THE STRUCTURE OF THE HELIUM ATOM.[1]

BY IRVING LANGMUIR.

SYNOPSIS.

Models for the Helium Atom.—(1) *Bohr's model* is unsatisfactory because it gives too great a value for the ionizing potential and is not in accord with the optical and magnetic properties of helium. Now the chemical evidence suggests that each electron in an atom has its own separate orbit, and that these orbits are closely interrelated. Accordingly two new models are considered. (2) In the *double circle model* the two electrons are assumed to move in two separate, parallel, circular orbits. The model, however, is unstable, for the ionizing potential, computed by applying the quantum theory, comes out negative. Another objection is that the magnetic moment is not zero. (3) In the *semi-circular model* each electron is assumed to oscillate back and forth along an approximately semi-circular path in accord with classical mechanics, each being brought to rest at the end of its path by the repulsion of the other. Assuming the maximum angular momentum of each equal to $h/2\pi$, the absolute dimensions of the model were computed. The total energy comes out less than for the Bohr model, and the ionizing potential, 25.62 volts, agrees closely with the experimental value. The magnetic moment is zero.

Application of the Quantum Theory to Coupled Electrons.—The success of the semi-circular model in giving the correct value for the ionizing potential of helium suggests that in the case of coupled electrons the quantum theory should be applied not to the momentum of the individual electrons according to the relation $\int pdq = h/2\pi$, but rather to the momentum which, by being relayed from one electron to another, passes in each direction around the nucleus.

BOHR'S models for the hydrogen atom and the positively charged helium ion are now almost universally accepted. The extraordinary success with which even the fine structure of the lines of the hydrogen and helium spectra have been worked out from these models by Bohr, Sommerfeld and others, is convincing proof that in these cases the structure is definitely known. On the other hand Bohr's models for the helium atom,[2] and for all the molecules considered by him, although they have seemed satisfactory in many respects, have not given quantitative agreement with the spectra or other properties. For example the ionizing potential of helium according to Bohr's model should be 28.8 volts while experiment gives 25.4 ± 0.25 volts.[3] Sommerfeld[4] states that this model is not in agreement with the optical proper-

[1] A preliminary notice giving some of the results of this paper has been published in Science, N. S., *51*, 605 (1920).

[2] Phil. Mag. *26*, 488 (1913).

[3] Frank and Knipping, Phys. Zeit., *20*, 481 (1919).

[4] Atombau, 1st edition, p. 70, Braunschweig, 1919.

ties of helium. Similarly, the molecular heat of dissociation of hydrogen into atoms should be 63,000 calories according to Bohr's model, while experiment gives about 84,000 calories.[1] From the behavior of the many-lined spectrum of hydrogen in a magnetic field Lenz[2] has concluded that the electrons in the hydrogen molecule can have no momentum about the axis of symmetry. In other words either each electron must have zero momentum, or the momenta of the two electrons must be equal and opposite.

From the chemical point of view Bohr's models are satisfactory only in the case of the hydrogen atom. Evidence was given by A. L. Parson[3] that in many atoms the electrons are arranged with cubic symmetry. G. N. Lewis[4] emphasized the importance of pairs of electrons and *octets* or groups of eight electrons. Kossel[5] showed that certain groupings of electrons in atoms, corresponding to the inert gases, are of unusual stability. These theories have been extended in a series of papers by the writer.[6] Similar conclusions regarding the structure and stability of the octet has been reached by M. Born[7] from evidence of an entirely different kind. Landé in a series of papers[8] has developed a mathematical theory of the motions of eight electrons about points in atoms distributed with cubic symmetry. Landé and Born, however, have not given consideration to the fact that the pairs of electrons in the helium atom and in the kernels of other atoms, exhibit a stability even greater than that of the octet.

Bohr's calculations,[9] based on coplanar, concentric, circular orbits gave no indications that a pair of electrons bound by a nucleus would show unusual stability. If $-W$ represents the total energy (kinetic plus potential) of an atom then the most stable model should be that which has the greatest value of W. For Bohr's model of the helium atom, in which the two electrons revolve about the nucleus in a single circular orbit, $W = 6.13W_0$, where W_0 is the value of W corresponding to the hydrogen atom. Bohr's model of the helium ion gives $W = 4W_0$. For a lithium atom, with the three electrons in a single circular ring, W

[1] Langmuir, Jour. Amer. Chem. Soc., *37*, 417 (1915).

[2] Verh. Deut. Phys. Ges., *21*, 632 (1919).

