Angular-Momentum Distributions of Residual Nuclei in Compound-Nuclear Reactions*

I. HALPERN, B. J. SHEPHERD,† AND C. F. WILLIAMSON‡
Department of Physics, University of Washington, Seattle, Washington
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A classical vector model is used to determine the changes which occur in the distribution functions for the angular momentum of nuclei when particles and photons are evaporated from these nuclei. Closed-form expressions are derived for angular-momentum distributions after a sequence of evaporations. It is found that at higher excitations one must expect the average angular momentum of a nucleus to decrease by a relatively small amount per evaporation. The mean-square angular momentum will increase or decrease depending on the ratio of the nuclear angular momentum to the equipartition value appropriate to the nuclear excitation. General expressions are given for the angular distributions of the evaporations. Some of the difficulties of applying statistical models to emissions at low excitation energies are briefly discussed.

I. INTRODUCTION

THE excited compound nuclei which are initially formed in a nuclear reaction have a distribution of angular momentum which often can be reliably estimated. The emission of particles and photons in the decay of these excited nuclei leads to changes in this initial distribution. These changes are the subject of the present paper.

The experimental observations which bear on this subject are the angular distributions of the evaporations and the distributions in angular momentum of identifiable states which appear in the de-excitation process. So far, the only identifiable states which have been studied are of very low excitation energy, e.g., isomeric states[1–4] and low-lying collective states.[5–9]

In the discussion of the angular-momentum changes that are brought about by evaporations, the angular momentum will be treated as an ordinary classical vector. That is, its quantization will in general be ignored. Whether one treats the angular momentum classically or quantum mechanically, the main mathematical problem is the coupling of angular momenta. The chief advantage of using a classical description is that the relevant mathematics is then the very familiar geometry of ordinary vectors in three dimensions. This mathematics is generally more transparent than the algebra of Clebsch-Gordan coefficients. It therefore lends itself more readily to the elucidation of general results which are basically geometrical in character. There is, moreover, little risk involved in using a classical description since it is unlikely that any of its implications would differ significantly from those of a quantum treatment as long as the angular momenta considered are fairly large and broadly distributed in value.

Section II is devoted to the derivation and justification of a simple statistical formula for the distribution of angular momentum in a system after a single evaporation. In Sec. III, the physical meaning of the parameters in the formula are discussed and a few of the more fundamental implications of the formula are explored. Section IV deals with the generalization of the formula to situations involving a sequence of evaporations of particles and photons. It also takes up briefly some of the difficulties encountered in the assignment of values to the parameters of the formula when it is applied to actual experiments. In particular, there is a discussion in Sec. IV of some of the problems which arise from the inapplicability of any statistical model (including the one developed in this paper) to transitions at the very end of an evaporation cascade. In Sec. V, the previous results are applied to typical compound-nuclear reactions at moderate bombarding energies. This is done by summing the final angular-momentum distributions associated with individual initial angular momenta $\mathbf{J}_0$ over the distribution function for $\mathbf{J}_0$. Section VI discusses the angular distributions of particles and especially of photons which are evaporated during the main part of the cascade. One finds that these photon distributions tend to be rather isotropic in contrast to the distributions expected for radiations from ground-state rotational bands which are excited in the same reactions.

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† Present address: Brookhaven National Laboratory, Upton, N. Y.
‡ Present address: Massachusetts Institute of Technology, Cambridge, Mass.

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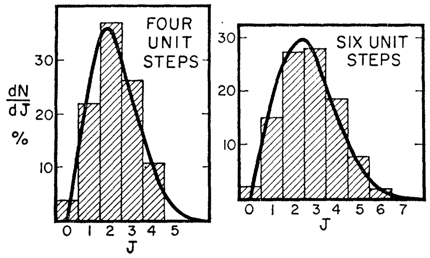

FIG. 1. Distributions of states of different angular momentum J generated by the compounding of N randomly oriented angular momenta of unit magnitude. The histograms give the exact results and the smooth curves give the distributions according to the classical vector model, Eq. (2).

A number of the problems taken up in this paper have been discussed before,[10–15] often in the context of the analysis of some particular experiment. Some of the results that are found to follow from the basic formula and the associated model of the present paper are not new. In presenting them again it is believed that there is some virtue in seeing how they are consequences of simple geometrical considerations. The main use of the model is probably conceptual in that it helps one to understand the relations of quantities that play a role in the angular-momentum history of a decaying nuclear system. One should, however, not overlook the fact that the model provides a very explicit and easy to use formula for estimating angular-momentum distributions. It makes it possible to obtain reasonably good answers to questions relating to angular momentum in compound-nuclear reactions with the help of a slide rule instead of a computer.

II. DERIVING AN EXPRESSION FOR THE DISTRIBUTION OF ANGULAR MOMENTUM IN A SYSTEM AFTER AN EVAPORATION

A. Introducing a Gaussian Form for the Distribution of Angular Momentum Removed

A particle or photon is evaporated from a system whose original angular momentum is $\mathbf{J}_0$. It is our purpose in the present section to derive a simple closed-form expression to represent the distribution for $\mathbf{J}_1$, the angular momentum of the residual system, after the evaporation. If $\mathbf{j}$ is the angular momentum of the emitted particle, then

$$\mathbf{J}_0=\mathbf{J}_1+\mathbf{j}. \tag{1}$$

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We shall now assume, and try to justify later, that (except for a generally small but important modification, to be discussed in Sec. II B), the distribution function for $\mathbf{j}$ can be represented by the Gaussian form

$$d^2N/j^2djd\omega=e^{-\alpha j^2}. \tag{2}$$

The denominator $j^2djd\omega$ is a volume element in j space, $d\omega$ being the element of solid angle specifying the orientation of $\mathbf{j}$. The form $\exp(-\alpha j^2)$ on the right side of (2) depends only on the magnitude of $\mathbf{j}$ and not upon its orientation, but the expression that will finally be used will also involve the orientation of $\mathbf{j}$.

