of the electrostatic capacity of the coil to ground.*

Equation (24*a*) gives the surface density of charge which maintains the surface of the coil at a potential $\frac{L}{2}\frac{di_1}{dt}$. Therefore, the electrostatic capacity of the coil is

$$\int_{v=0}^{v=\pi} \frac{\sigma_2 la \sin v dv}{\left(\frac{L}{2}\right)\frac{di_1}{dt}} = \frac{10^{-11}K}{17.978\log\left(\frac{16R}{a}\right)}.$$

But in (27) the distributed capacity due to grounding is

$$\frac{1}{4}\frac{10^{-11}c^{-2}Kl}{17.978\log\left(\frac{16R}{a}\right)}.$$

Therefore, the distributed capacity due to grounding is one fourth of the electrostatic capacity to ground.

This must be true as long as the distribution of charge introduced by grounding is the same as in the case of a coil grounded in free space, *i.e.*, removed from surrounding objects and grounded by a fine wire. Since the only difference between (14) and (20) is given by the term

$$\frac{L}{2}\frac{u_0 - u}{u_0}\frac{di_1}{dt}.$$

The distribution of charge introduced by the grounding of a coil in an elliptical shield is of the form $A/|\sin v|$, where A is a constant. Comparing this with (24) it is seen that *the distributed capacity introduced by grounding of the coil in an elliptical shield is one fourth of the electrostatic capacity of the coil to the shield.*

Further, even if the coil is not closely wound, the distributed capacity due to grounding must still be approximately equal to one fourth of the electrostatic capacity to ground as long as the winding is sufficiently close to make the average distribution of charge the same as that of a coil with close winding.

Neglecting minor corrections it is easy to measure the electrostatic capacity of a coil to ground in given surroundings. For this purpose a circuit L, C, Fig. 5, is tuned to the frequency of a generating set A and the reading of the shielded condenser C is noted. The shield of C is grounded and a wire BD extends permanently to the coil L' whose electrostatic capacity to ground is to be measured. By a slight motion of the end D, contact is made with L' and C is retuned. The difference between the capacity of C at the two settings gives the capacity of L' to ground. L' is of course not connected to ground directly during the measurement.

This has been verified experimentally. Thus for a coil for which the distributed capacity when ungrounded is 30 micromicrofarads the capacity when grounded came out 42 micromicrofarads while the measured electrostatic capacity to ground is 47 micromicrofarads. And it is seen that

$$30\ \mu\mu f + \frac{47\ \mu\mu f}{4} = 42\ \mu\mu f \text{ (approximately).}$$

The formulas derived are generally in satisfactory agreement with experiment if the capacity of leads is taken into account and the effect of condenser shields is made negligible. It is also of interest to note that formula (27) gives values for C_0 which are of the order of magnitude of the radius. This agrees with data accumulated at the Bureau of Standards.

Explanation of Constancy of C_0.

So far we have not explained why the number C_0 in (1) comes out a constant. In this section it will be shown that in the limit when the width of a coil becomes negligible compared to the diameter C_0 becomes a constant.

It was seen that the magnetic flux through all turns of the coil becomes the same under these conditions. Let then (L/N^2) be a constant such that the magnetic flux through a turn due to a current i in another turn is

$$\frac{L}{N^2}i,$$

N being the total number of turns. Then the flux through any one turn when i flows in all turns is

$$N\frac{Li}{N^2} = \frac{Li}{N}$$

and, therefore, the total flux through the coil is

$$N\frac{Li}{N} = Li$$

when the current is uniform. Thus L is the inductance of the coil.

Since the flux is the same through all turns, even if the current is not uniform the E.M.F. induced in a turn is

$$\frac{-L\left(\frac{d\bar{i}}{dt}\right)}{N},$$

where $\bar{i}$ is the average value of i, *i.e.*,

$$\bar{i} = \frac{i_1 + i_2 + \cdots + i_N}{N}, \tag{28}$$

$i_1, i_2 \cdots i_n$ being the currents in the 1st, 2d, 3d $\cdots$ Nth turns respectively. Thus the charge on the 1st, 2d, $\cdots$ Nth turns is

$$\frac{-f_1L}{N}\frac{d\bar{i}}{dt}, \qquad \frac{-f_2L}{N}\frac{d\bar{i}}{dt}\cdots\frac{-f_NL}{N}\frac{di}{dt},$$

where $f_1, f_2 \cdots f_N$ are independent of the frequency.