[3] Smithsonian Inst. Publ., Miscel. Collections, *65*, No. 11 (1915).

[4] Jour. Amer. Chem. Soc., *38*, 762 (1916).

[5] Ann. Physik, *49*, 229 (1916).

[6] Langmuir, Jour. Franklin Inst., *187*, 359 (1919); Jour. Amer. Chem. Soc., *41*, 868, 1543 (1919); *42*, 274 (1920); PHYS. REV., N. S., *13*, 300 (1919); Proc. Nat. Acad. Sci., *5*, 252 (1919); Nature, April 29 (1920).

[7] Verh. deut. Phys. Ges., *20*, 230 (1918).

[8] Verh. deut. Phys. Ges., *21*, 2, 644, 653 (1919); Zeit. f. Physik., *2*, 83, 380 (1920).

[9] Phil. Mag., *26*, 492 (1913).

$= 17.61W_0$, but if there are two electrons in an inner ring and a single electron in an outer ring $W = 16.02W_0$. For the lithium ion with two electrons in a single ring Bohr finds $W = 15.13W_0$. Therefore the work needed to remove an electron from a lithium atom (first model) is $2.48W_0$, while the work required to take an electron from a helium atom is only $2.13W_0$. It should thus be easier to ionize helium that lithium vapor. Bohr's theory is even less applicable to beryllium for the model with the greatest value of W is that in which all four electrons revolve in a single ring ($W = 37.04W_0$). Bohr's original theory would thus indicate that lithium and beryllium (and all subsequent elements) should be inert gases having even greater stability than helium.

The remarkable stability of the pair and the octet is not explainable on the basis of Bohr's theory. The chemical and physical properties of the elements prove that the electrons in atoms are coupled together in a rather complex manner, which seems quite inconsistent with the ordinary properties of the electron. I have tried to summarize the characteristics of this electron coupling in a series of postulates.[1] It is probable that an understanding of these postulates will require more definite knowledge of the mechanism underlying the quantum theory. Landé seems to have made progress in this direction, by postulating coupled orbits. Thus according to Landé, the eight electrons of an octet move in such a way that their positions are always symmetrically located with respect to each of three mutually perpendicular planes which pass through the nucleus. Landé offers no suggestions as to a cause or mechanism of this coupling.

The chemical evidence indicates that there is not only a definite coupling between the electrons in a single shell but also between electrons in different shells. Thus the octet exists only about nuclei which already have stable pairs. The magnetic properties of the elements furnish excellent proof of electron coupling between different shells. Such action is not taken into account in Landé's theory.

The most striking case of electron coupling seems to be that in the helium atom. Since the underlying mechanism must be essentially similar to that of the octet, it would seem that a study of the structure of the helium atom offers the most promising method for the attack on the problem of the mechanism of quantum phenomena.

Bohr's work on the structure of atoms and molecules has been based on the assumption that " the dynamical equilibrium of the systems in the stationary states is governed by the ordinary laws of mechanics " while the quantum laws apply to " the passing of the systems between

[1] See particularly Proc. Nat. Acad. Sci., *5*, 252 (1919).

different stationary states." For each degree of freedom there is an infinite number of solutions of the dynamical problem, but as Sommerfeld has shown, there are as many quantum conditions to be fulfilled as there are degrees of freedom. As a result, there is only a discrete number of possible solutions. These fundamental postulates have received such strong support from the work of Bohr, Sommerfeld and others that there seems to be no reason to doubt their general validity.

According to the theory which I advanced in 1919 (Postulates 1, 3 and 4, l. c.) the electrons in their most stable positions move only within certain limited regions or "cells" within the atom. In the atoms of the inert gases the cells were found to be arranged symmetrically with respect to an equatorial plane, no electrons however lying in this plane. From this viewpoint the two electrons in the helium atom should not move in the same orbit but in separate orbits located symmetrically with respect to the equatorial plane. Let us consider the most probable motions of this kind.

Sommerfeld has shown[1] that in general a single electron in an atom must fulfill three quantum conditions corresponding to its three degrees of freedom. Choosing coördinates like those used on the earth we may describe the position of an electron in terms of its latitude, longitude and distance from the center of the atom. In ordinary helium, with the atoms in their most stable state, we must assume that each electron has only one quantum, and that this quantum corresponds to some one of the three degrees of freedom. Radial motion such as rectilinear motion along the polar axis, is to be excluded[2] since the electrons would have to pass through the nucleus. We have then to consider motion either in longitude or in latitude.