One is motivated to choose a Gaussian form to represent

$$d^2N/j^2djd\omega$$

because Gaussians can readily be iterated. Thus, the total angular momentum removed in a sequence of evaporations, each of which is represented by a distribution of the form (2), will have a distribution which is also represented by this same form. The values of α for the separate steps in such a sequence need not, of course, be equal.

Before going on to elaborate this point about the mathematical convenience of the form (2), it is worth estimating how well the chosen form is likely to represent distributions for evaporated particles and photons. A particle of mass m evaporating from a Maxwell gas in a spherical enclosure, for example, has the following relative probability for coming out with velocity $v_\perp$ transverse to the surface normal:

$$P(v_\perp)\sim dN/v_\perp dv_\perp\sim e^{-mv_\perp^2/2T}. \tag{3}$$

Boltzmann's constant has been absorbed here into the definition of the temperature, as is customary in nuclear physics. If the gas-containing enclosure has radius R, the emission probability can be expressed in terms of the orbital angular momentum $l=mv_\perp R$ of the emitted particle:

$$dN/ldl\sim e^{-l^2/2mR^2T}. \tag{4}$$

It is reasonable that particles evaporated from a nucleus of radius R would have an orbital angular-momentum distribution close to that given by (4). It is seen that the form of (4) differs from the assumed Gaussian in that the latter involves an extra factor of the angular momentum. Yet the shapes of the two distributions differ sufficiently little to suggest that they might sometimes be used interchangeably, especially when only the very lowest moments of the distributions are of concern. Also, in applying the continuous distributions (2) or (4) to the evaporations of particles from nuclei, it is assumed that no significant complications arise from the quantization of the angular momentum. That is to say, one expects that the classical distribution (4) will be a good approximation to, say, the envelope of the actual discrete angular-momentum distribution, and in particular that its lower moments will represent reasonably well those of the actual emission.

It is harder to make these arguments for photon evaporations than for particle evaporations. The statistical photons tend to be mostly dipole photons. The distribution function for the angular momentum which they remove is therefore more nearly a δ function at $j=1$ than a Gaussian of the form $dN\sim j^2\exp(-\alpha j^2)dj$.

In summary, the distributions for $\mathbf{j}$ typically removed by evaporating particles and by photons do not resemble each other very closely, and the Gaussian form which we choose to represent them both is only a fair approximation to the distribution expected for particles and a poor approximation to that expected for photons. Nevertheless, the Gaussian form probably becomes a reasonably good approximation when one is trying to find the angular-momentum distribution after a moderately long *sequence* of emissions. It is well known that when one compounds similar distributions of whatever simple form the resultant distribution tends to look increasingly Gaussian as the number of iterations increases. For example, the distribution for the number of heads appearing in sequences of N coin tosses becomes quite Gaussian-looking when N is large despite the fact that the elementary distribution (for a single toss) is very non-Gaussian. To appreciate how quickly a resultant distribution tends to a Gaussian form, one can compare the distribution for the angular momentum removed in N uncorrelated emissions each removing unit angular momentum with a distribution of the form (2), where α is chosen to give the same mean-square resultant angular momentum. These comparisons are given in Fig. 1 for $N=4$ and $N=6$. In this figure the histograms give the relative numbers of states with the indicated value of J which can be constructed by adding N uncorrelated unit angular-momentum vectors. The smooth curves give the corresponding normalized curves of form (2). In constructing these smooth curves, the quantity j^2 corresponding to angular momentum J was taken as $J(J+1)$. This correspondence arises from the fact that j as it appears in (2) refers to an angular-momentum magnitude, i.e., to $\sqrt{j^2}$. In the quantum-mechanical calculation, the mean-square angular momentum after N uncorrelated steps is quite generally

$$\sum_i^N j_i(j_i+1)\,,$$

summed over the steps. Equation (2) has the property that for N uncorrelated steps the mean-square angular momentum is simply

$$\sum_i^N j^2.$$

Thus we see that j^2 in (2) must be associated with the quantities $j(j+1)$ when (2) is placed in correspondence with an exact calculation. Now the parameter α in (2) can be shown to be the reciprocal of $\frac{2}{3}\langle j^2\rangle$. It follows that the value of α to be used in the Gaussian expression (2) after N emissions of unit angular momentum is given by $\alpha^{-1}=N\times\frac{2}{3}(2)$ or $\alpha=3/4N$. In the same spirit, if one wants to know the probability for say $J=3$ in the resultant distribution according to (2), then one must determine the ordinate for the value of j^2 equal to $j(j+1)$ or to 12 instead of nine. In short, in applying and interpreting the classical formulas one must remember that the symbol j is to be assigned values $[j(j+1)]^{1/2}$.

It is seen that even for N as low as four the Gaussian form is a good approximation to the exact distribution in the region of large probability, i.e., around the peak. If, however, one is interested in relatively rare events (for example, the production of very-high-spin isomers in slow neutron capture), it will be the tail of the distribution that matters. Here the Gaussian idealization may depart significantly from reality.

Thus this problem of compounding angular momenta associated with evaporations resembles a typical random-walk problem. The angular-momentum steps are in reality quantized and the resulting distributions are therefore best represented by histograms. For purposes of computation it is, however, useful to have a continuous function to represent these histograms. For a random walk in D dimensions, the appropriate continuous function is of the form

$$dN/dj\sim j^{D-1}e^{-\alpha j^2}. \tag{5}$$

In the present problem, ordinary angular-momentum vectors are being compounded and $D=3$. The resemblance of the distribution (5) to the histogram that it represents improves as the number of iterated steps increases.

