SOME APPLICATIONS OF THE METHOD OF IMAGES—I.

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Synopsis.

Solution of Some Electrostatic Problems by the Method of Images. (*a*) *Charged Wire between Two Parallel Plates.*—In Part I. of this paper the authors obtain expressions for the potentials at any point between an infinitely long charged wire and two conducting infinite planes parallel to it, for the surface density of the induced charge at any point on the plates, and for the capacity per unit length of such a condenser. These expressions contain only circular and hyperbolic functions. (*b*) *Charged Wire Inside an Infinite Rectangular Tube.*—In Part II., the potential at any point inside of the tube is obtained by a single infinite summation of the potentials due to each of a singly infinite set of images as given by the expression in Part I. Thus, although the problem is essentially one of doubly periodic functions, the solution appears in circular and hyperbolic functions. Comparing a square tube with a circular tube of the same capacity, each with a small wire of a given size through its center, the square tube has the larger perimeter. Tables are given showing the variation of the capacity of a certain rectangular tube with the size of the wire at its center and with the position of a certain sized wire. (*c*) *Two or Four Charged Wires Inside a Rectangular Tube.*—In the case of certain symmetrical positions this problem can be solved immediately from the preceding results.

Introduction.

Green's theorem in the potential theory states that if the potential $V(x, y, z)$ in every point of a closed surface is given and also the value of ΔV in every point of the enclosed volume, then the potential V is uniquely determined in every point of that volume. This is the principle of the method of images, which we shall apply in the following cases:

I. The Capacity of an Infinite Wire between Two Infinite Parallel Planes.

Let a linear charge of e units per unit length be placed at $(a, 0)$ between the two earthed plates of infinite extent A and B of Fig. 1. Let the origin of the complex plane be taken at 0 and P be the point $z = x + iy$ at which the potential is required.

The images are shown at points z_1, z_{-1}, z_2, z_{-2}, etc., whose distances from P are r_1, r_{-1}, r_2, r_{-2}, etc. The potential at P due to the charge at a and the induced charges on the plates A and B will be calculated from the charge itself and its infinite set of images. The potential at P due to the charge at a is $V = c - 2e \log r_0$. The resultant potential at P is then the sum of such expressions, one for each charge.

$$V_P = C + 2e[-\log r_0 + \log r_1 + \log r_{-1} - \log r_2 - \log r_{-2} + \cdots],$$

$$V_P = C + 2e \log \frac{r_1 r_{-1} r_3 r_{-3} r_5 r_{-5} \cdots}{r_0 r_2 r_{-2} r_4 r_{-4} \cdots},$$

$$V_P = C + 2e \log \frac{\prod_{n=-\infty}^{+\infty} r_{2n-1}}{\prod_{n=-\infty}^{+\infty} r_{2n}}.$$

But $r_0 = |z - z_0|$, $r_1 = |z - z_1|$, $r_{-1} = |z - z_{-1}|$, $\cdots$, $r_n = |z - z_n|$, $r_{-n} = |z - z_{-n}|$; moreover $\log r = \log |z| = R \log z$, where R denotes the real part of the logarithm. For all images of positive charge

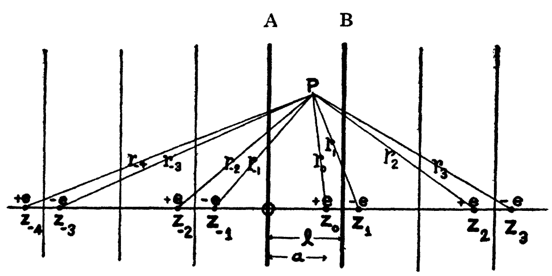

Fig. 1.