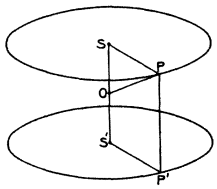

Fig. 1.
Double-circle model for the helium atom.

Motion in Longitude. Double Circle Model.—If the electrons move in longitude only, they will travel along parallels of latitude and rotate in the same direction in circular orbits in two parallel planes equidistant from the equatorial plane, as is shown in Fig. 1. Let r be the distance from the nucleus of the atom (at O) to one of the electrons P; let b be the distance OS or half the distance between the planes of the orbits; and let a be the radius SP of the orbits. Since in the helium atom the nucleus has the charge $2e$, the force of attraction between the nucleus and each electron is $2e^2/r^2$. The component of this force directed towards

[1] Atombau, 1st edit., p. 411 (1919).
[2] Atombau, p. 273.

the other electron is $2e^2b/r^3$ while the repulsive force between the electrons is $e^2/4b^2$. Equating these, we find $r = 2b$ as a condition for dynamical equilibrium; the latitude of the electrons must therefore be 30°.

If ω is the angular velocity of the electron in its orbit, the centrifugal force exerted by it is $ma\omega^2$, where m is the mass of the electron. The component of the attractive force of the nucleus in the direction towards the center of the orbit (PS) is $2e^2a/r^3$. Equating these forces we obtain:

$$m\omega^2 r^3 = 2e^2. \tag{1}$$

This is as far as we can proceed by means of the classical mechanics. We still have two unknown quantities and therefore cannot determine the constants of the model.

If we impose the quantum condition that the angular momentum of the electron about the center of its orbit shall be $h/2\pi$ we find

$$2\pi ma^2\omega = h. \tag{2}$$

These two equations, together with the relation $r = 2a/\sqrt{3}$ give:

$$a = \frac{h^2}{3\sqrt{3}\pi^2 me^2} = \frac{4}{3\sqrt{3}} a_0, \tag{3}$$

$$\omega = \frac{27\pi^3 me^4}{2h^3} = \frac{27}{16}\omega_0. \tag{4}$$

Here a_0 and ω_0 are the values of a and ω for the hydrogen atom as found by Bohr, namely:

$$a_0 = \frac{h^2}{4\pi^2 me^2}, \tag{5}$$

$$\omega_0 = \frac{8\pi^3 me^4}{h^3}. \tag{6}$$

Since Bohr has shown that the kinetic energy of the electrons is numerically equal to half the potential energy but opposite in sign, we may place the kinetic energy equal to W and thus find from (3) and (4):

$$W = \frac{27}{4}\frac{\pi^2 me^4}{h^2} = \frac{27}{8} W_0. \tag{7}$$

Here W_0 is the value of W corresponding to the hydrogen atom:

$$W_0 = \frac{2\pi^2 me^4}{h^2} = \frac{e^2}{2a_0}. \tag{8}$$

Taking $h = 6.554 \times 10^{-27}$ erg. second; $e = 4.774 \times 10^{-10}$ e.s. units; and $m = 9.002 \times 10^{-28}$ grams, we find the following numerical values:

$a_0 = 0.5304 \times 10^{-8}$ cm.,

$\omega_0 = 4.134 \times 10^{16}$ radians per second (6.580×10^{15} revolutions per second).

$W_0 = 2.157 \times 10^{-11}$ erg which corresponds to 312,000 g. calories per g. molecule. W_0/e is equivalent to 13.55 volts.

Bohr has shown that for the positively charged helium ion $W = 4W_0$. Comparing this with equation (7) we see that the helium model that we have been considering is unstable as it would lose one of its electrons and form a positively charged helium ion. The ionizing potential would thus be negative in sign and its value would correspond to $3.375W_0 - 4W_0 = -0.625W_0$ or -8.5 volts.[1]

This model therefore cannot represent the atom of ordinary helium, although it may well correspond to a possible disturbed state of the atom. There is the same objection to this model as to the Bohr model in regard to the momentum of the atom as a whole about it axis of symmetry, for the work of Lenz seems to show that there should be no momentum in the most stable atoms.

