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THE THERMIONIC CURRENT BETWEEN PARALLEL PLANE ELECTRODES; VELOCITIES OF EMISSION DISTRIBUTED ACCORDING TO MAXWELL'S LAW.

BY THORNTON C. FRY.

SYNOPSIS.

Thermonic Current between Parallel Plane Electrodes.—The electrical equations applying to this problem are developed without neglecting the distribution of initial velocities, which in the first place is allowed to be entirely general. *Maxwell's distribution of velocities* is then considered in detail and a complete solution obtained.

Curves from which to compute the space current when Maxwell's distribution applies are presented, together with an illustrative example of their use. Curves are also included showing the *deviation of the current-voltage relation from the 3/2-power law;* the variation of the *minimum potential between the electrodes* with plate voltage, and also the variation with plate voltage of *the distance between the cathode and the point at which this minimum potential occurs.*

1. INTRODUCTION.

A SIMPLE method of computing the thermionic current between parallel plane electrodes under circumstances such that the 3/2-power law published by Childs[1] and later by Langmuir[2] does not apply seems highly desirable. An attempt made several years ago to solve this problem led to a method of solution which was at once direct and general. It is the aim of this paper to present this method of solution as concisely as possible; and to apply it to a particular problem with a view to deriving quantitative results of a typical sort.

The solution requires the evaluation of a difficult integral which appears in equation (4). This evaluation was carried out, with the result shown in Figs. 2 and 4, by the use of the Integraph of Abdank-Abakanowicz. This integraph, which is not so well known in this country as its relatively low cost and very great accuracy seem to justify, is so constructed that when a pointer is passed along a curve $y = f(x)$ a ruling pen attached to the machine automatically draws the curve representing

[1] PHYS. REV., 32, p. 492, 1911. [2] PHYS. REV., 2, p. 350, 1913.

the indefinite integral, $\int f(x)dx$. Its accuracy when properly operated is such that successive integrations of the same function will seldom vary from one another by more than the width of the line drawn by the ruling pen.

2. Mathematical Formulation.

It is obvious that in any case of thermionic emission in which voltage saturation has not been reached there must be a region of negative potential gradient adjacent to the cathode. If this were not true every emitted electron would be drawn to the anode, and consequently no increase of current would result from an increase in the anode voltage. It therefore follows that the voltage curve must have the characteristic form shown in Fig. 1. A convenient notation denotes the potential at an arbitrary distance x from the cathode by V, the minimum potential by V' and the distance of this minimum potential from the cathode by x'. The region between O and x' may be called the region α, the remainder of the distance to the anode being denoted by β. If attention is fixed upon a particular electron its emission velocity may be denoted by v_0, its velocity at x' by v' and its velocity at an arbitrary point x by v. The number of electrons emitted per second per unit area with this particular velocity v_0 may be called $n(v_0)$ and the total emission N. Then

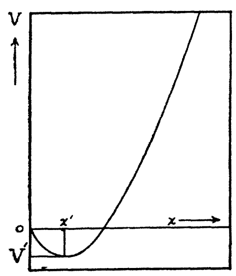

Fig. 1.
Characteristic Form of Voltage Curve.

$$N = \int_0^\infty n(v_0)dv_0.$$

In a current of electrons all of which travel with the speed v, the space charge is given by $e[n(v_0)/v]$, $n(v_0)$ being the number of them emitted per unit time. Hence where velocities differ the space charge is given by the equation

$$\rho = e\int \frac{n(v_0)}{v}dv_0,$$

the integration being performed over all the velocities of emission which are of such a magnitude that the electrons pass the point in question.

