# THEORY AND CALCULATION OF VARIABLE ELECTRICAL SYSTEMS.

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SYNOPSIS.

*Variable Electrical Systems* are defined as those in which either the circuit elements (resistance, inductance or capacity) are explicit time functions or those in which the relation between current and applied E.M.F. is non-linear. In the present paper, the *theoretical methods of solution* of "invariable systems" are extended to include variable systems by integral equations of the Volterra type. A number of representative problems are worked through to indicate the appropriate mathematical procedure and the applicability of the method to both transient and steady state phenomena.

THE symbolic or operational method of solution of problems in electric circuit theory is a highly developed and very serviceable mathematical tool which is responsible in considerable degree for the rapid developments of the more abstract side of electrical engineering where the problems encountered are of such character as to render essential an adequate theoretical guide in predicting and interpreting phenomena. The method, however, is explicitly limited in its application to those physical systems or networks which may be mathematically described by a set of linear differential equations in which the coefficients of the differential operators are constants. Physically this means two things; first the currents are proportional to the applied forces, and secondly the circuit elements of the system (resistance, inductance and capacity) are invariable.[1] A system or network in which these restrictions hold will be termed invariable. In the great majority of problems these restrictions are not serious from a practical standpoint, since the departures from the requirements of the ideal invariable systems are usually small, and can usually be taken into account by indirect methods. Theoretically, however, these restrictions are undesirable and furthermore an increasing number of problems is being encountered in which we are directly concerned with departures from the ideal requirements. Such systems, for example, are the vacuum tube where the current is not

[1] For an interesting discussion of variable electrical systems from the standpoint of dynamics, see "Theory of Variable Electrical Systems." PHYS. REV., Aug., 1917, by H. W. Nichols.

proportional to the applied electromotive force, and where the operation of the device as a detector or modulator is strictly conditioned by the fact that the characteristic is non-linear; the induction generator where the mutual inductance between primary and secondary is variable; and the microphone transmitter circuit where the resistance is varied by external means. The differential equations which describe variable systems of which the foregoing examples are typical, have, of course, been extensively studied from the standpoint of pure mathematics. At the same time curiously little application of general methods of solution appears to have been made to such physical problems, in spite of their great and increasing technical importance. In the present paper it will be shown that the differential equations of variable electrical systems can be thrown at once into the form of integral equations, by aid of which formulation the solution is quite simply expressible in terms of the solution of the corresponding invariable system. For a discussion of the application of integral equations to circuit theory the reader is referred to my paper on Transient Oscillations (March, 1919, Proc. A.I.E.E.).

The theoretical analysis of the present paper is based on the following fundamental theorem, which is derived and discussed in the paper referred to above:[1]

*If the current flowing in any branch (or mesh) of a network in response to a "unit E.M.F." (zero before, unity after, time $t = 0$) is denoted by $A(t)$, then the current $I(t)$ which flows in response to the arbitrary applied E.M.F. $f(t)$ is given by the formula:*

$$I(t) = \frac{d}{dt}\int_0^t f(y)A(t-y)dy. \tag{1}$$

The function $A(t)$ which characterizes the network will be termed the *indicial admittance*.[2]

It is important to observe that formula (1) is restricted in its direct applicability to *invariable systems* as hereinbefore defined; nevertheless, as will be shown, it enables us to deal successfully with systems which are not so restricted; that is ones which contain variable circuit elements, and ones in which the relation between current and voltage is non-linear.

Since the present paper is concerned exclusively with the extension of circuit theory to systems which include variable circuit elements it is assumed in the following that the indicial admittance $A(t)$ of the invari-

[1] See also a paper by T. C. Fry on the Solution of Problems in Circuit Theory, which appeared in the Physical Review, August, 1919.

[2] This terminology is suggested by the physical and mathematical significance of the function.

able or unvaried system is known. For a full discussion of the theory and calculation of invariable systems and of methods for determining the indicial admittance the reader is referred to Fry's paper, PHYSICAL REVIEW, August, 1919, and my previous papers, PHYSICAL REVIEW, September, 1917, and Proc. A.I.E.E., March, 1919.

Before developing the theory in more general terms we shall consider the simplest possible example which may be termed:

THE MICROPHONE TRANSMITTER PROBLEM.

Consider a circuit containing an invariable impedance denoted operationally by $Z$ in series with a variable resistance $rf(t)$, to which is applied the E.M.F. $E(t)$. The equation for the resultant current $I(t)$ may be written in operational notation as:

$$(Z + rf(t))I(t) = E(t)$$

or

$$ZI(t) = E(t) - rf(t)I(t). \tag{2}$$

Inspection of equation (3) shows at once that $I(t)$ is equal to the current flowing in an *invariable circuit* of impedance $Z$ in response to the applied E.M.F. $E(t) - rf(t)I(t)$; consequently if the *indicial admittance* of the *invariable* or *unvaried circuit* ($r = 0$) be denoted by $A(t)$ it follows at once from (1) that the current in the actual circuit may be written as:

$$I(t) = \frac{d}{dt}\int_0^t E(y)A(t-y)dy - r\frac{d}{dt}\int_0^t A(t-y)f(y)I(y)dy. \tag{3}$$