$z_{\pm 2n} = \pm 2nl + a$. For all images of negative charge $z_{\pm 2n-1} = \pm 2nl - a$. For a point taken on the plate A the potential is zero and the distances $r_0 = r_{-1}$, $r_1 = r_{-2}$, etc. Then $C = 0$; hence

$$V_P = 2e \log \frac{\Pi |z - z_{2n-1}|}{\Pi |z - z_{2n}|},$$

$$V_P = 2eR \log \frac{(z + a)(z + a - 2l)(z + a + 2l)(z + a - 4l)\cdots}{(z - a)(z - a + 2l)(z - a - 2l)(z - a + 4l)\cdots},$$

$$V_P = 2eR \log \frac{\frac{\pi}{2l}(z + a)\left[1 - \left(\frac{z + a}{2l}\right)^2 \frac{\pi^2}{\pi^2}\right] \times \left[1 - \left(\frac{z + a}{2l}\right)^2 \frac{\pi^2}{4\pi^2}\right]\left[1 - \left(\frac{z + a}{2l}\right)^2 \frac{\pi^2}{(3\pi)^2}\right]\cdots}{\frac{\pi}{2l}(z - a)\left[1 - \left(\frac{z - a}{2l}\right)^2 \frac{\pi^2}{\pi^2}\right] \times \left[1 - \left(\frac{z - a}{2l}\right)^2 \frac{\pi^2}{(2\pi)^2}\right]\left[1 - \left(\frac{z - a}{2l}\right)^2 \frac{\pi^3}{(3\pi)^2}\right]\cdots},$$

$$V_P = 2eR \log \frac{\sin \pi \left(\frac{z + a}{2l}\right)}{\sin \pi \left(\frac{z - a}{2l}\right)},$$

$$V_P = 2eR \log \frac{\sin \frac{\pi}{2l}(x + a) \cosh \frac{\pi}{2l} y + i \cos \frac{\pi}{2l}(x + a) \sinh \frac{\pi}{2l} y}{\sin \frac{\pi}{2l}(x - a) \cosh \frac{\pi}{2l} y + i \cos \frac{\pi}{2l}(x - a) \sinh \frac{\pi}{2l} y},$$

$$\text{(1)} \quad V_P = e \log \frac{\cosh \frac{\pi}{l} y - \cos \frac{\pi}{l}(x + a)}{\cosh \frac{\pi}{l} y - \cos \frac{\pi}{l}(x - a)}.$$

Now let us consider a wire of finite radius r having the same charge e per unit length. The potential of the surface of the wire is assumed to be the same as the potential at a point r units distant from z_0. This is equivalent to assuming that the equipotential surface at a distance r from z_0 is a circle and is quite accurately true for small values of r when the wire is not too near one of the plates. For the potential of the wire we choose

$$y = r; \qquad x = a;$$

$$V_w = e \log \frac{\cosh \frac{\pi}{l} r - \cos \frac{\pi}{l} 2a}{\cosh \frac{\pi}{l} r - \cos \frac{\pi}{l}(a - a)}.$$

The capacity per unit length of the system is therefore

$$C = \frac{e}{V_w} = \frac{1}{\log \frac{\cosh \frac{\pi}{l} r - \cos \frac{\pi}{l} 2a}{\cosh \frac{\pi}{l} r - 1}},$$

If $2a = l$, the capacity will be equal to:

$$C = \frac{1}{\log \frac{\cosh \frac{\pi}{l} r + 1}{\cosh \frac{\pi}{l} r - 1}}.$$

Expanding the cosh in infinite series we obtain approximately:

$$C = \frac{1}{2 \log \frac{4a}{\pi r}}.$$

From equation (1) we find the electric force in the x direction E_x as follows:

$$\frac{\partial V}{\partial x} = -E_x = e\frac{\pi}{l}\left\{\frac{\sin\frac{\pi}{l}(x+a)}{\cosh\frac{\pi}{l}y - \cos\frac{\pi}{l}(x+a)} - \frac{\sin\frac{\pi}{l}(x-a)}{\cosh\frac{\pi}{l}y - \cos\frac{\pi}{l}(x-a)}\right\}.$$

The surface density at any point on the plate A is found by

$$\sigma = -\frac{1}{4\pi}\left(\frac{\partial V}{\partial x}\right)_{x=0} = \frac{e}{2l}\,\frac{\sin\frac{\pi}{l}a}{\cos\frac{\pi a}{l} - \cosh\frac{\pi}{l}y}.$$