If this point lies in the region β all electrons pass it which had emission velocities greater than $\sqrt{2V'e/m}$, that is, high enough to pass the region of adverse gradient. Hence, denoting this critical emission velocity by v_0',

$$\rho = e\int_{v_0'}^\infty \frac{n(v_0)}{v}dv_0. \tag{1}$$

The space charge at a point x in α is not quite so simply expressed, since it is made up of those electrons which pass on to the anode and also of those which, although possessing enough energy when emitted to pass x, are brought to a stop before reaching x' and turn back. There are two equally dense streams of these latter electrons passing in opposite directions, each of which contributes to the space charge. If V is the potential of the point whose distance from the cathode is x, those electrons which are emitted with lower velocities than $\sqrt{2Ve/m}$ do not reach x, and those which are emitted with velocities higher than $v_0' = \sqrt{2V'e/m}$ do not return after passing. Hence,

$$\rho_a = 2e\int_{\sqrt{\frac{2Ve}{m}}}^{v_0'} \frac{n(v_0)}{v}\,dv_0 + e\int_{v_0'}^{\infty}\frac{n(v_0)}{v}\,dv_0. \tag{2}$$

These formulæ assume that all the electrons are shot out normally to the cathode. They are equally true, however, whatever the direction of emission may be, provided v_0 is understood to signify the *normal component* of the emission velocity and $n(v_0)$ the number of electrons shot off per second with this *normal component*. When later in the paper the Maxwell distribution of velocities is introduced the normal component only, and not the complete velocity, will be dealt with.

The current to the cathode is given by

$$\frac{i}{e} = \int_{v_0'}^{\infty} n(v_0)\,dv_0. \tag{3}$$

The only other equations necessary to determine the solution of the problem are the equation of energy,

$$v^2 = v_0^2 - \frac{2Ve}{m}, \tag{4}$$

and Poisson's equation,

$$\frac{\partial^2 V}{\partial x^2} = -4\pi\frac{\rho}{k}, \tag{5}$$

k being the dielectric constant.

Equations (1) and (5) of section 2 result in the differential equation:

$$\frac{d^2V}{dx^2} = -\frac{4\pi e}{k}\int_{v_0'}^{\infty}\frac{n(v_0)}{v}\,dv_0,$$

which formulates the potential distribution in the region β. A first integral of this equation is found in the customary manner by multiplying it, on both sides, by $2(dV/dx)$, and integrating the right-hand side under the sign of integration, the result being,

$$\left(\frac{dV}{dx}\right)^2 = \frac{8\pi m}{k}\int_{v_0'}^{\infty} n(v_0)(v - v')\,dv_0. \tag{6}$$

This result applies only to the region β. By using (2), (4) and (5) in exactly the same way a similar result may be obtained for the region α, provided the limits of integration, which are variable in this case, are properly dealt with when the order of integration is changed. This gives:

$$\left(\frac{dV}{dx}\right)^2 = \frac{8\pi m}{k}\left[\int_{v_0'}^{\infty} n(v_0)(v - v')dv_0 + 2\int_{\sqrt{\frac{2Ve}{m}}}^{v_0'} vn(v_0)dv_0\right]. \qquad (7)$$

So far, it has not been necessary to know the form of the function $n(v_0)$, but in order to proceed further it is desirable to introduce it at this time. The most logical expression is Maxwell's distribution law for the number of gas molecules having the normal velocity v_0 which pass a given plane in a second of time. This expression is

$$n(v_0) = \frac{\pi N v_0}{2\bar{v}_0^2}\,\epsilon^{-\left(\frac{\sqrt{\pi}v_0}{2\bar{v}_0}\right)^2}, \qquad (8)$$

in which $\bar{v}_0$ represents the average velocity of emission,

$$\bar{v}_0 = \frac{1}{N}\int_0^{\infty} n(v_0)v_0 dv_0.$$

It is somewhat simpler to use v instead of v_0 as the variable of integration in (6) and (7). When this is done and $n(v_0)$ is given the value (8) the equation