The first term on the right-hand side of (3) is by comparison with formula (1) simply the current which would flow in the unvaried circuit ($r = 0$) in response to the applied E.M.F. $E(t)$; denoting this current by $I_0(t)$ we have:

$$I(t) = I_0(t) - r\frac{d}{dt}\int_0^t A(t-y)f(y)I(y)dy. \tag{4}$$

Equation (4) is an integral equation of the Volterra type, methods for the formal solution of which are well known. The following series solution recommends itself by reason of the direct physical significance of each term of the series:

$$I(t) = I_0(t) - I_1(t) + I_2(t) - I_3(t) + \cdots + (-1)^n I_n(t), \tag{5}$$

where the successive terms of the series are defined by the relations:

$$\begin{aligned} I_1(t) &= r\frac{d}{dt}\int_0^t A(t-y)f(y)I_0(y)dy, \\ I_{n+1}(t) &= r\frac{d}{dt}\int_0^t A(t-y)f(y)I_n(y)dy. \end{aligned} \tag{6}$$

Referring to the fundamental formula (1) it will be observed that the successive terms of the series (5) as defined by (6) admit of direct physical interpretation as follows: $I_1(t)$ is equal to the current which would flow in the unvaried circuit of impedance $Z$ in response to the fictitious applied E.M.F. $rf(t)I_0(t)$; $I_2(t)$ is equal to the current in the same circuit in response to the fictitious applied E.M.F. $rf(t)I_1(t)$; etc. That is to say the product of the variable resistance $rf(t)$ into each component current of the series (5) acts like an additional component E.M.F. in the unvaried circuit to produce an additional component current.

The solution is of course complete in that it formulates the resultant current for all types of applied forces and all possible forms of resistance variations. In particular if the impressed E.M.F. and the resistance variation are both periodic or sinusoidal the solution includes both transient as well as steady states. In this case, if we are concerned only with the ultimate steady state of the network it is not necessary to evaluate the definite integrals of (6). All that is necessary in order to write down the steady state solution corresponding to the series solution (5) is to express the product of $rf(t)$ into each component current (starting with $I_0(t)$) as a periodic time function, and then to evaluate the succeeding component current by operating on the periodic function with the impedance $Z$ in accordance with usual operational rules. This is considered in greater detail below.

To illustrate the solution (5) and (6) the simplest possible case will be dealt with: into a circuit of unit resistance $r$ and inductance $L = 1/a$ in which a steady current $I_0$ is flowing a resistance $r$ is suddenly inserted at time $t = 0$; required the resultant current $I(t)$. In this case we have:

$$\begin{aligned} A(t) &= \text{indicial admittance of unvaried circuit} \\ &= 1 - e^{-at}, \\ f(t) &= 1, \end{aligned}$$

and the integral equation of the problem is:

$$\begin{aligned} I(t) &= I_0 - r\frac{d}{dt}\int_0^t (1 - e^{-ay})I(t-y)dy \\ &= I_0 - ra\int_0^t I(t-y)e^{-ay}dy. \end{aligned} \tag{7}$$

If the solution is carried out as indicated in (5) and (6), and if the notation $at = x$ is introduced, we get without difficulty

$$I(t) = I_0\left\{\begin{array}{l} 1 - r(1 - e_1(x)e^{-x}) + r^2(1 - e_2(x)e^{-x}) \\ \quad - r^3(1 - e_3(x)e^{-x}) + \end{array}\right. \tag{8}$$

Where the function $e_n(x)$ is defined as:

$$e_n(x) = 1 + x/1! + x^2/2! + x^3/3! + \cdots + x^{n-1}/(n-1)!$$
$$= \text{first } n \text{ terms of the exponential series.}$$

For all finite values of the resistance increment $r$ the series (8) can be summed by aid of the identity

$$1 - e_n(x)e^{-x} = \int_0^x dx e^{-x} x^{n-1}/(n-1)!$$

Substitution of this identity in (8) gives

$$I(t) = I_0(1 - r\int_0^x e^{-(1+r)x}dx)$$
$$= I_0 \frac{1 + re^{-(1+r)x}}{1+r}.$$

A more interesting example than the former is presented when the applied E.M.F. and the resistance variation are both sinusoidal time functions. In this case if the frequency of the applied E.M.F. be denoted by $F = q/2\pi$ and that of the resistance variation by $f = p/2\pi$, it is easy to show that the current $I_0(t)$ in the unvaried circuit is ultimately[1] a steady state current of frequency $F$. This follows from the fact that the definite integral of (3) which defines the current $I_0(t)$ is resolvable into the ultimate steady state current corresponding to an applied force of frequency $F$, and the accompanying transient oscillations which ultimately die away. The fictitious E.M.F. which may be regarded as producing the component current $I_1(t)$ is $rf(t)I_0(t)$; this is ultimately the product of the two frequencies $F$ and $f$, and therefore resolvable into two terms of frequency $F+f$ and $F-f$ respectively. Carrying through this analysis it is easy to show that each component current is ultimately a steady-state but poly-periodic oscillation, as indicated in the following table.

| Component Current | Frequency | |
|---|---|---|
| $I_0$ | $F$ | |
| $I_1$ | $F+f, F-f$ | |
| $I_2$ | $F+2f, F, F-2f$ | (10) |
| $I_3$ | $F+3f, F+f, F-f, F-3f$ | |
| $I_4$ | $F+4f, F+2f, F, F-2f, F-4f$ | |

It is of importance to observe that the component currents involve,

[1] It hardly seems necessary to remark that the reference time $t = 0$ is purely arbitrary and that the resistance variation may start at such a time thereafter that $I_0(t)$ may be regarded as steady state during the entire time interval in which we are interested. Going farther, if we confine our attention to sufficiently large values of $t$, the whole process may be treated as steady state.

from a mathematical standpoint, multiple integrals of successively higher orders, the $n$th component, $I_n(t)$, involving a multiple integral of the $n$th order with respect to $I_0(t)$. Consequently the successive currents require longer and longer intervals of time to build up to their proximate steady-state values, so that the time required for the resultant steady-state to be arrived at cannot be inferred from the time constant of the unvaried circuit.