Let i_0 be the current entering the first turn. Then

$$i_2 - i_0 = \frac{f_1L}{N}\frac{d^2\bar{i}}{dt^2},$$

$$i_3 - i_1 = \frac{f_2L}{N}\frac{d^2\bar{i}}{dt^2},$$

etc.

Consequently the excess of the current through any turn over i_0 is proportional to $d^2\bar{i}/dt^2$ and the current can be written as

$$i_n = -\varphi_n\frac{d^2i}{dt^2} + i_0 = i_0 + \omega^2\varphi_n\bar{i}.$$

Here φ_n is independent of the frequency. According to (28)

$$\bar{i} = i_0 + \omega^2\bar{i}\bar{\varphi},$$

where

$$\bar{\varphi} = \frac{\varphi_1 + \varphi_2 + \cdots + \varphi_N}{N};$$

$$\therefore \bar{i} = \frac{i_0}{1 - \omega^2\bar{\varphi}}. \tag{28a}$$

The E.M.F. induced in the coil is then

$$\frac{-\frac{di_0}{dt}L}{(1 - \omega^2\bar{\varphi})}.$$

If Q is the charge on the condenser

$$\frac{dQ}{dt} = - i_0.$$

Hence

$$\frac{1}{LC} = \frac{\omega^2}{1 - \omega^2\bar{\varphi}}$$

or

$$\omega^2 = \frac{1}{L\left(C + \frac{\bar{\varphi}}{L}\right)}.$$

One can now write

$$C_0 = \frac{\bar{\varphi}}{L},$$

C_0 being independent of ω. Then

$$\omega^2 = \frac{1}{L(C + C_0)},$$

where C_0 is a constant, which proves that the relation (1) is exact as long as the flux is the same through all the turns.

It may be worth mentioning here that if a coil is represented as a line with distributed constants C_0 is not independent of ω.

It must be mentioned also that in any of the applications mentioned the effective capacity is unchanged if the cross section is turned through an angle with reference to the axis of the coil. Thus the formula for a short single layer solenoid applies also to a pancake coil of small depth.

Current Distribution.

It is possible to study the current distribution experimentally in coils by inserting non-inductive resistors at different points when the coil is tuned to resonance. It can be shown that if R_1, R_2 are the values obtained for the resistance by the resistance variation method[1] at two points (1) and (2) of the coil then

$$\frac{i_1}{i_2} = \sqrt{\frac{R_2}{R_1}},$$

where i_1, i_2 are the currents at (1) and (2).

By this method the curves of Figs. 6, 7, 8 were obtained. The curves of Figs. 7 and 8 show the dissymmetry introduced into the circuit by a galvanometer or by grounding. The coil experimented on in the case of the curves on Fig. 6 was a 4-foot coil wound with 14 turns of number 19 double cotton covered wire with 1 cm. spacing on a wooden frame. The frame was so designed that as little as possible of the wood was in the field of the coil. The dissymmetry was avoided by using a very small galvanometer (Weston thermogalvanometer) and making the readings

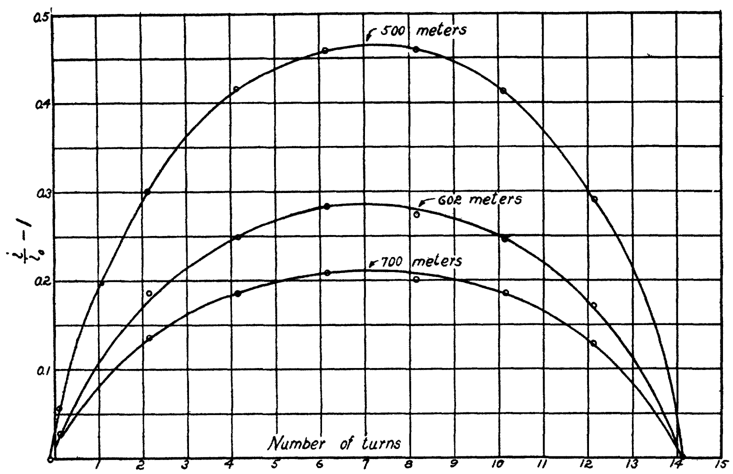

Fig. 6.
Current distribution in a coil determined experimentally at various wave-lengths.