$$\left[\frac{dV}{dx}\right]^2 = \frac{4\pi Nm}{k\bar{v}_0^2}\left[\int_0^{\infty} v^2\epsilon^{-\frac{\pi}{4\bar{v}_0^2}\left(v^2+\frac{2Ve}{m}\right)}dv - \int_{\sqrt{\frac{2e}{m}(V'-V)}}^{\infty} vv'\epsilon^{-\frac{\pi}{4\bar{v}_0^2}\left(v^2+\frac{2Ve}{m}\right)}dv \pm \int_0^{\sqrt{\frac{2e}{m}(V'-V)}} v^2\epsilon^{-\frac{\pi}{4\bar{v}_0^2}\left(v^2+\frac{2Ve}{m}\right)}dv\right] \qquad (9)$$

is obtained, in which the upper sign is to be used in the region α and the lower sign in the region β. The integrals involved in this expression are of well-known types and may easily be evaluated either in terms of algebraic functions or in terms of the normal error function. Denoting this latter function by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx,$$

(9) may be expressed as

$$\left(\frac{d\eta}{d\xi}\right)^2 = \epsilon^{\eta} - 1 \pm \left(\epsilon^{\eta}\operatorname{erf}\sqrt{\eta} - \frac{2}{\sqrt{\pi}}\sqrt{\eta}\right), \qquad (10)$$

where the quantities ξ and η are related to x and V by the formulæ

$$\xi = \epsilon^{-\frac{\pi e V'}{4\bar{v}_0^2 m}}\sqrt{\frac{2\pi^2 e^2 N}{km\bar{v}_0^3}}\,(x - x'),$$

$$\eta = \frac{\pi e}{2\bar{v}_0^2 m}(V - V').$$

It is to the introduction of these quantities ξ and η that the generality of the solution which we obtain must be attributed. They are both dimensionless, but, as the defining equations show, they are proportional to distance and potential difference respectively, both being measured from the point of minimum potential. Hence for any one state of the system the curve which represents η as a function of ξ will also represent V as a function of x; the only difference being that for the latter purpose the units of measure along the axes, and the location of the origin of reference must be changed. For this reason it is probably most convenient physically to think of ξ and η as length and potential respectively, remembering, however, that with each change in such quantities as cathode temperature and anode voltage the unit of measure must be readjusted.

From a purely mathematical point of view the reason it is desirable to use them rather than x and $\bar{v}$ is that they reduce equation (9) to the form (10) in which, aside from themselves, only absolute constants occur. Equation (9) defines one relation between V and x for every set of values of the constants $\bar{v}'$, $\bar{v}$., etc., and therefore to obtain a complete solution for it by graphical means would require the construction of a large number of curves. On the other hand equation (10) has only one solution which may be found once for all. The labor of obtaining a complete answer to our problem is, therefore, materially reduced by the use of the new symbols.

3. Solution of Equation (10).

The equation (10) applies to the region between the cathode and the point of minimum potential when the upper sign is used and to the region beyond the point of minimum potential when the lower sign is used. In either case, however, the boundary condition to be applied is that $V = V'$, and hence $\eta = 0$ when $x = x'$, and hence $\xi = 0$. The right-hand side of this equation has been computed for a sufficient range of values of η. These values are shown in the accompanying table, where it is denoted by $\phi(\eta)$.

The equation (10) may now be written in the form

$$\xi = \int_0^\eta \frac{d\eta}{\sqrt{\phi(\eta)}}, \qquad (11)$$

which represents the general solution of the problem under discussion provided the integral can be evaluated. This could easily be accomplished by the use of the integraph if it were not for the fact that the integrand becomes infinite at $\eta = 0$. However, no mechanical device is ever capable of handling infinities, and it is necessary to perform the

TABLE I.

Values of the Functions $\phi(\eta)$.