σ is a maximum for $y = 0$

$$\sigma_{y=0} = \frac{e}{2l}\,\frac{\sin\frac{\pi}{l}a}{\cos\frac{\pi a}{l} - 1}$$

$$= \frac{e}{2l}\operatorname{cotg}\frac{\pi a}{2l}.$$

II. The Capacity of a Circular Wire in a Rectangular Cylinder.

We shall begin with the potential due to a linear charge. Let the cylinder have a height of l cm. and breadth of l' cm. (Fig. 2). The charge

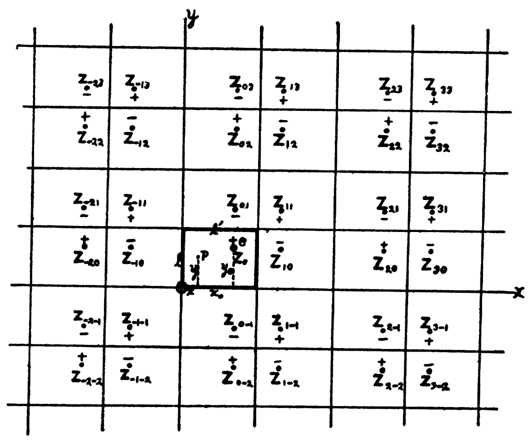

Fig. 2.

is at z_{00} or simply $z_0 = x_0 + iy_0$ and has e units per unit length. We shall remove the walls with their induced charges and consider in their stead all the infinite sets of images, which are distributed so that one of them is in each rectangle throughout the whole complex plane. The potential at a point $P(z = x + iy)$ will first be calculated by obtaining formula (1) in generalized form applying to any row of images and then taking a single summation for all rows.

The expression for the potential due to any row of images is given by formula (1) when y is replaced by the perpendicular distance from the point P to the row of images. In Fig. 2 such a distance is represented for any row by one of the expressions:

$$(a)\ y - y_0 + 2nl,$$

$$(b)\ y + y_0 + 2nl,$$

where n is any positive or negative integer. The distances $x + a$, $x - a$, and l in Fig. 1 become $x + x_0$, $x - x_0$, and l' when applied to Fig. 2. The potential at P is the algebraic sum of the potentials due to all such rows of images. Noting that the potentials due to the rows whose distances are given by (b) are negative, we have

$$(2)\quad V_P = e\sum_{n=-\infty}^{+\infty}\log\frac{\left[\cosh\frac{\pi}{l'}(2nl + y - y_0) - \cos\frac{\pi}{l'}(x + x_0)\right]\times\left[\cosh\frac{\pi}{l'}(2nl + y + y_0) - \cos\frac{\pi}{l'}(x - x_0)\right]}{\left[\cosh\frac{\pi}{l'}(2nl + y - y_0) - \cos\frac{\pi}{l'}(x - x_0)\right]\times\left[\cosh\frac{\pi}{l'}(2nl + y + y_0) - \cos\frac{\pi}{l'}(x + x_0)\right]}.$$

This problem is essentially that of the conformal transformation of the rectangle with a singular point within of character log $1/r$ and is treated in many texts on elliptic functions. See Greenhill, Elliptic Functions, §§ 273–275; and Kneser, Die Integral Gleichungen und ihre Anwendung in der Math. Physik, p. 137. The usual solution of this problem in elliptic functions treats the whole set of images at once and is expressible in sigma or theta functions. The same expressions may be derived as follows by associating with each image a factor of a sigma function.

The coördinates of the images (Fig. 2) are given as follows: for positive images

$$z_{\pm 2m,\ \pm 2n} = z_0 \pm 2ml' \pm i2nl,$$

$$z_{\pm 2m-1,\ \pm 2n-1} = -\ z_0 \pm 2ml' \pm i2nl,$$

where $z_0 = x_0 + iy_0$ and $\bar{z}_0 = x_0 - iy_0$; for negative images

$$z_{\pm 2m,\ \pm 2n-1} = \bar{z}_0 \pm 2ml' \pm i2nl,$$

$$z_{\pm 2m-1,\ \pm 2n} = -\ \bar{z}_0 \pm 2ml' \pm i2nl,$$

then

$$V_P = 2eR \log \prod_{\substack{m=-\infty \\ n=-\infty}}^{+\infty} \frac{(z + \bar{z}_0 + \Omega)(z - \bar{z}_0 + \Omega)}{(z + z_0 + \Omega)(z - z_0 + \Omega)}, \tag{2a}$$