From table (10) it will be seen that the ultimate steady-state current is obtained by rearranging the series $I_0 + I_1 + I_2$ and is of the form

$$\sum_{n=-\infty}^{+\infty} A_n \cos (q + np)t + B_n \sin (q + np)t.$$

It is interesting to note that this series comes within the definition of a Fourier series only when $q = 0$ or an exact multiple of $p$. The steady-state solution is of very considerable importance and is considered in more detail in a succeeding section.

So far nothing has been said regarding the convergence of the formal series solution (5). For the case of variable resistance, however, it can be shown that the sufficient conditions for the absolute convergence of the series correspond to the physical restrictions imposed in a large and important class of problems. In the first place the series is absolutely convergent when $A(o) = 0$, since in this case the successive terms are related by the equation

$$I_{n+1}(t) = r\int^{t} A'(t - y)f(y)I_n(y)dy,$$

where

$$A'(t - y) = \frac{d}{dt}A(t - y).$$

In physical terms this restriction means that the branch of the network in which the variable resistance is located contains inductance also. In a large number of problems this condition is satisfied.

In the second place the solution is a power series in the parameter $r$, which fixes the size of the resistance variation. In general therefore the series will be absolutely convergent for some restricted range of values of $r$, which will, however, depend on the particular network under consideration. For the very important case of periodic resistance variations physical considerations restrict the maximum value of the variable resistance to a value less than the invariable resistance in the same branch. It is easy to see from physical considerations that in this case series (5) is absolutely convergent.

However a perfectly general restriction, imposed by physical conditions, is that $[1 + rA_{11}(0)f(t)] > 0$, since otherwise the branch would contain negative resistance. In view of this restriction an absolutely convergent series solution of equation (4) is obtainable by aid of the transformation.

$$J(t) = I(t)[(1 + rA(0)f(t)],$$
$$\phi(t) = f(t)/(1 + rA(0)f(t)).$$

In terms of $J$ and $\phi$ the integral equation (4) becomes

$$J(t) = I_0(t) - r\int_0^t A'(t-y)\phi(y)J(y)dy.$$

This integral equation has the absolutely convergent solution:

$$J(t) = J_0(t) - J_1(t) + J_2(t) - \cdots,$$

where the terms of the series are defined by the relations:

$$J_0(t) = I_0(t)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$J_{n+1}(t) = r\int_0^t A'(t-y)\phi(y)J_n(y)dy.$$

For the sake of its physical interpretation this last equation may be written as:

$$J_{n+1}(t) = -rA(0)\phi(t)J_n(t) + r\frac{d}{dt}\int_0^t A(t-y)\phi(y)J_n(y)dy.$$

Inspection of this equation shows that the term $J_{n+1}(t)$ may be physically interpreted as the difference in the currents flowing in the unvaried network and in a resistance $1/A(0)$ in response to the fictitious E.M.F. $r\phi(t)J_n(t)$. This interpretation is of value in enabling one to write down immediately the corresponding steady-state current in the important case of periodic applied forces and periodic resistance variations.

In the light of the foregoing example the extension of the method of solution to more complicated networks and to the case of simultaneous impedance variations in a plurality of branches of the network should present no difficulties. The appropriate procedure, however, will be briefly illustrated by an example of some practical importance which may be termed:

## The Induction Generator Problem.

In a sufficiently general form, this problem, which includes the fundamental theory of the dynamo, may be stated as follows:

Given an invariable primary and secondary circuit with a variable mutual inductance $Mf(t)$ which is an arbitrary but specified time function, and let the primary be energized by an E.M.F. $E(t)$ impressed in the circuit at the reference time $t = 0$; required the primary and secondary currents.

In operational notation the problem may be formulated by the equations:

$$
\begin{aligned}
Z_{11}I_1 - pMf(t)I_2 &= E(t) \\
- pMf(t)I_1 + Z_{22}I_2 &= 0
\end{aligned}
\tag{11}
$$

in which $Z_{11}$ and $Z_{22}$ are the self impedances of the primary and secondary respectively; $Mf(t)$ is the variable mutual inductance; $E(t)$ is the applied E.M.F. in the primary, and $p$ denotes the differential operator $d/dt$. By aid of the fundamental formula (1) these equations may be written down as the following simultaneous integral equations:

$$
\begin{aligned}
I_1(t) &= \frac{d}{dt}\int_0^t dyA_{11}(t-y)\left(E(y) + M\frac{d}{dy}[f(y)I_2(y)]\right), \\
I_2(t) &= M\frac{d}{dt}\int_0^t dyA_{22}(t-y)\frac{d}{dy}[f(y)I_1(y)].
\end{aligned}
\tag{12}
$$

In these equations $A_{11}(t)$ and $A_{22}(t)$ denote the indicial admittances of the primary and secondary circuits respectively (when $M = 0$); that is the currents in these circuits in response to a unit E.M.F. (zero before, unity after time $t = 0$). We of course assume that they are known or can be determined by usual methods.