η	$\phi(\eta)$. Upper Sign.	$\phi(\eta)$. Lower Sign.	$[\phi(\eta)]^{-1/2}$. Upper Sign.	$[\phi(\eta)]^{-1/2}$. Lower Sign.
.1	.129	.0808	+2.79	−3.52
.2	.293	.1488	1.85	2.59
.3	.489	.2107	1.43	2.19
.4	.716	.2681	1.06	1.93
.5	.977	.3214	1.01	1.76
.6	1.271	.3726	.887	1.64
.7	1.607	.4213	.789	1.54
.8	1.983	.4688	.710	1.46
.9	2.408	.5118	.644	1.40
1	2.878	.5575	.590	1.34
1.2	4.002	.6377	.500	1.25
1.4	5.384	.716	.431	1.18
1.6	7.116	.790	.375	1.12
1.8	9.24	.862	.329	1.08
2	11.85	.931	.291	1.04
2.5	21.27	1.09	.217	.962
3	37.13	1.24	.164	.901
3.5	62.85	1.38	.126	.855
4	105.68	1.52	.097	.813
4.5	176.41	1.63	.075	.781
5	293.07	1.75	.058	.758
9		2.56		0.625
16		3.65		0.525
36		5.86		0.413
64		8.09		0.352
100		10.34		0.311
225		15.9		0.251
400		21.6		0.215
900		32.8		0.174
1600		45.2		0.149
2500		56.4		0.133

integration for small values of η in some other way. This is done by noting that when η is small

$$\phi(\eta) = \eta \pm \frac{2}{\sqrt{\pi}}\eta^{3/2} + \frac{1}{2}\eta^2.$$

Substituting this value in (11) and performing the integration the approximate relation

$$\xi \doteq 2\sqrt{\eta} \mp \frac{1}{\sqrt{\pi}}\eta + \left(\frac{3}{\pi} - \frac{1}{2}\right)\eta^{3/2} \qquad (12)$$

is obtained.

This curve is shown plotted as curve (1) of Fig. 2. Curve (2) is $[\phi(\eta)]^{-1/2}$ for the region between the cathode and the point of minimum potential while curve (3) represents the same function beyond this point. Curves (4) and (5) are the integrals of (2) and (3) as drawn by the integraph, so constructed as to meet the approximation curve (1) near the origin. From the fact that (4) and (5) join on to (1) smoothly, it is evident that the approximation given by (12) is sufficiently accurate for all physical purposes.

4. Discussion of Results.

There are three distinct types of potential distribution curves, for the case of pure electron discharge between plates. The first of these is met when the space current is saturated; that is, when every electron which frees itself from the cathode is pulled across to the anode. It is at once obvious that the potential gradient in this case cannot be negative; hence this curve must have a positive slope throughout its entire range. It is typified by curve (1) of Fig. 3. As the plate voltage is increased,

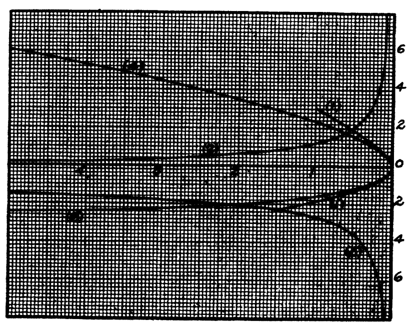

Fig. 2.
Voltage Distribution Curve, Approximation Curve, and Derivatives.

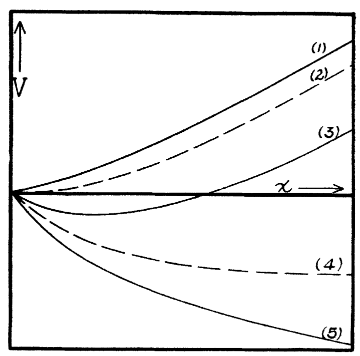

Fig. 3.
Three Types of Voltage Distribution.

it straightens out more and more, and the angle at which it meets the cathode becomes more and more acute. Similarly, as the plate voltage decreases toward the critical voltage which will just saturate the current the curve rounds off, and the angle at which the cathode is met becomes more and more nearly a right angle. When the plate voltage is made just sufficient to saturate the current a limiting distribution of potential is met that separates the first type of curve from the second. For this distribution the gradient at the surface of the cathode is zero, while at all other points of space it is positive. Such a curve is shown by the dotted line (2) of Fig. 3.