where $\Omega = 2ml' + i2nl$. Dividing each term by Ω and multiplying and dividing by a proper exponential factor, we may associate each term with a sigma function

$$\sigma(z) = z\Pi'\left(1 + \frac{z}{\Omega}\right)\epsilon^{z/\Omega + \frac{1}{2}(z/\Omega)^2}.$$

Reduction leaves

$$V_P = 2e \log \frac{\sigma(z + \bar{z}_0)\sigma(z - \bar{z}_0)}{\sigma(z + z_0)\sigma(z - z_0)},$$

a well-known formula which with proper interpretation has been applied to vertex motion and the flow of heat.

The sigma functions may be expanded into a single convergent product as follows:

$$\sigma(z) = \epsilon^{\eta_1 z^2/2\omega_1} \frac{2\omega_1}{\pi} \sin \frac{\pi z}{\omega_1} \prod_{n=1}^{\infty} \frac{\sin \pi\left(\dfrac{2n\omega_2 - z}{2\omega_1}\right) \sin \pi\left(\dfrac{2n\omega_2 + z}{2\omega_1}\right)}{\sin^2 \pi n \dfrac{\omega_2}{\omega_1}},$$

where

$$\eta_1 = \frac{\pi^2}{2\omega_1}\left[\frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{\sin^2\left(i\pi n \dfrac{\omega_2}{\omega_1}\right)}\right],$$

$\omega_1 = l'$, and $\omega_2 = il$. After considerable reduction we get identically equation (2).

It was suggested to the authors that for purposes of computation it is sometimes easier to use theta functions instead of the sigma function. The functions are connected in the following way:

$$\sigma(z) = A\epsilon^{2\eta_1\omega\left(\frac{z}{2\omega_1}\right)^2} \vartheta_1\left(\frac{z}{2\omega_1}\right).$$

Hence,

$$\frac{\sigma(z + \bar{z}_0)\sigma(z - \bar{z}_0)}{\sigma(z + z_0)\sigma(z - z_0)}$$

$$= \epsilon^{2\eta_1\omega_1\left[\left(\frac{z+\bar{z}_0}{2\omega_1}\right)^2 + \left(\frac{z-\bar{z}_0}{2\omega_1}\right)^2 - \left(\frac{z+z_0}{2\omega_1}\right)^2 - \left(\frac{z-z_0}{2\omega_1}\right)^2\right]} \frac{\vartheta_1\left(\dfrac{z + \bar{z}_0}{2\omega_1}\right)\vartheta_1\left(\dfrac{z - \bar{z}_0}{2\omega_1}\right)}{\vartheta_1\left(\dfrac{z + z_0}{2\omega_1}\right)\vartheta_1\left(\dfrac{z - z_0}{2\omega_1}\right)}$$

$$= \epsilon^{\frac{-4\eta_1}{\omega_1} i x_0 y_0} \frac{\vartheta_1\left(\dfrac{z + \bar{z}_0}{2\omega_1}\right)\vartheta_1\left(\dfrac{z - \bar{z}_0}{2\omega_1}\right)}{\vartheta_1\left(\dfrac{z + z_0}{2\omega_1}\right)\vartheta_1\left(\dfrac{z - z_0}{2\omega_1}\right)}.$$

Substituting this expression in equation (2) and using the classical expansion of the theta functions we find essentially the same result as before.