B. Introducing the Spin-Cutoff Factor

The distribution (2) cannot, as it stands, represent the angular momentum removed in an evaporation. Since the removed angular momentum $\mathbf{j}$ is assumed to be isotropic with respect to the initial angular momentum $\mathbf{J}_0$, it follows from (1) that

$$\langle J_1^2\rangle=\langle J_0^2\rangle+\langle j^2\rangle. \tag{6}$$

This equation implies that the mean-square angular momentum always increases in an evaporation. Normal systems do not behave in this way. As a system cools by evaporation it must eventually lose its angular momentum. The quantity $\langle\mathbf{j}\cdot\mathbf{J}_0\rangle$ is not zero, as has been assumed in the derivation of (6). It is positive because the evaporating particle tends to remove a share of $\mathbf{J}_0$. In other words, the $\mathbf{j}$ distribution is not isotropic. The distribution (2) can be modified easily to take this effect into account by the introduction of an additional factor $\exp(-\beta J_1^2)$. Thus,

$$d^2N/j^2djd\omega\simeq e^{-\alpha j^2}e^{-\beta J_1^2}\simeq e^{-\alpha j^2}\exp[-\beta(\mathbf{J}_0-\mathbf{j})^2]. \tag{7}$$

The chosen form of the new factor can be appreciated on physical grounds. The factor is simply a Boltzmann factor. If $\mathcal{I}_f$ is the moment of inertia of the system after the emission, then

$$E=J_1^2/2\mathcal{I}_f \tag{8}$$

is the energy tied up in the rotation of the system in the final state. This energy is therefore unavailable for the other degrees of freedom in this state. Its ratio to T must therefore appear in the exponent of a Boltzmann factor in expressions which compare the probabilities of various final states. It follows from (7) and (8) that

$$\beta=1/2\mathcal{I}_f T. \tag{9}$$

In the literature, β is often written $(2\hbar^2\sigma^2)^{-1}$, where σ is the so-called spin-cutoff factor.[11]

The introduction of $\exp(-\beta J_1^2)$ into (2), although necessary on physical grounds, apparently destroys the feature of the distribution, $\exp(-\alpha j^2)$, that was emphasized in Sec. II A. The distributions are now no longer isotropic Gaussians and therefore cannot be compounded (for a sequence of evaporations) in the usual trivial way to obtain the distribution for the resultant.

C. Restoring the Gaussian Form for Distribution of the Angular Momentum Removed

For purposes of compounding evaporation distributions, it is so convenient to be dealing with Gaussians that it suggests itself to ask whether a coordinate transformation exists that will restore (7) to Gaussian form. There is such a transformation and it can be found as follows.

We begin by writing (7) in terms of the residual angular momentum $\mathbf{J}_1$ instead of the removed angular momentum $\mathbf{j}$.

$$d^2N/J_1^2dJ_1d\Omega\simeq e^{-\alpha(\mathbf{J}_0-\mathbf{J}_1)^2}e^{-\beta J_1^2}. \tag{10}$$

Here $J_1^2dJ_1d\Omega$ is a differential volume element in $\mathbf{J}_1$ space, $d\Omega$ being the element of solid angle specifying the orientation of $\mathbf{J}_1$. In converting (7) to (10), use has been made of the fact that the Jacobian for the transformation from $\mathbf{j}$ to $\mathbf{J}_1$ space is just unity (the transformation is the same as that involved in the familiar nonrelativistic conversion from laboratory to c.m. velocities).

By completing the square in the exponent of (10), this equation may be rewritten

$$d^2N/J_1^2dJ_1d\Omega\simeq\exp[-(\alpha+\beta)J_1^2+2\alpha\mathbf{J}_1\cdot\mathbf{J}_0-\alpha J_0^2]$$
$$=\exp\left[-(\alpha+\beta)\left\{\frac{\alpha}{\alpha+\beta}\mathbf{J}_0-\mathbf{J}_1\right\}^2-\frac{\alpha\beta}{\alpha+\beta}J_0^2\right] \tag{11}$$

or introducing

$$\alpha'\equiv\alpha+\beta, \tag{12}$$

$$\gamma\equiv\alpha/(\alpha+\beta), \tag{13}$$

and

$$\mathbf{j}'\equiv(\gamma\mathbf{J}_0-\mathbf{J}_1), \tag{14}$$

(11) becomes

$$d^2N/J_1^2dJ_1d\Omega\simeq\exp[-\alpha'j'^2-\beta\gamma J_0^2]. \tag{15}$$

The last factor, $\exp(-\beta\gamma J_0^2)$, may be ignored. It is a constant, independent of J_1, and disappears anyhow when

$$d^2N/J_1^2dJ_1d\Omega$$

is normalized. Thus one has

$$d^2N/J_1^2dJ_1d\Omega\simeq\exp[-\alpha'j'^2], \tag{16}$$

which can now be transformed to $\mathbf{j}'$ space. Again, the Jacobian for the transformation is unity and

$$d^2N/j'^2dj'd\omega'\simeq\exp[-\alpha'j'^2]. \tag{17}$$

This has the sought-for simple Gaussian form.

It is useful to explain in words the nature of the transformation from (7) to (17). If an *anisotropic* array of vectors having the form (7) is added to some fixed vector $\mathbf{J}_0$, then the distribution of resultants $\mathbf{J}_1$ can be reproduced *exactly* if one adds a certain *isotropic* Gaussian distribution to a fixed vector $\gamma\mathbf{J}_0$ which is slightly shorter than $\mathbf{J}_0$. This is shown schematically in Fig. 2, where for simplicity the Gaussian distributions in length have been ignored. All vectors have been drawn equally long. Since the drawings in the figure resemble flowers, one can say that the actual unsymmetrical flower has been shown to be equivalent to a shorter flower with shorter petals, but where the petals are symmetrically arrayed.

III. CHANGES IN A SYSTEM'S AVERAGE ANGULAR MOMENTUM AND IN ITS RMS ANGULAR MOMENTUM BROUGHT ABOUT BY EVAPORATION

A. Average Angular Momentum

A system with initial angular momentum $\mathbf{J}_0$ has average angular momentum $\langle\mathbf{J}_1\rangle=\gamma\mathbf{J}_0$ after an evaporation. This follows from (14) when we remember that $\mathbf{j}'$ is isotropically distributed. Thus γ, a number less than unity, gives a foreshortening factor (per evaporation) for the average angular momentum. The average angular momentum always decreases because the escaping particle or photon removes, on the average, some small share of it.