Motion in Latitude. Oscillating, or Semi-circular Model.—Let us consider the case that the electrons have no component of motion in longitude. The motion is thus confined to the plane of a meridian of our system of coördinates. In Fig. 2 let FF' represent the polar axis and the circle $BEB'E'B$ a meridian circle. The intersection of the equatorial plane with the plane of the meridian is the line AOA'. If we imagine two electrons at P and P' symmetrically placed with respect to the line AA', we can conceive of a series of orbits consistent with the ordinary mechanics. We are concerned here with those in which there is as little radial motion as possible. We shall now proceed to show that a motion of the electrons back and forth along the lines CBD and $C'B'D'$ respectively is in accord with the laws of mechanics. This motion corresponds to that of the two electrons in the circular ring of Bohr's model for the helium atom except that the electrons revolve in opposite directions about the nucleus. If we were dealing with electrons having more than one quantum, we should have to consider paths which bear the same relation to elliptical orbits that the paths CBD and $C'B'D'$ bear to the circular orbits of Bohr's model.

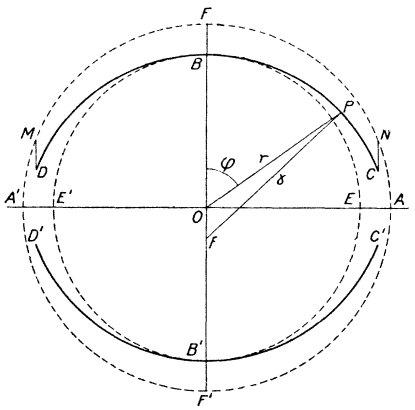

Fig. 2.

Oscillating or semi-circular model for the helium atom.

[1] By a misprint this appeared as -5.8 volts in the preliminary notice in SCIENCE.

The electron at P is attracted by the nucleus at O and repelled by the electron at P'. Let N be the atomic number of the nucleus at O; it thus has the positive charge Ne. In moving the two electrons from infinity to the positions P and P' the work done by the attraction of the nucleus is $2Ne^2/r$ while the work that must be done against the repulsion of the electrons is $e^2/(2r\cos\varphi)$, where r is the distance PO and φ is the angle BOP. Hence the potential energy E_p of *one electron* is

$$E_p = -\frac{e^2}{r}\left(N - \frac{1}{4\cos\varphi}\right) \tag{9}$$

and the kinetic energy E_k is

$$E_k = \frac{m}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right]. \tag{10}$$

The equations of motion for the electrons are most easily obtained by Hamilton's method. If q is any coördinate, we place $q' = dq/dt$ and define p by[1]

$$p = \frac{\partial E_k}{\partial q'}. \tag{11}$$

Letting q be successively equal to r and to φ we find from (11) and (10)

$$p_r = m\frac{dr}{dt}, \qquad p_\phi = mr^2\frac{d\varphi}{dt}. \tag{12}$$

The Hamilton function H is obtained by expressing $E_p + E_k$ in terms of q and p. Thus (9), (10) and (12) give

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{e^2}{r}\left(N - \frac{1}{4\cos\varphi}\right). \tag{13}$$

The equations of motion are then found by means of the relation[1]

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{14}$$

from (12) and (13) as follows (after dividing by m and mr^2 respectively):

$$\frac{d^2r}{dt^2} = r\left(\frac{d\varphi}{dt}\right)^2 - \frac{e^2}{mr^2}\left(N - \frac{1}{4\cos\varphi}\right), \tag{15}$$

$$\frac{d^2\varphi}{dt^2} = -\frac{e^2}{4mr^3}\frac{\sin\varphi}{\cos^2\varphi} - \frac{2}{r}\frac{d\varphi}{dt}\frac{dr}{dt}. \tag{16}$$

Since it is impossible, by the classical mechanics alone, to find the absolute dimensions of this model, it will be convenient to express the above equations in terms of variables which depend only on relative dimensions. It is evident from (16) and from considerations of sym-

[1] See Sommerfeld, Atombau, p. 216.

metry that the electrons move with the maximum angular velocity when they are at the mid-points of their paths that is, when $\varphi = 0$. Let us then choose the unit of time so that this maximum angular velocity φ_0' shall be unity. If we let θ be the time expressed in these new units we have

$$\theta = \varphi_0' t. \tag{17}$$

Similarly let us take as our unit of length, r_0 the radius vector BO at the mid-point of the path. Let ρ represent the radius vector at the point P expressed in the new units. Then

$$\rho = \frac{r}{r_0}. \tag{18}$$

By making these substitutions in equations (15) and (16) we are able to reduce them to the form

$$\frac{d^2\rho}{d\theta^2} = \rho\left(\frac{d\varphi}{d\theta}\right)^2 - \frac{1}{\alpha\rho^2}\left(N - \frac{1}{4\cos\varphi}\right), \tag{19}$$