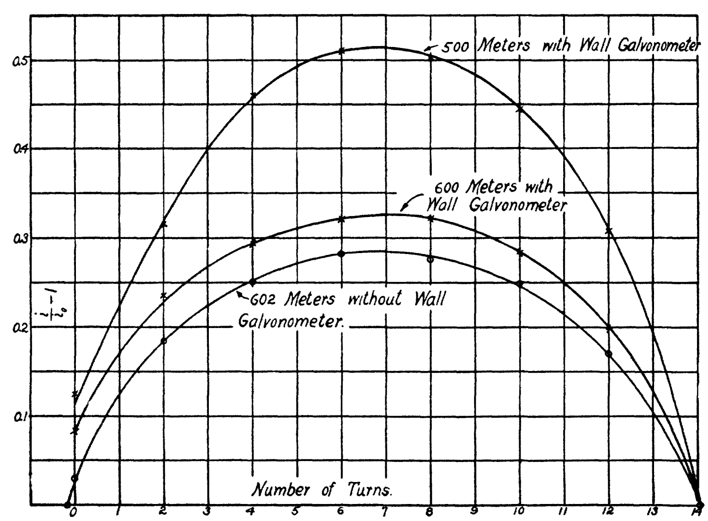

Fig. 7.
Current distribution in a coil determined experimentally at various wave-lengths with slight dissymmetry caused by wall galvanometer.

from a large distance with a telescope. In the case of Fig. 8, i_0 was used for the current at the shielded condenser terminal.

[1] See Bureau of Standards Circular No. 74, p. 180.

It is interesting to see how this measurement (obtained during the course of some work at the Johns Hopkins University) verifies the theory. In the first place, since the width of the coil is small compared to the

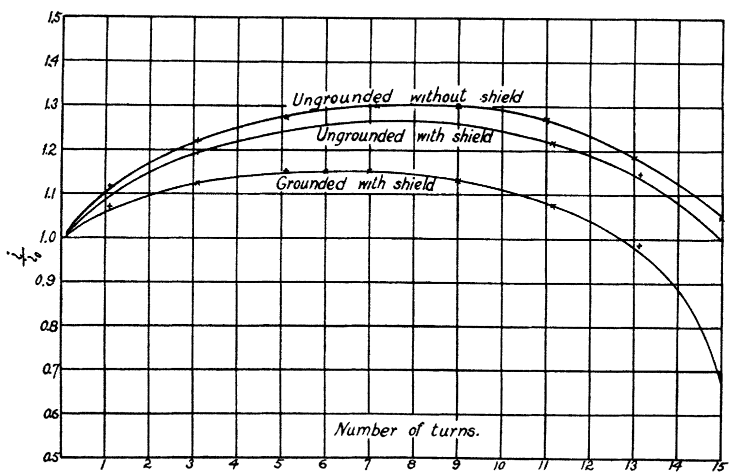

Fig. 8.

Effect of grounding and of condenser shield on current distribution.

diameter $(1/i_0) - 1$ must be nearly the same for all frequencies. Taking the experimental values for $(1/i_0) - 1$ at 700, 602, and 500 meters the following table is obtained