This can be simply illustrated if we consider as an example the evaporation of a (spinless) particle from a classical gas enclosed in a sphere. The foreshortening factor γ is, according to (13), simply $(1+\beta/\alpha)^{-1}$, where $\beta=1/2\mathcal{I}_f T$. From the way in which α appears in (7) it can be shown that

$$\alpha=\tfrac{3}{2}(1/\langle j^2\rangle), \tag{18}$$

provided that $\langle j^2\rangle$ is computed under the condition that $\beta=0$. For a spinless particle evaporating from a classical gas, this value of $\langle j^2\rangle$ can be obtained, for example, from (4), where it is seen to be $2mR^2T$. Thus γ is $[1+(\frac{2}{3}mR^2/\mathcal{I}_f)]^{-1}$. But $\frac{2}{3}mR^2$ is just the average value of the moment of inertia associated with a particle of mass m evaporating from the surface of a sphere of radius R. Thus if $\mathcal{I}_i$ is the system's moment of inertia before evaporation, we may set

$$\tfrac{2}{3}mR^2=\mathcal{I}_i-\mathcal{I}_f, \tag{19}$$

leading to

$$\gamma(\text{spinless particle})=\mathcal{I}_f/\mathcal{I}_i. \tag{20}$$

That this is the expected result for evaporation from a classical gas can be appreciated by rewriting it in the form

$$\gamma=\mathcal{I}_f\omega/\mathcal{I}_i\omega, \tag{21}$$

where $\boldsymbol{\omega}$ is the initial angular velocity. Since γ is $\langle J_1\rangle/\langle J_0\rangle$, (21) can be interpreted to mean that the *average* angular velocity of the system does not change upon evaporation. This constancy of $\boldsymbol{\omega}$ is precisely what one would expect. The velocity of an escaping particle can be compounded of two parts: (1) a thermal velocity (randomly directed with respect to a coordinate system rotating with the body) and (2) the surface speed at the point of evaporation. Neither of these motions gives rise to recoil effects which change the average value of the angular velocity $\boldsymbol{\omega}$ of the rotating system.

The quantity γ is involved not only in the foreshortening of the fixed vector from which $\mathbf{J}_1$ must be constructed. It also appears in the foreshortening of the average length of the distributed vector (j'). From (12) and (16) it is seen that the introduction of β changes $\langle j'^2\rangle$ from $\frac{3}{2}\alpha^{-1}$ to $\frac{3}{2}(\alpha+\beta)^{-1}$. Thus $\langle j'^2\rangle$ is multiplied by γ [or $(j')_{\text{rms}}$ is multiplied by $\gamma^{1/2}$] when one takes account of β. This reduction in the rms value of $\mathbf{j}'$ comes about because the departure of a particle or photon with angular momentum $\mathbf{j}$ gives to the residual system a recoil angular momentum of the same amount. It is the inclusion of the energy associated with this recoil in the Boltzmann factor that is responsible for a reduction in the rms value of the angular momentum of evaporated particles and photons. In the limit of infinitely large nuclei where β and γ approach 0 and 1, respectively, this recoil effect disappears.

B. RMS Angular Momentum

We have seen that the average angular momentum of a system decreases when there is an evaporation. We shall now consider what happens to the rms angular momentum. From (14) and the fact that $\mathbf{j}'$ is isotropic it follows that

$$\langle J_1^2\rangle=\langle(\gamma J_0)^2\rangle+\langle j'^2\rangle \tag{22a}$$

or

$$\langle J_1^2\rangle-\langle J_0^2\rangle=(\gamma^2-1)\langle J_0^2\rangle+\langle j'^2\rangle. \tag{22b}$$

The quantity on the right may be negative or positive.

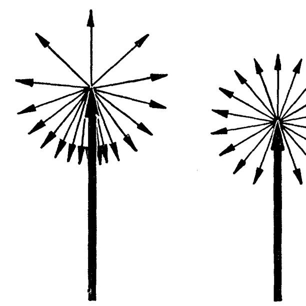

FIG. 2. The heavy arrow in the figure at the left represents the initial angular momentum of a nucleus. The lighter arrows show the anisotropic distribution of angular momenta given to the nucleus by an evaporating particle or photon. For simplicity, these added vectors have all been drawn with fixed length. The distribution of resultants is obtained by joining the tail of the heavy arrow to the heads of the light ones. It is shown in the text that if the distribution of light arrows is given by Eq. (7), the distribution of resultants for the figure at the left can be reproduced *exactly* by adding a certain isotropic Gaussian distribution of vectors to a fixed vector which is slightly shorter than the original fixed vector. This is illustrated schematically on the right.

Clearly, if J_0 happens to be zero, it is positive. Then the rms angular momentum necessarily increases upon evaporation. But given a j', one can find values of J_0 so large that the first (negative) term dominates the second and $\langle J_1^2\rangle$ is less than $\langle J_0^2\rangle$.

It is useful to explore the implications of (22b) by once again specializing to the simple example of evaporation from a classical Maxwell gas enclosed in a sphere. We have considered the fact in Sec. III A that $\langle j'^2\rangle$ is γ times the value that one would obtain when spin-cutoff effects are ignored (i.e., when β is set equal to zero). This $(\beta=0)$ value of $\langle j'^2\rangle$ was determined to be $2mR^2T$. Thus

$$\langle j'^2\rangle=\gamma\cdot 2mR^2T=\gamma\times 3(\mathcal{I}_i-\mathcal{I}_f)T, \tag{23}$$

where we have used (19). Making use of (23) and of (20), Eq. (22b) may be rewritten

$$\frac{\langle J_1^2\rangle}{2\mathcal{I}_f}-\frac{\langle J_0^2\rangle}{2\mathcal{I}_i}=\frac{\mathcal{I}_i-\mathcal{I}_f}{\mathcal{I}_i}\left[\tfrac{3}{2}T-\frac{\langle J_0^2\rangle}{2\mathcal{I}_i}\right]. \tag{24}$$

This equation states that the average rotational energy of the evaporating system will increase when $J_0^2/2\mathcal{I}_i$ is less than $\frac{3}{2}T$, the equipartition value for the energy which is connected with the system's three rotational degrees of freedom. If the rotational energy happens to exceed the equipartition value, the rotational energy will tend to decrease.