$$\frac{d^2\varphi}{d\theta^2} = -\frac{1}{4\alpha\rho^3}\frac{\sin\varphi}{\cos^2\varphi} - \frac{2}{\rho}\frac{d\rho}{d\theta}\frac{d\varphi}{d\theta}, \tag{20}$$

where

$$\alpha = \frac{mr_0^3(\varphi_0')^2}{e^2}. \tag{21}$$

The total energy of the molecule is $2(E_p + E_k)$ and we may place this equal to $-W$ and thus obtain from (9), (10), (17), (18), and (21):

$$\frac{Wr_0}{e^2} = \frac{2}{\rho}\left(N - \frac{1}{4\cos\varphi}\right) - \alpha\left[\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2\left(\frac{d\varphi}{d\theta}\right)^2\right]. \tag{22}$$

At the mid-point of the path (B in Fig. 2) we may place $\varphi = 0$, $\rho = 1$, $d\rho/d\theta = 0$, and $d\varphi/d\theta = 1$. Since the total energy W is constant we may substitute these values in (22) and obtain

$$\frac{Wr_0}{e^2} = 2N - \frac{1}{2} - \alpha. \tag{23}$$

Combining this with (22) gives

$$\alpha\left[\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2\left(\frac{d\varphi}{d\theta}\right)^2\right] = \frac{2}{\rho}\left(N - \frac{1}{4\cos\varphi}\right) + \alpha - 2N + \frac{1}{2}. \tag{24}$$

This equation together with (19) defines the motion of the electrons in this model. It is necessary to resort to numerical calculations in order to obtain ρ and θ as functions of φ. Table I. gives the results of such calculations. The procedure was as follows:

As a first approximation ρ was assumed constant and equal to unity, so that $d\rho/d\theta$ and $d^2\rho/d\theta^2$ were taken to be zero. Equations 19 and 24

then take simple forms. Placing $N = 2$ for helium, equation (19) gives for $\varphi = 0$, the result $\alpha = 7/4$. By using this value of α approximate values for $d\varphi/d\theta$ as a function of φ can then be calculated from equation (24). By substituting these in (19) approximate values for $d^2\rho/d\theta^2$ can be found. By dividing these by $d\varphi/d\theta$ and by approximate integration with respect to φ we obtain the radial velocity $d\rho/d\theta$ as a function of φ. The end of the path (C in Fig. 2) is determined by finding the point at which $d\varphi/d\theta$ is zero. If the electrons are to return back along the same path it is evidently necessary that $d\rho/d\theta$ and $d\varphi/d\theta$ shall vanish for the same value of φ. Because of the approximate nature of the calculations this condition will not be met and it is therefore necessary to adjust the values of $d\rho/d\theta$ so that they become zero at the point C. By dividing these adjusted values by $d\varphi/d\theta$, and by approximate integration with respect to φ, we obtain ρ as a function of φ. The values of θ as a function of φ are obtained by integrating the reciprocal of $d\varphi/d\theta$ with respect to φ. From the adjusted values of $d\rho/d\theta$ the second derivative $d^2\rho/d\theta^2$ at the point $\varphi = 0$ is calculated. Substituting this in (19) gives a second approximation for α. Using this and the values of ρ and $d\rho/d\theta$ already obtained, we then proceed by (24) and (19) to obtain second approximations for $d\varphi/d\theta$ and $d^2\rho/d\theta^2$ in a similar manner. This process may be repeated until sufficiently accuracy has been obtained.

The numerical calculations were carried out with the aid of a calculating machine to five or six figures. The integrations were done by Simpson's rule or more often by Weddle's rule and special methods were adopted to increase the accuracy in the region close to the end of the path. Four series of approximations were carried out and it is believed that the results which are recorded in Table I. are accurate to about one tenth per cent.

By plotting ρ and φ as polar coördinates as in Fig. 2, it is seen that the path DBC of the electron is very nearly an arc of circle whose center lies at some point F on the line OB'. If we let γ be the distance from F to the moving point P and if ϵ is the distance OF then we have

$$\gamma = \rho^2 + \epsilon^2 + 2\rho\epsilon \cos \varphi. \tag{25}$$

Let us choose the point F so that γ is the radius of curvature of the path DBC at the point B. Then the first and second derivatives of γ with respect to θ will be zero when φ is zero, and for this point we may also place $\rho = 1$;

$$\frac{d\rho}{d\theta} = 0 \qquad \text{and} \qquad \frac{d\varphi}{d\theta} = 1.$$

We thus obtain

$$\left[\frac{d^2\rho}{d\theta^2}\right]_{\phi=0} = \frac{\epsilon}{1+\epsilon}. \tag{26}$$

TABLE I.