It follows at once that the formal solution of these equations is the infinite series:

$$
I_1(t) = X_0(t) + X_2(t) + X_4(t) + \cdots + X_{2n}(t) + \cdots \tag{13}
$$

$$
I_2(t) = Y_1(t) + Y_3(t) + Y_5(t) + \cdots \tag{14}
$$

in which the successive terms of the series are defined as follows:

$$
\begin{aligned}
X_0(t) &= \frac{d}{dt}\int_0^t dyA_{11}(t-y)E(y) = I_0(t), \\
Y_1(t) &= M\frac{d}{dt}\int_0^t dyA_{22}(t-y)\frac{d}{dy}[f(y)X_0(y)], \\
X_2(t) &= M\frac{d}{dt}\int_0^t dyA_{11}(t-y)\frac{d}{dy}[f(y)Y_1(y)], \\
Y_3(t) &= M\frac{d}{dt}\int_0^t dyA_{22}(t-y)\frac{d}{dy}[f(y)X_2(y)], \qquad \text{etc.}
\end{aligned}
\tag{15}
$$

In the light of formula (1) the physical interpretation of the series solutions (13) and (14) follows at once. Thus $X_0(t)$ is equal to the current $I_0(t)$ flowing in the *isolated* primary in response to the applied E.M.F. $E(t)$; the first component current $Y_1(t)$ in the secondary is equal to the current which would flow in the isolated secondary in response to the applied E.M.F. $M(d/dt)f(t)X_0(t)$; $X_2(t)$, the second component current in the primary is equal to the current in the isolated primary in response to the applied E.M.F. $M(d/dt)f(t)Y_1(t)$; etc. The resultant currents are thus represented as built up by a to-and-fro interchange of energy between primary and secondary or by a series of successive reactions. In the important case where the applied E.M.F. and the variation of mutual inductance are both sinusoidal time functions of frequency $F$ and $f$ respectively, it is easy to show that each component current becomes ultimately equal to a set of periodic steady-state currents. Thus the component $X_0$ is ultimately singly periodic of frequency $F$; $Y_1$ is ultimately doubly periodic of frequencies $F + f$ and $F - f$; $X_2$ triply periodic of frequencies $F + 2f$, $F$ and $F - 2f$; $Y_3$ quadruply periodic of frequencies $F = 3f$, $F + f$, $F - f$, $F - 3f$; etc.

In connection with this solution attention should be called to a discussion of the same problem in a paper by Liebowitz (Proc. Inst. Radio Engineers, Dec., 1915). The method of solution there employed and credited by the author to M. I. Pupin is based entirely on steady-state concepts and is limited to sinusoidal impressed forces and inductance variations. With these restrictions the solution arrived at corresponds term by term to the steady-state part of the complete solution given above. While the two solutions thus become ultimately identical in the region where Pupin's solution is convergent and valid[1] the present treatment is of broader scope in that transient as well as steady-states and arbitrary forces and inductance variations are included in the complete solution.

It is beyond the scope of the present paper to go into a consideration of the energy relations of the induction generator, but a few deductions may be noted. In the case of resistance variations the ignored force which controls the variable resistance element neither supplies nor abstracts energy; consequently in this case all the energy consumed in the system is furnished by the electrical source. In the case of the induction generator, on the other hand, the ignored force which controls the variation of mutual inductance may either supply or abstract energy from the electrical system; in other words the system may act either as an induction generator or an induction motor, and the ignored force correspondingly as a source or sink of energy. In the simple case where the series is so rapidly convergent as to make the component currents of frequencies $F$ and $F + f$ alone of importance the system acts like an

[1] In the paper referred to, incorrect physical conclusions are deduced from the fact that the steady-state series diverges under certain conditions. As explained in the present paper no physical significance can in general be attached to the divergence of this series.

electrical generator or motor according as the frequency $f$ of inductance variation is greater or less than the frequency $F$ of the electrical supply. This agrees with the well-known fact that the induction motor acts like a generator when driven above synchronism. When the circuits are appropriately tuned to emphasize the higher harmonics the device is essentially a Goldschmidt generator which has been proposed for the production of radio frequency currents.

The formal series solutions are absolutely convergent in the majority of actual circuits. In special circuit arrangements, however, such as the Goldschmidt generator, where the primary and secondary are tuned to emphasize the higher harmonic currents, the series may be divergent. In this case some transformation, such as that discussed in connection with the preceding example must be introduced. For example, if the equations (12) are cleared of the differential operator, and if we introduce the functions

$$J_1(t) = I_1(t)/(1 - \mu_1\mu_2 f(t)), \qquad \mu_1 = MA_{11}'(0),$$
$$J_2(t) = I_2(t)/(1 - \mu_1\mu_2 f(t)), \qquad \mu_2 = MA_{22}'(0),$$

it is easily shown that the series solutions in $J_1$ and $J_2$ are absolutely convergent.