$\left(\frac{i}{i_0} - 1\right)_{700}$	$\left(\frac{i}{i_0} - 1\right)_{602}$	$\left(\frac{i}{i_0} - 1\right)_{500}$	$\frac{\left(\frac{i}{i_0} - 1\right)_{602}}{\left(\frac{i}{i_0} - 1\right)_{700}}$	$\frac{\left(\frac{i}{i_0} - 1\right)_{500}}{\left(\frac{i}{i_0} - 1\right)_{700}}$
0.135	0.186	0.297	1.38	2.20
0.186	0.247	0.414	1.33	2.22
0.208	0.284	0.459	1.36	2.20
0.207	0.273	0.461	1.36	2.29
0.184	0.247	0.413	1.34	2.24
0.127	0.174	0.291	1.37	2.29
	Mean:		1.36	2.24

Measurements by the resistance-variation method are difficult to make with an accuracy higher than 1 per cent. This would give an accuracy of 0.5 per cent. in i/i_0 and worse than 10 per cent in $(i/i_0) - 1$. The largest deviation in the table is 2.3 per cent. in the case of the comparison between 602 and 700 meters; and 2.5 per cent. in the comparison between 500 and 700 meters. The agreement must thus be regarded as satisfactory.

Further a knowledge of the natural period of the coil enables one to calculate the ratios tabulated. In fact from (28*a*)

$$i - \dot{i}_0 = \frac{\dot{i}_0 \bar{\varphi} \omega^2}{1 - \bar{\varphi}\omega^2} = \frac{\dot{i}_0}{\dfrac{1}{\bar{\varphi}\omega^2} - 1}. \tag{29}$$

But

$$C_0 = \frac{\bar{\varphi}}{L}, \qquad i.e., \qquad \bar{\varphi} = LC_0.$$

Hence

$$\bar{i} - \dot{i}_0 = \frac{\dot{i}_0}{\dfrac{\omega_0^2}{\omega^2} - 1}, \tag{30}$$

where

$$\omega_0^2 = \frac{1}{LC_0}$$

and is the value of ω which would correspond to the natural frequency of a pure inductance L connected in series with a capacity C_0. In terms of wave-length λ

$$\bar{i} - \dot{i}_0 = \frac{\dot{i}_0}{\dfrac{\lambda^2}{\lambda_0^2} - 1}, \tag{31}$$

where

$$\lambda_0 = 2\pi \sqrt{LC_0}.$$

For the coil in question the value of l in (18) is 470 cm. Hence using the formula (19)

$$C_0 = 32.6\ \mu\mu f \text{ when } K = 1.$$

The measured value of C_0 with leads was 38 $\mu\mu f$ and the measured value of L_0 was 547 μh. The capacity of the leads was calculated as 3.4 $\mu\mu f$. Thus the measured value without leads is 34.6 $\mu\mu f$. The lead correction being uncertain 33.6 $\mu\mu f$ can be taken as the value of C_0. The corresponding value of λ_0 is 255 meters. Then for

	700 meters	602 meters	500 meters
$\dfrac{1}{\dfrac{\lambda^2}{\lambda_0^2} - 1} =$	0.153	0.219	0.350

Here

$$\frac{0.350}{0.153} = 2.29,$$

$$\frac{0.219}{0.153} = 1.43.$$

According to the theory (Eq. 31) these must be the same as the measured quantities; viz.,

2.24 and 1.36.

It is seen that a difference exists between the measured and theoretical values. However, the difference is not large. It becomes less as the absolute value of the ratio increases and in the overall comparison between 500 and 700 meters it is 2.5 per cent. Although this is a serious discrepancy if we are concerned with current distribution, its effect on the constancy of the effective capacity of the coil is very small because 2.5 per cent. in 38 $\mu\mu f$ is barely 1 $\mu\mu f$ and is just within the limits of ordinary measurement.

Summary.

1. The effective capacity of an inductance coil is defined as the constant C_0 in the equation

$$L(C + C_0) = \frac{1}{\omega^2},$$

which is known to be true experimentally.

2. A general formula is derived for the effective capacity. This is formula (6) of the paper.

$$C_0 = \int_{x_1}^{x_2} \frac{M(x)}{L} \left\{ \int_{x_1}^{x} \frac{\alpha(x)}{L} dx \right\} dx$$

Here x is an arbitrary parameter along the wire

$$\alpha(x) = \frac{Q(x)}{\left(\dfrac{di_1}{dt}\right)},$$

where $Q(x)dx$ is the charge between x and $x + dx$.
i_1 is the current at the terminal,
L is the self-inductance,
$M(x)dx$ is the mutual inductance of the element dx to the rest of the coil.