Data for the Motion of the Electrons in the Second Model of the Helium Atom, $\alpha = 2.0920$.

φ Radians.	θ.	$\frac{d\varphi}{d\theta}$.	ρ.	$\frac{d\rho}{d\theta}$.
0.00	0.0000	1.0000	1.0000	0.0000
0.10	0.1001	0.9979	1.0008	0.0165
0.20	0.2006	0.9912	1.0033	0.0325
0.30	0.3020	0.9802	1.0074	0.0480
0.40	0.4048	0.9646	1.0131	0.0628
0.50	0.5095	0.9448	1.0204	0.0765
0.60	0.6168	0.9201	1.0292	0.0887
0.70	0.7271	0.8913	1.0396	0.0992
0.80	0.8417	0.8558	1.0515	0.1074
0.90	0.9612	0.8162	1.0647	0.1131
1.00	1.0881	0.7633	1.0792	0.1146
1.10	1.2245	0.6974	1.0947	0.1114
1.20	1.3784	0.6019	1.1110	0.0995
1.25	1.4659	0.5305	1.1193	0.0885
1.30	1.5712	0.4425	1.1276	0.0701
1.31	1.5957	0.3925	1.1293	0.0650
1.32	1.6224	0.3578	1.1310	0.0591
1.33	1.6520	0.3164	1.1326	0.0521
1.34	1.6864	0.2648	1.1342	0.0434
1.35	1.7300	0.1944	1.1359	0.0318
1.355	1.7596	0.1434	1.1367	0.0234
1.36	1.8110	0.0513	1.1375	0.0085
1.3607	1.8394	0.0000	1.1376	0.0000

By comparing this with equation (19) we find since $N = 2$

$$\gamma_0 = 1 + \epsilon = \tfrac{4}{7}\,\alpha, \tag{27}$$

where γ_0 is the value of γ corresponding to $\varphi = 0$ (BF in Fig. 2). Placing $\alpha = 2.0920$ this gives $\epsilon = 0.1954$ and $\gamma_0 = 1.1954$. Taking this value of ϵ we can now calculate γ for other values of φ by equation (25). Table II. shows that γ, the radius vector from the point F, is almost exactly constant or in other words the path DBC is nearly an arc of a circle with F as its center.

TABLE II.

Radius Vector from the Point F.

φ. . . .	0.00	0.50	1.00	1.20	1.25	1.36073
γ. . . .	1.1954	1.1954	1.1959	1.1956	1.1953	1.1937

To determine the absolute dimensions of this model we must now apply the quantum theory. The completely successful applications of the quantum theory to atomic structure have been limited to atoms having only one electron. There is therefore no certainty as to how this

theory should be formulated for the case of the more complicated atoms. Bohr assumes[1] that "the 'permanent' state of any atomic system is determined by the condition that the angular momentum of every electron round the center of its orbit is equal to $h/2\pi$," but this is supposed to apply only to circular orbits. In the present case the momentum of each electron varies from zero to a maximum so that it might seem at first sight that Bohr's assumption is not applicable. However it is an attractive hypothesis to assume that in the case of coupled electrons, the quantum theory is concerned not with the angular momentum possessed by one electron but rather with the angular momentum which, by being transferred from electron to electron, circulates in each of two directions about the nucleus. The discrete nature of quanta suggests that units of angular momentum have something analogous to independent physical existence. In the model we are considering we may imagine that when the electrons are at the midpoints of their paths, each possesses one quantum of angular momentum. As the electrons move, the velocity decreases, but we may consider that the momentum, instead of disappearing, is transferred from one electron to the other. Thus when the electrons have reached the ends of their paths, the momentum is not zero but each electron has one half quantum of clockwise and one half quantum of counter-clockwise angular momentum. One advantage of this viewpoint is that it indicates a very close analogy between Bohr's model and the model under consideration.