Now let us replace the linear charge by a wire of finite radius r. If the surface of the wire coincided with an equipotential surface, then the potential could be calculated exactly from formula (2). This is very nearly the case when the dimensions l and l' are large compared with r and when the wire is not too near the walls of the cylinder. At the point P of Fig. 3 we have:

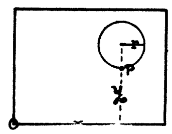

Fig. 3.

$$x + x_0 = 2x_0,$$
$$x - x_0 = 0,$$
$$y - y_0 = -r,$$
$$y + y_0 = 2y_0 - r.$$

Then

$$(3)\quad V_w = e\sum_{-\infty}^{+\infty}\log\frac{\left[\cosh\frac{\pi}{l'}(2nl - r) - \cos\frac{\pi}{l'}2x_0\right]\times\left[\cosh\frac{\pi}{l'}(2nl + 2y_0 - r) - 1\right]}{\left[\cosh\frac{\pi}{l'}(2nl - r) - 1\right]\times\left[\cosh\frac{\pi}{l'}(2nl + 2y_0 - r) - \cos\frac{\pi}{l'}2x_0\right]}.$$

By definition the capacity C of the system per unit length is equal to

$$C = \frac{e}{V_w}.$$

Instead of (3) we may write:

$$(4)\quad V_w = e\left\{\begin{aligned}&\sum_{n=0}^{\infty}\log\frac{\cosh\frac{\pi}{l'}(2nl + r) - \cos\frac{\pi}{l'}2x_0}{\cosh\frac{\pi}{l'}(2nl + r) - 1}\\ &\quad + \sum_{n=1}^{\infty}\log\frac{\cosh\frac{\pi}{l'}(2nl - r) - \cos\frac{\pi}{l'}2x_0}{\cosh\frac{\pi}{l'}(2nl - r) - 1}\\ &\quad + \sum_{n=0}^{\infty}\log\frac{\cosh\frac{\pi}{l'}(2nl - 2y_0 + r) - 1}{\cosh\frac{\pi}{l'}(2nl - 2y_0 + r) - \cos\frac{\pi}{l'}2x_0}\\ &\quad + \sum_{n=1}^{\infty}\log\frac{\cosh\frac{\pi}{l'}(2nl + 2y_0 - r) - 1}{\cosh\frac{\pi}{l'}(2nl + 2y_0 - r) - \cos\frac{\pi}{l'}2x_0}.\end{aligned}\right.$$

Calculations from formulæ (1) to (5) are made simple by use of the Smithsonian Institute Tables of Hyperbolic Functions. Where more than three or four significant figures are desired, the hyperbolic functions can be built up by means of the tables of exponentials in the same book.

We shall next compare the relative dimensions of circular and square cylindrical condensers of the same capacity. The capacity per unit length of a condenser formed of two concentric cylinders is given by

$$C = \frac{1}{2\log\frac{R}{r}},$$

where R is the radius of the outer cylinder and r that of the inner cylinder. Now let the outer cylinder have a square cross-section and the inner cylinder be a wire of the same radius r as before. In the square cylinder $l = l' = 2x_0 = 2y_0$. If r/l is small in comparison to unity, it may be dropped from each term of equation (4) except in the denominator of the first term when $n = 0$. We obtain:

$$(5)\quad V_0 = e\left\{\begin{array}{c}\log 2 - \log\left(\cosh\frac{\pi r}{l} - 1\right) + 2\log\frac{\cosh\pi - 1}{\cosh\pi + 1} \\ + 2\sum_{n=1}^{\infty}\log\frac{\cosh 2\pi n + 1}{\cosh 2\pi n - 1} + 2\sum_{n=1}^{\infty}\log\frac{\cosh(2n-1)\pi - 1}{\cosh(2n-1)\pi + 1}\end{array}\right\}.$$

All the terms except the second are purely numerical quantities, which may be represented by $\log N$, hence

$$V_0 = e\log\frac{N}{\cosh\frac{\pi r}{l} - 1}.$$

The capacities of the square and circular condensers are equal if

$$2\log\frac{R}{r} = \log\frac{N}{\cosh\frac{\pi r}{l} - 1}$$

$$= \log\frac{N}{\frac{1}{2!}\left(\frac{\pi r}{l}\right)^2 + \frac{1}{4!}\left(\frac{\pi r}{l}\right)^4 + \cdots}.$$

Neglecting powers above the second:

$$\frac{R^2}{r^2} = \frac{2N}{\left(\frac{\pi r}{l}\right)^2} \quad \text{or} \quad R = \frac{\sqrt{2N}}{\pi}l,$$

$$N = 1.43555 \quad \text{and} \quad R = 0.539364l.$$

The circular cylinder lies partly outside the square cylinder. The perimeter of the circular cylinder is $2\pi R = 3.36l$, which is smaller than that of the square cylinder $4l$. This might have been expected from the fact that the average distance from the center to all points on the square is less than the radius of a circle of the same perimeter.