3. The formula is illustrated in special cases. These are: the short single-layer solenoid, used when grounded in elliptical shield, insulated in elliptical shield, insulated and grounded in free space.

The formulas for the cases discussed are:

(I.) Short single-layer solenoid one turn of which has a perimeter l or pancake of small depth used in elliptical shield of major semi-axis α and minor semi-axis β and insulated from shield

$$C_0 = 0.06952Kl\frac{\alpha}{\beta}\mu\mu f = \frac{Kl}{16}\frac{\alpha}{\beta}\text{C.G.S. electrostatic units.}$$

(II.) Same coil connected to shield at terminal

$$C_0 = 0.06952Kl\left[\frac{\alpha}{\beta} + \frac{2}{\log\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}}\right]\mu\mu f$$

$$= \frac{Kl}{16}\left[\frac{\alpha}{\beta} + \frac{2}{\log\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}}\right] \text{ C.G.S. electrostatic units.}$$

(III.) Same coil used outside shield and insulated

$$C_0 = 0.06952Kl\mu\mu f = \frac{Kl}{16}\text{ C.G.S. electrostatic units.}$$

(IV.) Same coil used outside shield and grounded

$$C_0 = 1.1124\left[\frac{1}{16} + \frac{1}{8\log\left(\frac{16R}{a}\right)}\right]Kl\mu\mu f$$

$$= \left[\frac{1}{16} + \frac{1}{8\log\frac{16R}{a}}\right] \text{ C.G.S. electrostatic units.}$$

R being the radius. All the formulas with the exception of the last apply not only to coils having circular turns but also to other shapes such as square, rectangular, etc.

4. An explanation is given of the remarkable constancy of C_0 as found by experiment in the case of short coils.

5. An experimental verification of the theory is given by direct measurement of capacity and by investigating the current distribution in a coil at various frequencies.

The author is very grateful to Dr. F. W. Grover, Dr. C. Snow, Dr. J. H. Dellinger, Mr. L. E. Whittemore, Dr. J. S. Ames, Dr. H. L. Curtis, and Mr. C. N. Hickman for reading the manuscript.

Appendix.

The approximation used here is

$$F(k) = \log\frac{4}{\sqrt{1-k^2}}.$$

In order to see the truth of this expression use can be made of Maxwell's formula for the mutual inductance of two coaxial circles whose radii are both approximately a and whose shortest distance apart is b. This is in electromagnetic units in air.

$$M = 4\pi a\left[\log\left(\frac{8a}{b}\right) - 2\right].$$

The above expression is only approximately true. The expression will now be equated to Maxwell's exact formula in elliptic integrals in its limiting form and hence the value of $F(k)$ for small $k - 1$ will be derived. This exact formula for two concentric circles of radii a_1, a_2 is in the same units.

$$M = 4\pi\left[\frac{a_1^2 + a_2^2}{a_1 + a_2} F\left(\frac{2\sqrt{a_1a_2}}{a_1+a_2}\right) - (a_1 + a_2)E\left(\frac{2\sqrt{a_1a_2}}{a_1+a_2}\right)\right].$$

If a_1 is nearly a_2 it becomes approximately

$$M = 4\pi a\left[F\left(\frac{2\sqrt{a_1a_2}}{a_1+a_2}\right) - 2E\left(\frac{2\sqrt{a_1a_2}}{a_1+a_2}\right)\right].$$

Also, since

$$\frac{2\sqrt{a_1a_2}}{a_1+a_2}$$

is nearly 1,

$$E\left(\frac{2\sqrt{a_1a_2}}{a_1+a_2}\right)$$

is nearly 1. Identifying the two expressions for M the approximation for $F(K)$ follows immediately.

A different proof is found in Whittaker and Watson, Modern Analysis, pp. 514–515, Sections 22–737.

Radio Laboratory,
Bureau of Standards,
Washington, D. C.