For all distributions of potential belonging to the first class, the space current is constant and determined solely by the emissivity of the cathode. Whatever the design of the tube or the voltage applied, so long as it exceeds the saturation voltage, this current remains constant. When, however, the voltage is reduced below the limiting value of this class the gradient at the cathode, and for some distance x' beyond, becomes negative. At x' it changes sign and thereafter remains positive, as shown in Fig. 3 (3).

At x' there is a minimum on the potential distribution curve. The value of V' at this minimum point, and the distance x' between it and the cathode vary with the anode potential. When this is so high that nearly all of the emitted electrons are drawn across both V' and x' are very small; but as it is lowered more and more V' becomes greater and greater, and moves further and further from the cathode. During these changes of plate voltage the current is determined solely by the number of electrons which are capable of passing the restraining voltage V', the relation between them being that given in equation (3).

When the anode voltage is made zero the potential throughout the space between the plates is everywhere negative, and this remains true as the anode voltage is still further reduced. A second limiting condition, which separates the second and third classes of potential distributions is reached when the anode potential is so low that x' becomes equal to the distance between the plates. When this condition is reached the potential gradient at the anode is zero; while throughout the remainder of the space between the electrodes it is negative. The voltage on the anode is then V', and the space current is determined by the number of electrons emitted with sufficient energy to overcome this potential drop. This limiting condition is shown in Fig. 3 by a dotted line (4).

When the potential on the anode is still further reduced, the gradient becomes everywhere negative, and curves of type (5) result. When potential distributions of this type occur the space current is determined solely by the number of electrons which are able to move against the adverse potential $-V$.

The second condition is the one most frequently met in practice, and is the only one to which consideration is given in this paper. The curve obtained in Fig. 2 when drawn for a sufficient range of values of ξ and η as in Fig. 4 applies to all distributions of this type and makes possible computations regarding them. In order to make use of it most conveniently, however, it is desirable to express ξ and η in terms of quantities which are not so difficult to determine as V' and $\bar{v}_0$. The current i, the saturation current i_s and the potential change $\overline{V}_0$ which would give to an electron an energy equal to the average energy of those shot out from the cathode are satisfactory in this respect, and it is not difficult

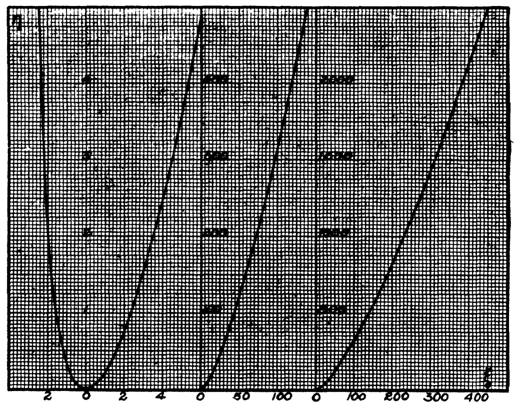

Fig. 4.

The ξ, η Curves.

in the light of the preceding equations to see that they are related and the quantities $\bar{v}_0$ and V' by means of the equations

$$i_s = Ne,$$

$$\overline{V}_0 = -\frac{2m}{e}\bar{v}_0^2,$$

$$i = i_s \epsilon^{V'/\overline{V}_0}.$$

The last of these equations furnishes the relation between V' and i, which can be written in the form

$$V' = \overline{V}_0 \log_\epsilon \frac{i}{i_s}. \tag{13}$$

Upon introducing these new symbols in the equations defining ξ and η and inserting the numerical values of such of the quantities as are universal constants, they become

$$\xi = 820 \frac{i^{1/2}}{\overline{V}_0^{3/4}}(x - x'), \qquad \eta = \frac{V - V'}{\overline{V}_0}, \tag{14}$$

where current and potential are expressed in amperes and volts.