The evolution of the angular momentum of a system in the course of the evaporation of a *series* of particles and photons will be examined in detail in Sec. IV. At this point it is useful to discuss this evolution in a qualitative way, making use of the ideas embodied in (24). For this purpose one may plot the equipartition value of the system's angular momentum J_{rms} against the system's excitation energy (Fig. 3). Since we mean to apply these results to evaporations from nuclei, where at higher excitations the temperature is generally as-

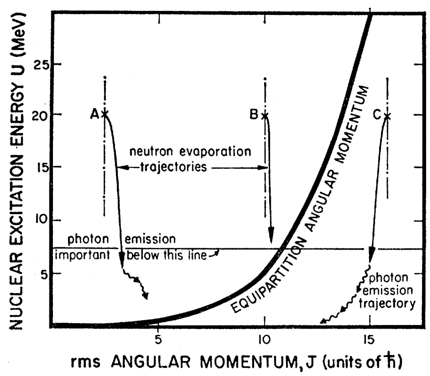

FIG. 3. When a particle or photon evaporates from a system of nuclei, both the average excitation energy U and the rms angular-momentum J_{rms} change. By connecting the points (J_{rms},U) which represent the state of the system at each stage of an evaporation sequence, one can construct a "trajectory" for the motion of the system in (J_{rms},U) space. Several such trajectories are plotted. At higher excitation energies in heavy nuclei, neutron evaporations occur first and they typically reduce the value of U without appreciably changing J_{rms}. At lower excitation energies, photon emission becomes possible, and here the trajectory may begin to move toward the equipartition curve (see text).

sumed to be proportional to the square root of the excitation energy U, the quantity J_{rms} (which is proportional to $T^{1/2}$) has been drawn proportional to $U^{1/4}$ in Fig. 3.

The initial state of an excited nucleus can be represented by a point in the (J_{rms},U) plane. This point will tend to move toward the equipartition line as particles and photons are evaporated. For particle emission, which occurs first, this tendency is, however, quite weak. Because particle binding energies are rather larger than typical evaporation kinetic energies, the state point moves almost straight down toward the J_{rms} axis when a particle is emitted. One can appreciate just how vertical the trajectory is for particle evaporations by applying Eq. (22b) to a typical example. To make the tendency for J to change as large as possible, consider that J_0 is so large to begin with that the first term in (22b) dominates the second. Thus the change in J_{rms}^2 per evaporation is

$$\Delta J_{\text{rms}}^2 \simeq (\gamma^2-1)J_{\text{rms}}^2.$$

Using (19) and (20) and the so-called rigid-body value $\frac{2}{5}AmR^2$ for $\mathcal{J}$ (where A is the atomic weight of the emitting nucleus and m is a nucleon mass), one finds that

$$\Delta J_{\text{rms}}^2/J_{\text{rms}}^2 = -10/3A.$$

Thus for $A\simeq 160$, there is only a 1% decrease in J_{rms} per evaporation. If the nuclear excitation energy is, say, 40 MeV, about four neutrons will, in general, be evaporated and U will change by 10 MeV or 25% in the first evaporation. The motion of the state point in Fig. 3 is therefore essentially vertically downward. Thus particle (mostly neutron) evaporations do not significantly distort the angular-momentum spectrum originally produced in a bombardment.

These same remarks do not apply, however, to the photon emissions which follow the particle emissions. This is in part because photons have no binding energies. Moreover, evidence is accumulating[7] that decreases in J_{rms}^2 due to photon emissions often can be significantly larger than one would expect on the basis of Eq. (22b) using conventional parameters. In any event, the relative yields of low-lying states with different spins which are produced in nuclear bombardments reflect, mainly, the course of the final (photon) stages in an evaporation cascade. The preceding (particle) stages change the input angular-momentum distributions very little.

IV. ANGULAR-MOMENTUM DISTRIBUTION AFTER A SEQUENCE OF EVAPORATIONS

A. Derivation of a General Formula for This Distribution

Section III dealt with the problem of determining the angular-momentum distribution of an ensemble of systems all of which started in the same angular-momentum state and all of which evaporated a single particle or photon. The expression developed for this distribution, Eq. (17), had a Gaussian form and it was indicated that this form would prove to be useful for the problem of determining the angular-momentum distribution after a *sequence* of evaporations.

We assume that the ith emission in the sequence can be characterized by a pair of parameters α_i and β_i which are defined as in Sec. III. In general, the constants α_i and β_i may be different for the different emissions in the sequence.

If the starting angular momentum of the system is $\mathbf{J}_0$, then Eq. (17) allows one to determine a distribution for $\mathbf{J}_1$, the angular momentum after the first emission. To determine the distribution for $\mathbf{J}_2$, the angular momentum after the second emission, let us begin by selecting some particular $\mathbf{J}_1$ that occurs after the first emission. From Sec. III it follows that the distribution for $\mathbf{J}_2$ starting with this particular $\mathbf{J}_1$, say, $\mathbf{J}_1{}^p$, can be generated by adding to the fixed vector

$$[\alpha_2/(\alpha_2+\beta_2)]\mathbf{J}_1{}^p \equiv \gamma_2 \mathbf{J}_1{}^p$$

the isotropic distribution

$$d^2N/j_2'dj_2'd\omega_2' = e^{-(\alpha_2+\beta_2)(j_2')^2}. \tag{25}$$

Now according to (14) we may write

$$\gamma_2\mathbf{J}_1{}^p = \gamma_2[\gamma_1\mathbf{J}_0 - \mathbf{j}_1'{}^p], \tag{26}$$

where $\mathbf{j}_1'{}^p$ stands for the particular $\mathbf{j}_1'$ which leads to $\mathbf{J}_1{}^p$. The total $\mathbf{J}_2$ distribution must be obtained by summing over all possible vectors $\mathbf{j}_1'$. This summation is straightforward as long as γ_2 does not depend on $\mathbf{j}_1'$, i.e., as long as the parameters which characterize the second evaporation do not depend on the angular momentum actually emitted in the first emission. According to (26), the distribution in $\mathbf{J}_2$ can then be looked upon as the sum of the fixed vector $\gamma_2\gamma_1\mathbf{J}_0$, the Gaussian distribution (25), and the Gaussian distribution in $-\gamma_2\mathbf{j}_1'$. The latter is the same as the distribution for $-\mathbf{j}_1'$, except for the foreshortening factor γ_2. It is, therefore,

$$\frac{d^2N}{j_1'^2dj_1'd\omega_1'}=\exp\left[-\frac{(\alpha_1+\beta_1)}{\gamma_2^2}j_1'^2\right]. \tag{27}$$