Let us therefore place the angular momentum of the electrons at the midpoint of their orbits equal to $h/2\pi$:

$$2\pi m r_0^2 \varphi_0' = h. \tag{28}$$

Whence by (21) and (5) we find

$$r_0 = \frac{a_0}{\alpha}. \tag{29}$$

The energy of the atom is then found by substituting this and $N = 2$ in (23) and combining with (8):

$$W = W_0 \alpha (7 - 2\alpha). \tag{30}$$

From (28), (29), (5) and (6) we obtain

$$\varphi_0' = \alpha^2 \omega_0. \tag{31}$$

Placing $\alpha = 2.0920$ gives us the absolute dimensions of the helium atom according to this model:

$$\begin{aligned} r_0 &= 0.4780 a_0 &&= 0.2535 \times 10^{-8} \text{ cm.}, \\ W &= 5.8911 W_0 &&= 1.2707 \times 10^{-10} \text{ erg.}, \\ \varphi_0' &= 4.3765 \omega_0 &&= 1.8094 \times 10^{17} \text{ radians per sec.} \end{aligned} \tag{32}$$

[1] Phil. Mag., *26*, 875 (1913).

From these results we can now calculate from Table I. the data for the motion of the electrons in absolute units. Thus to express time coördinate in seconds we divide θ by the above value of φ_0'. The angular velocity at any point is found by multiplying $d\varphi/d\theta$ by φ_0'. The radius vector is expressed in cm. by multiplying ρ by r_0 and the radial velocity is obtained by multiplying $d\rho/d\theta$ by $r_0\varphi_0'$ which is the linear velocity of the electrons as they pass the midpoint of their paths (4.587×10^8 cm. per sec.). The time taken for one complete oscillation of the electrons is thus 4.067×10^{-17} seconds, while the number of oscillations per second is 2.459×10^{16}.

The energy of the singly charged helium ion according to Bohr is $W = 4W_0$. The difference between this and the value of W which we have found for our model ($1.8911 \times W_0$) corresponds to the energy required to ionize the helium atom. Expressing this in volts we obtain for the ionizing potential of helium the value 25.62 volts. This agrees with Frank and Knipping's value 25.4 ± 0.25 volts within the probable experimental error. Horton and Davies[1] found the ionizing potential to be 25.7 volts, and K. T. Compton, in a very recent paper,[2] finds 25.5 volts.

In Bohr's model for the helium atom the distance between the electrons remains constant. For this case, we may place in (19) $\cos\varphi = 1$; $\rho = 1$, and $d\varphi/d\theta = 1$, and thus obtain

$$\alpha = N - \tfrac{1}{4}.$$

For the helium atom placing $N = 2$ we find $\alpha = 7/4$. By substituting this value of α in equations (29), (30) and (31) we obtain that same values[3] for r, W and ω that Bohr found for the helium atom,[4] namely:

$$\begin{aligned} r_B &= \tfrac{4}{7}\, a_0 &&= 0.3031 \times 10^{-8} \text{ cm.}, \\ W_B &= \tfrac{4}{8}\tfrac{9}{}\, W_0 &&= 1.321 \times 10^{-10} \text{ erg.}, \\ \omega_B &= \tfrac{4}{1}\tfrac{9}{6}\, \omega_0 &&= 1.266 \times 10^{17} \text{ radians per sec.}, \\ & & & \quad (2.015 \times 10^{16} \text{ revolutions per sec.}). \end{aligned} \tag{33}$$

According to Bohr's model, the energy needed for ionization is $W_B - 4W_0$ and the ionization potential is thus 28.79 volts. This is 3.4 volts higher than Frank and Knipping's value and is thus not to be reconciled with their experiments.

We have seen from Table II. that the path of an electron in the oscillating model is nearly an arc of a circle whose radius γ_0 is given by

[1] Proc. Royal Soc., A, *95*, 408 (1919).

[2] Phil. Mag., *40*, 553 (1920).

[3] In a similar manner we may obtain Bohr's data for the hydrogen atom from (29), (23) and (31) by placing $\alpha = 1$ and $N = 1$.

[4] Phil. Mag., *26*, 489 (1913).

equation (27). Multiplying this by the value of r_0 as given by equation (29) we obtain $(4/7)a_0$ as the absolute value of this radius. By comparison with (33) we see that this is identical with r_B the radius of the orbit of the electrons in Bohr's model. We thus obtain the extraordinary result that notwithstanding the complicated nature of the interaction between the two electrons in the oscillating model, the paths of the electrons consist of unaltered portions of the circular orbits of the Bohr model which are merely displaced bodily along the diameter of the Bohr orbit as indicated in Fig. 2. The circle $AFA'F'$ represents the orbit of the Bohr model. The path CBD of the oscillating model (drawn to scale) is obtained by taking a portion NFM of the Bohr orbit and displacing it by the distance FB in the direction of the radius FO. The distances MD and NC are thus equal to FB, which is 0.1635 of the distance FO. The angle φ corresponding to the end of the path (at C) is 1.3607 radians or 77° 57′ 45″.