In order to illustrate the procedure to be followed in making use of Fig. 4 a particular example may be considered. For this purpose take a pair of electrodes $\frac{1}{2}$ cm. apart, one of which is emitting electrons in such a fashion that[1] i_s is 0.16 amperes and $\overline{V}_0$ is 0.3 volts. In this case if ξ' and η' are the values of ξ and η at the cathode equations (14) become

$$\eta' = \log \frac{i_s}{i},$$
$$V = 0.3(\eta - \eta'), \qquad (15)$$
$$\xi = \xi' + 1010\sqrt{i}.$$

In Table II. a set of values of i has been chosen ranging from 0.1 per cent. to 100 per cent saturation. From these values η' has been found

TABLE II.

The Application of Fig. 4 to a Specific Problem.

i/i_s.	i amps.	η'.	ξ'.	$1010\sqrt{i}$.	ξ.	η.	V Volts.	x' cm.	V' Volts.
0.001	0.00016	6.91	−2.4	12.8	10.4	13	2	0.094	−2.1
0.002	0.00032	6.21	−2.4	18.0	15.6	23	5	0.067	−1.9
0.005	0.0008	5.30	−2.4	28.4	26.0	47	13	0.042	−1.6
0.01	0.0016	4.61	−2.40	40.4	38.0	80	23	0.030	−1.4
0.02	0.0032	3.91	−2.35	57.1	54.8	133	39	0.021	−1.2
0.05	0.008	3.00	−2.24	90.4	88.2	263	78	0.012	−0.9
0.1	0.016	2.30	−2.09	128	126	435	130	0.008	−0.7
0.2	0.032	1.61	−1.88	180	178	695	208	0.005	−0.5
0.4	0.064	0.92	−1.56	255	253	1130	338	0.003	−0.3
0.6	0.096	0.51	−1.22	313	312	1515	453	0.002	−0.2
0.8	0.128	0.22	−0.86	361	360	1840	550	0.001	−0.1
1.0	0.16	0.00	−0.00	404	404	2150	645	0.000	−0.0

by the first of equations (15) and then ξ' taken from the curve of Fig. 4. Knowing the values of ξ', it is easy to compute the value of ξ corresponding to each value of i. Using these values of ξ, η is picked off from Fig. 4 and entered in the table. Finally V is obtained by substituting in the second of equations (15) the values already found for η and η'.

The relation between V and i is represented graphically in Fig. 5, where it has been plotted to logarithmic scales in order to facilitate comparison with the 3/2-power law, which gives the accompanying straight line.

In the process of obtaining this curve it has been necessary to find the values of ξ' and η', from which x' and V' can easily be obtained by the use of the relations $V' = -0.3\eta'$ and

$$x' = -\frac{\xi'}{2020\sqrt{i}}.$$

Accordingly these quantities, which have a certain amount of interest in themselves, have been entered in the last two columns of Table II. and

[1] This corresponds roughly to a tungsten cathode at 2400° K.

represented by Figs. 6 and 7. Of course the absolute magnitudes of x' and V' corresponding to any given voltage are very largely dependent upon such things as the electrode separation and the saturation current,

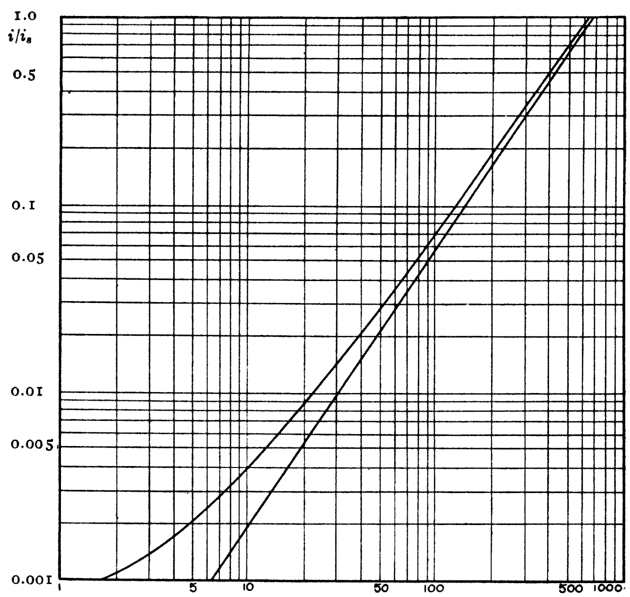

Fig. 5.