The distribution for $\mathbf{J}_2$ can be expressed more simply by folding together the two Gaussian distributions (25) and (27). One makes use of the well-known property of Gaussians that they can be compounded to yield other Gaussians. The mean-square magnitude in the resultant distribution is the sum of the mean-square magnitudes of the constituent distributions. Thus the resultant Gaussian which is to be added to the fixed vector $\gamma_1\gamma_2\mathbf{J}_0$ in order to give the distribution for $\mathbf{J}_2$ is

$$\frac{d^2N}{(j')^2(dj')d\omega'}=\exp\left[\frac{-j'^2}{\gamma_2^2/(\alpha_1+\beta_1)+1/(\alpha_2+\beta_2)}\right]. \tag{28}$$

It should be clear that this folding procedure iterates and that the form of the residual angular-momentum distribution after any number of emissions remains essentially the same, i.e., an isotropic Gaussian distribution added to a certain fixed vector.

It is easy to establish that the appropriate length of the fixed vector (which represents the average residue of the initial $\mathbf{J}_0$ left behind by N evaporating particles and photons) is

$$\mathbf{J}_F=\gamma_N\gamma_{N-1}\cdots\gamma_2\gamma_1\mathbf{J}_0. \tag{29}$$

For example, if all of the emissions involve classical spinless particles, then according to (20), (29) becomes

$$\mathbf{J}_F=(\mathcal{I}_N/\mathcal{I}_i)\mathbf{J}_0 \quad \text{(particles)}. \tag{30}$$

This expression is consistent with the fact, previously discussed, that for such emissions the average angular velocity $\boldsymbol{\omega}$ remains constant as evaporation proceeds.

It can also easily be shown that in the Gaussian distribution

$$d^2N/j_F^2dj_Fd\omega=e^{-\alpha_Fj_F^2}, \tag{31}$$

which must be added to $\mathbf{J}_F$, the quantity α_F is given by

$$\frac{1}{\alpha_F}=\tfrac{2}{3}\langle j_F^2\rangle=\frac{\gamma_N^2\gamma_{N-1}^2\cdots\gamma_2^2}{\alpha_1+\beta_1}+\frac{\gamma_N^2\gamma_{N-1}^2\cdots\gamma_3^2}{\alpha_2+\beta_2}+\cdots+\frac{1}{\alpha_N+\beta_N}. \tag{32}$$

Equation (32) is a generalization of the denominator in the brackets of (28) which applies to a sequence of only two emissions.

Since the final resultant angular momentum $\mathbf{J}_R$ is $\mathbf{J}_F+\mathbf{j}_F$, Eq. (31) may be written

$$d^2N/J_R^2dJ_Rd\Omega_R=\exp[-\alpha_F(\mathbf{J}_F-\mathbf{J}_R)^2]. \tag{33}$$

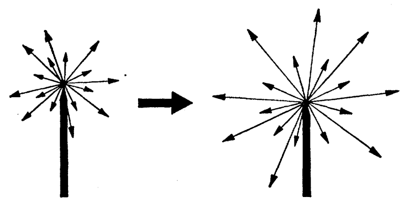

FIG. 4. The evolution of an angular-momentum distribution in a series of evaporations. As in Fig. 2, the resultant distribution at any stage is constructed by joining the tail of the fixed vector to the heads of the random vectors. In each evaporation the fixed vector becomes somewhat shorter and the rms random vector becomes somewhat longer.

Once again use has been made of the fact that the Jacobian of the coordinate transformation from $\mathbf{j}_F$ to $\mathbf{J}_R$ is unity [see the derivation of Eq. (10)]. Equation (33), used in conjunction with (32) and (30), gives the distribution of the resultant angular momentum after a series of N evaporations from a system with initial angular momentum $\mathbf{J}_0$. It is to be emphasized that this distribution depends on only two parameters, $\mathbf{J}_F$ and $(j_F)_{\text{rms}}=[\tfrac{3}{2}(1/\alpha_F)]^{1/2}$ [see Eq. (32)]. Both of these parameters can easily be computed using the explicit algebraic expressions for these quantities which have been derived. One important use of Eqs. (30) and (32) is to estimate the sensitivity of $\mathbf{J}_R$ to the values of the parameters on which $\mathbf{J}_F$ and $(j_F)_{\text{rms}}$ depend. Thus if one wants to learn whether quadrupole radiations in place of certain assumed dipole radiations would significantly change the distribution in $\mathbf{J}_R$, one has only to see how much the implied changes in the relevant α_i's would alter $\mathbf{J}_F$ and $(j_F)_{\text{rms}}$.

As the evaporation of particles and photons proceeds, $|\mathbf{J}_F|$ becomes smaller and $(j_F)_{\text{rms}}$ increases, as illustrated in Fig. 4. It is seen that the spread of the resultant angular-momentum vectors in angle about the direction of $\mathbf{J}_0$ necessarily increases with each evaporation. This spread is important for the interpretation of angular distributions of any radiations observed from the resultant states, and will be discussed in Sec. VI C.

B. Problems in the Application of the General Formula for Angular-Momentum Distributions

In order to use Eq. (33), together with (30) and (32), to compute an angular-momentum distribution after a series of evaporations, one must assign values to a number of parameters. To begin with, one must pick $\mathbf{J}_0$ and then one must decide how many steps to assume in the evaporation sequence. Finally, one must pick values for each of the parameters α_i and β_i. Every one of these choices raises questions, some of which will now be briefly discussed.