If we take a square of diagonal 2 cm. and a wire in the center of radius $r = 0.001$ cm. the formulæ (4) and (5) give the same result to within 0.2 per cent.

By means of formula (3) or (4) the capacity of a square outer cylinder of side $l = l' = 2\pi$ was calculated per unit length for different sizes of wire placed at the center. These values are given in Table I.

Table I.

r in cm.	0.0005	.001	.002	.01	.02	.06	.1
C in E.S.U.	.300	.326	.3565	.455	.518	.657	.755

Table II. contains the values of the capacity of a rectangular cylinder π cm. high ($l' = \pi$) 2π cm. wide ($l = 2\pi$) with a wire of radius $r = 0.01$ cm. half way between the upper and lower walls. Values of C are given for different positions of the wire as it is moved toward the side wall. The capacity changes but little for positions of the wire near the center.

Table II.

$y_0 = \pi/2$; $l = 2\pi$, $l' = \pi$, $r = .01$ cm.

x_0 in cm.	π	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{19}{12}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$
C in E.S.U.	.501	.503	.508	.514	.523	.580

The electric force in the x direction $E_x = -(\partial V_P/\partial x)$ in any point P, and the surface density of the induced charge in any point of the side wall ($x = 0$) is equal to

$$\sigma = -\frac{1}{4\pi}\left(\frac{\partial V}{\partial x}\right)_{x=0}.$$

From (2) we deduce:

$$-E_x = \frac{\partial V_P}{\partial x} = e\frac{\pi}{l'}\sum_{n=-\infty}^{+\infty}\left\{\frac{\sin\frac{\pi}{l'}(x+x_0)}{\cosh\frac{\pi}{l'}(2nl+y-y_0)-\cos\frac{\pi}{l'}(x+x_0)}\right.$$
$$\left.-\frac{\sin\frac{\pi}{l'}(x-x_0)}{\cosh\frac{\pi}{l'}(2nl+y-y_0)-\cos\frac{\pi}{l'}(x-x_0)}\right.$$

$$+ \frac{\sin \frac{\pi}{l'}(x - x_0)}{\cosh \frac{\pi}{l'}(2nl + y + y_0) - \cos \frac{\pi}{l'}(x - x_0)} - \frac{\sin \frac{\pi}{l'}(x + x_0)}{\cosh \frac{\pi}{l'}(2nl + y + y_0) - \cos \frac{\pi}{l'}(x + x_0)} \Bigg\}.$$

We shall finally apply the previous results to the two and four wire cable in rectangular conduits. From the distribution of images in Fig. 2 it is evident at once, that the capacity of a system represented by Fig. 4 can be found as follows: Let the charges per unit length be $+ e$ and $- e$ and placed at equal distances from the walls AF and CD and on a line

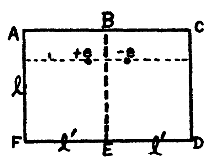

Fig. 4.

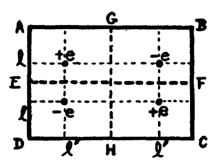

Fig. 5.

parallel to AC. The distribution of the field is not changed by inserting a conducting partition BE. This plate would become charged positively on the right-hand side and negatively on the left-hand side and the system appears as two equal condensers connected in series. Then if equation (3) is written in the form $V_w = ef(x_0, y_0, l, l', r)$, the capacity of $ABEF$ is equal to $(e/V_w) = (1/f)$ and the capacity of $ACDF$ containing both wires is equal to $1/2f$.

If four wires are placed symmetrically as shown in Fig. 5, two walls, EF and GH, may be inserted without disturbing the original field of force. We have two sets of condensers joined in series, each set consisting of two condensers in parallel. This combination has the capacity of any single condenser.

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