## The Solution for the Steady-State Oscillations.

For the very important case of periodic applied forces and periodic variations of circuit elements we are often concerned exclusively with the ultimate steady-state of the system, and not at all with the mode in which the steady-state is approached; that is, attention is restricted to the periodic oscillations which the system executes after transient disturbances have died away. In this case, if the periodic variations of circuit elements are sufficiently small the required steady-state is obtained in the form of a series by replacing each term of the complete series solution by its ultimate steady-state value; a process which is very simple in view of the physical significance of each term of the latter series. The procedure will be briefly illustrated in connection with the *microphone transmitter* problem, which is formulated and solved in equations (1)–(6). The variable resistance element will be taken as $r \cos pt$, and the current $I_0(t)$ as the *real part* of $I_0e^{iqt}$ where $I_0$ is in general complex, and the symbol $i$ denotes the imaginary operator $\sqrt{-1}$. The frequency $F$ of the impressed electric force is therefore $q/2\pi$ and the frequency $f$ of resistance variation is $p/2\pi$. The symbolic notation commonly used in the theory of alternating current will be employed and the symbolic impedances of the unvaried network at frequency $(q + np)/2\pi$ will be denoted by $Z_n$. This is obtained by methods long employed in alternating current calculations by replacing the operator $d/dt$ by $i(q + np)$ in the differential equations which describe the network or system. Similarly the impedance of the unvaried network at frequency $(q - np)/2\pi$ will be denoted by $Z_n'$.

If reference is now made to equations (6) and their physical significance kept in mind it is evident at once that the component current $I_1(t)$ is equal to the current in the unvaried network in response to the applied E.M.F.

$$rf(t)I_0(t) = (r/2)I_0(e^{i(q+p)t} + e^{i(q-p)t}).$$

Consequently, after transient effects have died away, the component current $I_1(t)$ is replaceable by

$$I_1 = (r/2)I_0\left(\frac{e^{i(q+p)t}}{Z_1} + \frac{e^{i(q-p)t}}{Z_1'}\right).$$

It will be understood, of course, that the real part of this complex expression is the actual solution, and that the imaginary part is to be discarded. Proceeding in precisely the same way with the second component current $I_2(t)$, it is ultimately replaceable by

$$I_2 = (r/2)^2 I_0\left\{\frac{e^{i(q+2p)t}}{Z_1Z_2} + \frac{e^{i(q-2p)t}}{Z_1'Z_2'} + \frac{e^{iqt}}{Z_0}\left(\frac{1}{Z_1} + \frac{1}{Z_1'}\right)\right\}.$$

Similarly,

$$I_3 = (r/2)^3 I_0\left\{\begin{aligned}&\frac{e^{i(q+3p)t}}{Z_1Z_2Z_3} + \frac{e^{i(q+p)t}}{Z_1}\left(\frac{1}{Z_1Z_2} + \frac{1}{Z_0Z_1} + \frac{1}{Z_0Z_1'}\right)\\ &\qquad + \frac{e^{i(q-3p)t}}{Z_1'Z_2'Z_3'} + \frac{e^{i(q-p)t}}{Z_1'}\left(\frac{1}{Z_1'Z_2'} + \frac{1}{Z_0Z_1'} + \frac{1}{Z_0Z_1}\right).\end{aligned}\right.$$

In this way the steady-state series solution is built up term by term, the component currents being poly-periodic as indicated in (10).

For sufficiently small impedance variations this method of solution works very well, and leads to a rapidly convergent solution. In other cases, however, the solution so obtained may be divergent, even when the complete series solution from which it is derived is absolutely convergent. The explanation of this lies in the fact that the steady-state series so obtained is the *sum of the limits* (as $t$ approaches infinity) of the terms of the complete series solution, whereas the actual steady-state is the *limit of the sum*. These are not in general equal; in particular the former may be and often is divergent when the latter is convergent.

In view of the foregoing considerations it is of great importance to develop another method of investigating the steady-state oscillations which avoids the difficulties in the formal series solution. The following method has suggested itself to the writer and works very well in cases where the previous form of solution fails. It should be stated at the outset, however, that the absolute convergence of the solution to be discussed, while reasonably certain in all physically possible systems, has not been established by a rigorous mathematical investigation which appears to present very considerable difficulties.

The method of solution will be elucidated in connection with the example discussed above under the title of the *microphone transmitter problem* and formulated in equations (1)–(6); the extension of the method of solution to more involved problems will be obvious. For the case of an applied E.M.F. of frequency $F = q/2\pi$ and a resistance variation of frequency $f = p/2\pi$ the formal series solution shows that the ultimate steady-state oscillations are of frequency $F$, $F \pm f$, $F \pm 2f$, $F \pm nf$. If the variable resistance is taken as $r \cos pt$ and the current $I_0(t)$ as the *real part* of $I_0 \exp (iqt)$ (where $I_0$ is in general complex), the following tentative solution suggests itself:

$$I(t) = A_0 e^{iqt} + \sum_{j=1}^{n} A_j e^{i(q+jp)t} + A_j' e^{i(q-jp)t} + R_n(t).$$

In this expression the coefficients $A_0$, $A_j$, $A_j'$, which are to be determined, are in general complex and the real part of the expression is alone to be retained in the final solution. The foregoing is of course equivalent to a trigonometric series but the exponential form is much more convenient to handle. In the summation the upper limiting index $n$ is a finite positive integer which may be assigned any desired value. The "remainder" $R_n(t)$ and the coefficients $A_0$, $A_j$, $A_j'$ are to be determined. If the steady-state solution is convergent the remainder $R_n(t)$ must approach zero as the index $n$ is indefinitely increased. The practical value of the solution will therefore depend on the rate of convergence of the series.