J_0. Generally, in nuclear reactions the initially formed compound states are broadly distributed in angular momentum. One must integrate Eq. (33) over this $\mathbf{J}_0$ distribution to determine the distribution for $\mathbf{J}_R$ after a sequence of evaporations from the originally formed

excited nuclei. This is done in a simple but typical situation in Sec. V.

Number of Emissions. In an actual nuclear reaction, say an α, $3n$ reaction, it is likely that the number of photons that are emitted to reach some particular low-lying state of the residual nucleus is not always the same. The simplest thing that one can do in view of the generally variable multiplicity is to calculate for the estimated *average* number of emitted photons. It might improve the calculations if one sums over distributions where the photon chains have different multiplicities. Note that in either procedure the final results may be incorrect if in actual decays the multiplicity is correlated with the history of a cascade. For example, it may be true that whenever the angular momentum happens to increase at first rather than decrease, the photon cascades are longer than average. In deriving Eq. (33) it was explicitly assumed that the parameters describing, say, the second evaporation in a cascade do not depend on which of the many allowed angular-momentum changes actually occurred in the first evaporation. This assumption cannot be exactly right, but it is a reasonable starting point for a statistical model. It is a separate matter to decide to what extent radiation cascades in nuclei can be described by such a model.

Values of α_i and β_i. It is relatively easy to assign reasonable values to α_i and β_i for neutron evaporations (and presumably for other particle evaporations when these occur in sufficient number). Besides, as we have seen, they do not very much influence the final $\mathbf{J}_R$ distribution. It is somewhat more difficult to assign parameters to the photon cascade. It is harder to be sure about multipolarities and about values of β, especially for radiative transitions which connect lower-lying excited states.

The basic reason for these difficulties is that the model developed in this paper, or indeed any statistical model, must break down at low excitation energies. It is clear that the angular-momentum distribution of the ensemble of nuclei does not continue to develop *ad infinitum*. After a finite number of emissions, we may be sure that the distribution has collapsed to a single spin (that of the ground state). Toward the end of the photon cascade, the choices of possible final states for radiations from any given state must become severely circumscribed. Since the emissions, according to a statistical model, must (at least on the average) have available an ample distribution of final states, a statistical model is a poor approximation for radiations between the lowest states.

A striking example of this kind of breakdown is given by the radiations of the ground-state rotational band in a distorted even-even nucleus. If J is the spin of a state in the band, the residual spin after a photon emission is definitely $J-2$, and not sometimes $J-1$, J, $J+1$, or $J+2$. Moreover, there are no nearby states which have an extreme tendency to radiate upward in spin to compensate for the behavior of the states in the band. To a lesser and variable degree, these same difficulties will apply to radiations from excited bands. Here there will be some options for decays from each state (e.g., interband transitions) but to the extent that a photon cascade has any tendency to run down a band, a statistical description may prove inappropriate.

Let us assume that despite these difficulties one wants to treat the photon cascade with a common statistical formalism. One must then see whether it is possible to choose values of the important parameters in the statistical model to "mock-up" the cascade in a rotational band. It is not hard to choose a value of α that represents quadrupole radiation, but how should the β's be chosen? If the transition is from the state of spin J and one picks β to give the correct average final J (namely, $J-2$), the statistical distribution will give a wrong value for the rms value of the final J. If β is chosen to match the rms value, the average will come out wrong. The statistical Gaussian form is simply unsuited to represent rotational transitions.

But let us persist nonetheless (in the hope that the errors made by forcing the statistical model to describe a rotational cascade will not be serious enough to destroy the approximate validity of the calculation). Let us decide to pick β to give the average spin of the final state correctly. Then $\gamma J=J-2$, where γ is $(1+\beta/\alpha)^{-1}$, Eq. (13). For quadrupole radiation, α is $(2\hbar)^{-2}$ and we are led to the relation $\beta\simeq(2J\hbar^2)^{-1}$. It is to be emphasized that the value of β that is needed here is spin-dependent. Moreover, if one now substitutes typical values for J, it is found that β comes out rather larger than the β's ($\sim 0.03\hbar^{-2}$) familiar from radiations at somewhat higher energies. One can describe this result by saying that the effective moment of inertia appears to be abnormally small. It has, in fact, often been remarked that effective moments of inertia seem to become small for the low-lying radiations in evaporation cascades. It is the point of the present discussion to indicate how the presence of low-lying collective excitations can give rise to an apparently small moment of inertia in a statistical model of the compound-nuclear decay. One must not, however, expect that the values of β (or $\mathcal{I}$ or σ^2) so obtained are a property of the nucleus alone. They must depend also on the magnitudes of the spins which are populated in the bombardment being studied.

The foregoing paragraphs indicate that a statistical model should not be expected to represent the decay of, say, an excited nucleus in the deformed region all the way down into the ground-state band. But it is hard to estimate the extent of the transition region between the point at which a statistical model becomes inappropriate and the ground-state band. If this region is small enough, statistical models may give useful first orientations to spin distributions of low-lying states which are excited in nuclear reactions. At least they should allow one to follow a good part of the de-excitation of a compound nucleus. To some degree such models should also help with the qualitative and semiquantitative understanding of the angular distributions as well as the yields of radiations from the low-lying states.