The Relation of Space Current to Plate Voltage: Comparison of Maxwell Distribution and 3/2-Power Law.

so that Figs. 6 and 7 can be regarded as typical only in the sense that curves of the same general shape are obtained for other values of these quantities.

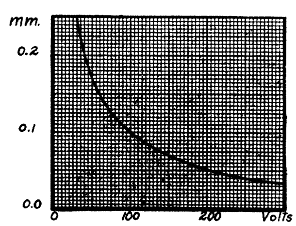

Fig. 6.

The Relation between Plate Voltage and Distance of Minimum Potential from Cathode.

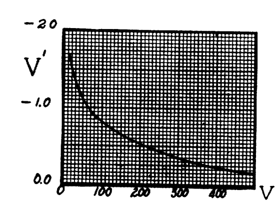

Fig. 7.

Relation between Minimum Potential and Plate Voltage. The Scales are in Volts.

5. Conclusion.

Perhaps the most interesting comment which may be made on these results concerns the magnitude of the variations from the 3/2-power law introduced by the finite velocities of emission. These variations are frequently quite large, and, as is evident from Fig. 5, may be close to 50 per cent. for voltages as high as 40 or 50 volts. At still lower voltages the 3/2-power law need not even give a rough approximation to the values to be expected from a Maxwell distribution; so that at these voltages a more accurate means of computation is needed. This means is furnished by the curves of Figs. 2 and 4, the use of which, it is believed, will be found sufficiently simple and accurate for most laboratory computations.

Research Laboratories of the American Telephone &
Telegraph Company and the Western Electric Company, Inc.,
February, 1920.

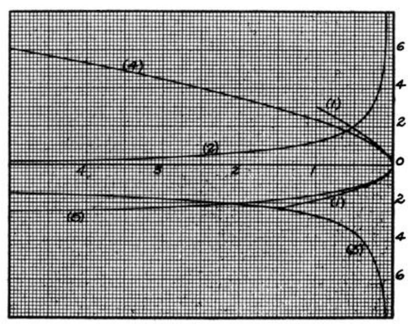

Fig. 2.

Voltage Distribution Curve, Approximation Curve, and Derivatives.

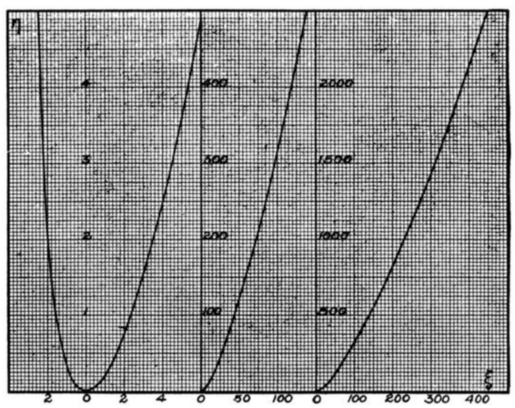

Fig. 4.

The ξ, η Curves.

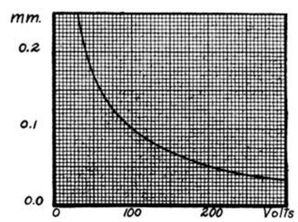

Fig. 6.

The Relation between Plate Voltage and Distance of Minimum Potential from Cathode.

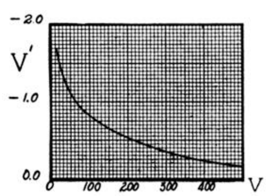

Fig. 7.

Relation between Minimum Potential and Plate Voltage. The Scales are in Volts.