If the coefficients $A_0$, $A_j$, $A_j'$ are determined in accordance with the process developed below, it may be shown that the remainder $R_n(t)$ for large values of $t$ satisfies the integral equation

$$R_n(t) = -(r/2)\left(\frac{A_n}{Z_{n+1}} e^{i(q+(n+1)p)t} + \frac{A_n'}{Z'_{n+1}} e^{i(q-(n+1)p)t}\right) - r\frac{d}{dt}\int_0^t A(t-y)\cdot f(y)\cdot R_n(y)dy.$$

In this expression $Z_{n+1}$ and $Z'_{n+1}$ are the complex expressions for the impedance of the unvaried network at frequencies $(q + (n + 1)p)/2\pi$ and $(q - (n + 1)p)/2\pi$ respectively. They are obtained in accordance with well-known rules from the differential equations of the unvaried network by replacing the differential operator $d/dt$ by $i(q + (n + 1)p)$ and $i(q - (n + 1)p)$ respectively. The imaginary part of the foregoing expression is of course to be discarded.

The coefficients $A_0$, $A_j$, $A_j'$ are now determined by the following set of equations which are obtained by substitution of the assumed solution in the integral equation of the problem and then letting the time $t$ become indefinitely large.

$$\begin{aligned} A_n &= -h_n A_{n-1}, \\ A_n' &= -h_n' A'_{n-1}, \\ A_j &= -h_j(A_{j-1} + A_{j+1}), \\ A_j' &= -h_j'(A'_{j-1} + A'_{j+1}), \qquad j = (n-1), (n-2), \cdots, 2, \\ A_1 &= -h_1(A_2 + A_0), \\ A_1' &= -h_1'(A_2' + A_0), \\ A_0 &= I_0 - h_0(A_1 + A_1'). \end{aligned}$$

In these equations the symbol $h_j$ denotes $r/2Z_j$; similarly $h_j'$ denotes $r/2Z_j'$.

It will be observed that starting with $A_n$, $A_n'$ each coefficient is determinable in terms of the coefficient of next lower order. Thus from the first and second set of equations:

$$A_{n-1} = -h_{n-1}A_{n-2}\frac{1}{1 - h_{n-1}h_n},$$

$$A'_{n-1} = -h'_{n-1}A'_{n-2}\frac{1}{1 - h'_{n-1}h_n'}.$$

Continuing this process it is easy to show that

$$\begin{aligned} A_j &= -h_j C_{jn} A_{j-1}, \\ A_j' &= -h_j' C'_{jn} A'_{j-1}, \qquad j = n, (n-1), \cdots, 2, \end{aligned}$$

where $C_{jn}$ denotes the terminating continued fraction

$$C_{jn} = \cfrac{1}{1 - h_j h_{j+1}\cfrac{1}{1 - h_{j+1}h_{j+2}\cfrac{1}{1 - h_{j+2}h_{j+3} \cdot \cdots \cfrac{1}{1 - h_{n-1}h_n}}}}$$

and $C_{jn}'$ the corresponding expression in $h_j'$, $h_n'$.

Finally, therefore

$$A_1 = -h_1 C_{1n} A_0,$$
$$A_1' = -h_1' C_{1n}' A_0$$

and

$$A_0 = I_0 - h_0(A_1 + A_1')$$
$$= I_0/(1 - h_0 h_1 C_{1n} - h_0 h_1' C_{1n}').$$

The coefficients are thus all determined in terms of $I_0$ and the remainder $R_n(t)$ is given by an integral equation. It follows therefore that, provided the series converges, the coefficients $A_0$, $A_j$, $A_j'$ are the limits of the foregoing expressions as the index $n$ is made indefinitely large and the terminating continued fractions become infinite continued fractions. The complete solution therefore involves the evaluation of infinite continued fractions.

The practical value of this method of solution will depend, of course, on the rate of convergence of the continued fractions. While no rigorous proof has been obtained, it is believed that they are absolutely convergent for all physically possible systems, but this question certainly requires fuller investigation. Nevertheless any doubt regarding the convergence of the solution need not prevent the use of the method in a great many problems where physical considerations furnish a safe guide. For example this method of solution, when applied to the problem of the induction generator, discussed above leads to the usual simplified engineering theory of the induction generator and motor, besides exhibiting effects which the usual treatment either ignores or fails to recognize.

It seems worth while pointing out that the method of solution just discussed does not exclude the investigation of the transient disturbances which exist when the electrical forces are impressed on the system or when the impedance variations are initiated. To show this in connection with the variable resistance problem, let the steady-state current, as derived above, be denoted by $S(t)$ and the total current $I(t)$ by $S(t) + T(t)$. Substitution in equation (4) gives:

$$T(t) = I_0(t) - S(t) - r\frac{d}{dt}\int_0^t A(t-y)f(y)S(y)dy - r\frac{d}{dt}\int_0^t A(t-y)f(y)T(y)dy,$$

which determines the transient disturbance $T(t)$.

## Non-Linear Circuits.

In the previous examples discussed the variations of the variable circuit elements are assumed to be specified time functions, which is the same thing as postulating that these variations are controlled by ignored forces which do not explicitly appear in the statement and equations of the problem. We distinguish another type of variable circuit element where the variation is not an explicit time function, but rather a function of the current (and its derivatives) which is flowing through the circuit. For example the inductance of an iron-core coil varies with the current strength as a consequence of magnetic saturation. The equation of a circuit which contains such a variable element may be written down in operational notation

$$ZI + \phi(I) = E(t),$$

or

$$ZI = E(t) - \phi[I(t)]. \qquad (16)$$

In this equation $Z$ is, of course, to be taken as the impedance of the invariable part of the circuit, the indicial admittance of which is denoted by the usual symbol $A(t)$.