The above close relationship between the oscillating model and the Bohr model seems to constitute additional justification for the quantum assumption we have made. It is of interest to enquire, however, what results are obtained if the quantum condition is formulated in other ways?

Sommerfeld and others express the quantum condition by the equation

$$\int p dq = nh, \tag{34}$$

where q is a generalized coördinate and p is defined by equation (11). For the kind of model we are considering I have not been able to find any general method for determining what coördinates should be used. Let us try, however, to calculate the ionizing potential of helium by two different quantum assumptions, viz:

1. Let $q = \varphi$, so that we apply (34) to the angular momentum of the electrons about the nucleus. Then p is given by (12) and we then find by (17), (18) and (34), placing $n = 1$:

$$2\pi m r_0^2 \varphi_0' P = h, \tag{35}$$

where P is equal to P_1 defined by the equation

$$2\pi P_1 = \int \left(\frac{d\varphi}{d\theta}\right) \rho^2 d\varphi. \tag{36}$$

The integration is to be carried over a complete cycle. Equation (36) reduces to (28) if $P = 1$.

2. Let q represent the distance between P and B along the path DBC (Fig. 2). The quantum condition then becomes

$$\int mv ds = h, \tag{37}$$

where v is the linear velocity of the electron at the point P and s is the distance along the path that separates the point P from the midpoint B. This can be reduced to (35) if we place P equal to P_2 which is defined by

$$2\pi P_2 = \int \left[\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \left(\frac{d\varphi}{d\theta} \right)^2 \right] d\theta. \tag{38}$$

From equation (35), by methods similar to those used in the derivation (29) and (30), we obtain

$$r_0 = \frac{a}{\alpha P^2}, \tag{39}$$

$$W = \alpha(7 - 2\alpha)P^2 W_0. \tag{40}$$

The values of P can be found by approximate integration from the data of Table I. but P_2 can also be calculated in terms of α as follows. By comparison of (38) with (10) and (22) we find that the quantity in brackets is equal to $2E_k r_0/\alpha e^2$. By reference to (17) and (23), placing $N = 2$ we then obtain

$$2\pi P_2 = \frac{(7 - 2\alpha)\varphi_0'}{\alpha W} \int E_k dt.$$

But

$$\int E_k dt = 4\bar{E}_k t_c,$$

where $\bar{E}_k$ is the time average of the kinetic energy of each of the electrons, and t_c is the time taken by an electron to move from B to C. Bohr and Sommerfeld have shown that the total energy $(-W)$ of any atom is equal to the mean kinetic energy with its sign reversed. Therefore $2\bar{E}_k = W$ and by considering (17) we then find

$$P_2 = \frac{(7 - 2\alpha)\theta_c}{\pi\alpha}. \tag{41}$$

Placing $\alpha = 2.0920$ and $\theta_c = 1.8394$ (Table I.), we thus obtain $P_2 = 0.7881$. By approximate integration of equation (38) by the data of Table I. we find $P_2 = 0.7886$. The closeness of this agreement affords a check on the accuracy of the data in Table I. Substituting the value of P_2 in (40) we find W and thus calculate the ionizing potential as -4.58 volts.

In a similar way we find from (36) $P_1 = 0.7807$ which gives an ionizing potential of -5.55 volts.

The oscillating model for the helium atom is thus not compatible with the formulation of the quantum theory given by equation (34). On the other hand the success in calculating the ionizing potential by assuming the angular momentum at the midpoint of the path to be $h/2\pi$, as well as the

remarkable relationships with the Bohr model which were developed, lend strong support to the oscillating type of model and suggest new directions in which the quantum theory may be applied to atoms and molecules containing more than one electron.

The real test for the new model will probably be obtained through a study of the spectrum of helium in the neighborhood of 590 Å. for this spectrum should make it possible to calculate the ionizing potential with very high accuracy. It is also desirable to test the model by calculating other atomic properties such as refractive index, absorption and ionization by X rays. The mathematical difficulties appear rather serious but by no means insuperable. It is hoped that the data of Table I. may prove to be of service in such calculations.

RESEARCH LABORATORY,
GENERAL ELECTRIC CO.,
SCHENECTADY, N. Y.,
November 28, 1920.