Equation (16) may be interpreted as the equation of the current $I(t)$ in a circuit of invariable impedance $Z$ when subjected to an applied E.M.F. $E(t) - \phi[I(t)]$; consequently by aid of formula (1), $I(t)$ is given by

$$I(t) = \frac{d}{dt}\int_0^t A(t-y)E(y)dy - \frac{d}{dt}\int_0^t A(t-y)\phi[I(y)]dy. \qquad (17)$$

The first integral is simply the current in the invariable circuit of impedance $Z$ in response to the applied E.M.F. $E(t)$; denoting this by $I_0(t)$ we have

$$I(t) = I_0(t) - \frac{d}{dt}\int_0^t A(t-y)\phi[I(y)]dy. \qquad (18)$$

This is a *functional integral equation*, the solution of which is gotten by some process of successive approximations. For example, provided the sequence converges, $I(t)$ is the limit as $n$ approaches infinity of the *sequence*

$$I_0(t),\ I_1(t),\ I_2(t),\ \cdots,\ I_n(t), \qquad (19)$$

where the successive terms of the sequence are defined by the relations:

$$\begin{aligned} I_1(t) &= I_0(t) - \frac{d}{dt}\int_0^t A(t-y)\phi[I_0(y)]dy \\ &\cdots\cdots\cdots\cdots \\ I_{n+1}(t) &= I_0(t) - \frac{d}{dt}\int_0^t A(t-y)\phi[I_n(y)]dy. \end{aligned} \qquad (20)$$

With this brief sketch, the method will now be applied to *The Problem of the Three-Element Vacuum Tube.*[1]

[1] The reader is assumed in the following to be acquainted with the theory of the vacuum tube in its broad outlines. For a very able account of the theory and physics of the tube see Van der Bijl's papers, PHYSICAL REVIEW, Sept., 1918, and Proc. Inst. Radio Engineers, April, 1919. For some mathematical methods of dealing with the device see my paper, Proc. Inst. Radio Engineers, April, 1919.

In this device the output circuit or plate current is assumed to be a known function of the grid-filament and plate-filament potential differences. It is further assumed that the output or plate circuit is closed through an impedance $Z$ (whose circuit elements are invariable) and that a specified potential difference $E(t)$ is applied between the grid and filament. We denote by $V(t)$ the potential difference between plate and filament and by $I(t)$ the unknown plate current[1] which we are to determine. We assume that $I(t)$ is a known function of $E(t)$ and $V(t)$ and therefore write

$$I(t) = F[E(t),\ V(t)], \tag{21}$$

But since this same current flows into the output circuit impedance $Z$ (of indicial admittance $A(t)$) across whose terminals the potential difference is $V(t)$, we have also

$$I(t) = \frac{d}{dt}\int_0^t A(t-y)V(y)dy. \tag{22}$$

Equating (20) and (21) we get the functional equation:

$$F[E(t),\ V(t)] = \frac{d}{dt}\int_0^t A(t-y)V(y)dy. \tag{23}$$

Now in the actual tube the work of Van der Bijl and others has shown that over a considerable part of the characteristic the current is given by the approximate relation

$$I(t) = \frac{\mu E(t) - V(t)}{R}, \tag{24}$$

where $\mu$ is a physical parameter of the tube commonly termed the amplification factor and $R$ is the "internal resistance" of the tube. This suggests that the characteristic function be written as:

$$F(E,\ V) = \frac{\mu E - V}{R} + \phi(E,\ V). \tag{25}$$

Whence by substitution in (23) and rearrangement we get

$$V(t) = \mu E(t) + R\phi[E(t),\ V(t)] - R\frac{d}{dt}\int_0^t A(t-y)V(y)dy. \tag{26}$$

Which is a functional integral equation in the unknown potential difference $V(t)$. With this function determined the current $I(t)$ is given by (20) or (21).

The solution of this functional integral equation is obtained by a process of successive approximations; in the present problem the most con-

[1] The potential differences $E(t)$, $V(t)$ and the current $I(t)$ are to be understood as denoting not the total values but rather their variations from the normal or steady condition. Thus $I(t)$ does not include the steady d.c. current.

venient and physically significant mode of approximation is to take $V(t)$ as the limit (as the index $n$ increases indefinitely) of the sequence

$$V_0(t),\ V_1(t),\ V_2(t),\ \cdots,\ V_n(t), \tag{27}$$

where the successive approximations are defined by the equations:

$$\begin{aligned} V_0(t) &= \mu E(t) - R\frac{d}{dt}\int_0^t A(t-y)V_0(y)dy \\ &\cdots\cdots\cdots \\ V_{n+1}(t) &= \mu E(t) + R\phi[E(t),\ V_n(t)] - R\frac{d}{dt}\int_0^t A(t-y)V_{n+1}(y)dy. \end{aligned} \tag{28}$$

The physical significance of these approximations, which is of importance in the following discussion, may be seen as follows: Consider a circuit of resistance $R$ in series with an impedance $Z$ (of indicial admittance $A(t)$) with a potential difference $E(t)$ impressed on the terminals of the circuit, and let $V(t)$ denote the unknown voltage developed across the impedance $Z$. The following expressions for the current $I(t)$ in the circuit can be written down at once:

$$\begin{aligned} I(t) &= \frac{1}{R}(E(t) - V(t)) \\ &= \frac{d}{dt}\int_0^t A(t-y)V(y)dy. \end{aligned}$$

Equation of these two expressions gives the following integral equation for $V(t)$:

$$V(t) = E(t) - R\int_0^t A(t-y)V(y)dy. \tag{29}$$

Comparison of this equation with (28) shows at once that the first term $V_0(t)$ of the sequence (27) is simply equal to the voltage developed across the output circuit impedance $Z$ on which an E.M.F. $\mu E$ is impressed through a resistance $R$. Similarly $V_{n+1}(t)$ is equal to the potential developed across $Z$ when an E.M.F. $\mu E + R\phi(E,\ V_n)$ is impressed thereon through a resistance $R$. It follows, therefore, that the resistance $R$ is to be regarded as the "internal resistance" of the tube. In the engineering theory of the amplifier this is usually defined by the relation $1/R = \partial F(E,\ V)/\partial V$, but in the present discussion its value is not so limited. A second consequence of the above is that, so long as the characteristic is approximately a straight line the equivalent circuit of the tube is simply a resistance $R$ in series with an impedance $Z$ on which an E.M.F. $\mu E$ is impressed. This agrees with the engineering theory of the vacuum tube amplifier[1] and shows that the simple theory is equivalent to the first approximation of the complete solution.

[1] See papers by Van der Bijl and the writer in Proc. I. R. E., April, 1919, and also a paper by H. W. Nichols, PHYSICAL REVIEW, June, 1919.

This physical interpretation is of substantial advantage in the following discussion of the convergence of the sequence. From the physics of the vacuum tube and the way in which the $\phi$ function is formed from the characteristic function $F(E, V)$ of the tube, it may be shown that

$$\phi(E, V) \leqq (\mu E - V)/R$$

and

$$\phi(E, V_n) - \phi(E, V_{n+1}) \leqq (V_{n+1} - V_n)/R.$$

If $V_{n+1}(t) - V_n(t)$ is denoted by $d_{n+1}(t)$, subtraction of the $n$th from the $(n + 1)$st equation of (28) gives

$$d_{n+1}(t) + R\frac{d}{dt}\int_0^t A(t - y)d_{n+1}(y)dy \leqq d_n(t).$$

In the light of the physical interpretation of the integral equation of this type, it is immediately evident that $d_{n+1}(t)$ is related to $d_n(t)$ as the voltage across an impedance $Z$ is to the potential impressed thereon through a resistance $R$. It follows therefore from physical considerations that $d_{n+1} \angle d_n$ in all circuits of practical importance and that the sequence is convergent.

Referring to equations (28) it will be observed that each stage of the approximation formally requires the solution of an integral equation. As a matter of fact this operation can be dispensed with and the solution written as

$$V_0(t) = \frac{d}{dt}\int_0^t K(t - y)\mu E(y)dy$$

$$\cdots\cdots\cdots\cdots$$

$$V_{n+1}(t) = V_0(t) - R\frac{d}{dt}\int_0^t dyK(t - y)\phi[E(y), V_n(y)].$$

The function $K(t)$ may be itself evaluated from the integral equation

$$K(t) = 1 - R\frac{d}{dt}\int_0^t A(t - y)K(y)dy$$

Since this equation, physically interpreted, states that $K(t)$ is the voltage developed across the impedance $Z$ when a "unit E.M.F." (zero before, unity after, time $t = 0$) is impressed thereon through a resistance $R$, it is legitimate to regard $K(t)$ as a datum of the problem, like the indicial admittance $A(t)$, since it can be evaluated by processes applicable to the theory of invariable circuits.

In view of the physical significance of the successive terms of the sequence, the actual evaluation of the integral equations and the definite integrals may be dispensed with if we are concerned only with steady-state phenomena, and the integral equations of (28) may be replaced by equivalent operational formulæ which are calculable by usual methods. Thus if $E(t)$ is a periodic time function and we ignore transient states, we have

$$V_0 = \mu E \frac{Z}{R + Z}, \tag{30}$$

which follows at once from the physical interpretation of the sequence. $V_0$ is thus a periodic function of the same frequency as that of the impressed E.M.F. $E$. Having calculated $V_0$, we have in operational notation:

$$V_1 = (\mu E + R\phi(E,\ V_0)) \frac{Z}{R + Z}. \tag{31}$$

This can be solved for $V_1$ by usual steady-state methods provided that $\phi(E,\ V_0)$ is expanded either as a Fourier series, which is always possible, or preferably as a power series in $E$ and $V_0$ which is usually possible over the operating range of the characteristic. Higher approximations $V_2$, $V_3$, $\cdots$ follow by straightforward operations.

## Conclusion.

The purpose of the present paper has been to illustrate in a few representative problems the application of integral equations to the solution of those problems in electric circuit theory in which variable circuit elements are involved. Integral equations have been employed for some time by the writer in the solution, both formal and numerical, of practical problems in circuit theory and have proved to be a serviceable instrument. In the present paper the emphasis has been placed on general methods and the physical interpretation of the solutions, and no attempt has been made to discuss the appropriate methods of numerical solution. A considerable experience, however, has convinced the writer that the application of integral equations to the problems of circuit theory is attended by marked advantages in actual engineering calculations.

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