V. DISTRIBUTION IN MAGNITUDE OF RESIDUAL ANGULAR MOMENTUM IN TYPICAL COMPOUND-NUCLEAR REACTIONS

A. Derivation of an Expression for This Distribution

Despite some of the reservations about the validity of Eq. (33) which were mentioned in Sec. IV, we shall in this and the following sections deduce some of the implications of this equation. If one is interested in the distribution of *magnitudes* of the final angular momentum $\mathbf{J}_R$ given some particular $\mathbf{J}_0$, then one must integrate (33) over all possible orientations of $\mathbf{J}_R$'s having the same size. This integration can readily be shown to give

$$\frac{dN}{dJ_R}(J_F)=\left(\frac{\alpha_F}{\pi}\right)^{1/2}\frac{J_R}{J_F}\left[e^{-\alpha_F(J_R-J_F)^2}-e^{-\alpha_F(J_R+J_F)^2}\right]. \quad (34)$$

The quantity on the left is indicated to depend on J_F, which in turn is proportional to the magnitude of the initial angular momentum $\mathbf{J}_0$. If the starting ensemble involves a distribution of values of $\mathbf{J}_0$, then one must average over $\mathbf{J}_0$ or, alternatively, over $\mathbf{J}_F$ in order to obtain an over-all dN/dJ_R. To facilitate such averaging, (34) has been written in properly normalized form. That is,

$$\int \frac{dN}{dJ_R}(J_F)dJ_R=1. \quad (35)$$

If $p(J_F)$ and $P(J_0)$ give, respectively, the relative probabilities for the different values of J_F or the corresponding J_0, then the over-all distributions can be written

$$\frac{dN}{dJ_R}=\int \frac{dN}{dJ_R}(J_F)p(J_F)dJ_F \quad (36a)$$

or, in terms of J_0,

$$\frac{dN}{dJ_R}=\int \frac{dN}{dJ_R}(J_0)P(J_0)dJ_0. \quad (36b)$$

The distributions $P(J_0)$ must be obtained from the conditions involved in the bombardments which produce the starting set of excited compound nuclei.

B. Residual Angular-Momentum Distribution for Bombardments of Moderate Incident Energy

Although one might generally think of obtaining the function $P(J_0)$ from an optical-model description of the bombardment in question, one may instead use the familiar classical approximation for $P(J_0)$. For sufficiently high bombarding energies, this approximation should not differ too much from that implied by an optical model. Moreover, in view of its mathematical simplicity (it depends on but one parameter) it leads to expressions for dN/dJ_R which are easy to evaluate.

The classical distribution for $P(J_0)$ for spinless incident particles and a spinless spherical target nucleus has the form

$$P(J_0)=2J_0/J_{0M}^2,\quad J_0<J_{0M}$$
$$=0,\quad J_0>J_{0M} \quad (37)$$

where the maximum angular momentum is

$$J_{0M}=[2mR^2(E-V_c)]^{1/2}. \quad (38)$$

Here m and E are the mass and energy of the incident particle, R is the nuclear radius, and V_c is the Coulomb-barrier height for charged incident particles. Because of uncertainties in the values to be assigned to R and V_c, it may be best to use an optical model to determine a value for J_{0M} even when one uses (37) instead of the optical model in Eq. (36a).

Putting (37) into (36a) one finds that

$$\frac{dN}{dJ_R}=\frac{2}{\sqrt{\pi}}\frac{J_R}{J_{FM}^2}\left[\int_{-A}^{A}e^{-x^2}dx-\int_{B-A}^{B+A}e^{-x^2}dx\right], \quad (39)$$

where

$$A^2=\alpha_F J_R^2 \quad (40)$$

and

$$B^2=\alpha_F J_{FM}^2. \quad (41)$$

In these expressions J_{FM} is the value of J_F that corresponds to J_{0M}, the maximum value of J_0. Equation (39) can be written in terms of error functions,

$$dN/dJ_R=(J_R/J_{FM}^2)[2\,\mathrm{erf}A-\mathrm{erf}(B+A)+\mathrm{erf}(B-A)]. \quad (39')$$

It is interesting to examine (39) in the limits where B is much larger or much smaller than unity. Where B is very small, the total rms angular momentum which is removed by evaporating particles and photons is much greater than the angular momentum originally in the system. Under these circumstances, (39) can be shown to approach

$$\frac{dN}{dJ_R}\underset{B\to 0}{=}\frac{4}{\sqrt{\pi}}\alpha_F^{3/2}J_R^2e^{-\alpha_F J_R^2}. \quad (39'')$$

This equation is of the simple random-walk, or Gaussian, form. It gives the final angular-momentum distribution when the nuclei excited in the original bombardment all have negligible spin.

In the other extreme where $B\gg 1$, it is found that, aside from slight distortions near its limits, (39) approaches

$$\frac{dN}{dJ_R}\underset{B\to 0}{=}\frac{2J_R}{J_{FM}^2}. \quad (39''')$$

This is, of course, as it should be. When the random

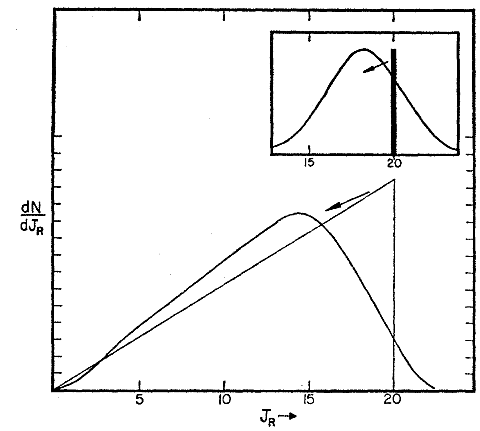

FIG. 5. The angular-momentum distributions in an idealized α, $3n$ reaction in a nucleus with $A=160$. The triangle gives the classical distribution for the originally formed compound nuclei and the curve toward its left gives the calculated distribution after the evaporation of the three neutrons and three or four photons. The inset shows the final distribution for a group of nuclei with initial spin equal to 20. It is seen that not much angular momentum is lost through the sequence of evaporations.

angular momenta contributed by the evaporations are negligible compared to the initial angular momenta, the final angular-momentum distribution will retain the initial shape (37). It will only shrink somewhat as the evaporations remove their small share of the system's angular momentum.

Since the assumed initial distribution, Eq. (37), ignores the possible existence of spin for either the projectile or the target nucleus, it is necessary to indicate how such spins may be taken into account. One can attribute these isotropic angular-momentum distributions (assuming unpolarized beams and targets) to fictitious isotropic first emissions. This will have the effect of decreasing the value of α_F that one must use. It will not affect the value of J_{FM} since the appropriate γ for the fictitious emissions is equal to unity.

As an illustration, Fig. 5 shows the angular-momentum distribution that one would expect after the evaporation of three neutrons and three or four photons in a 40-MeV α-particle bombardment of a nucleus of atomic weight $A\simeq 160$. The critical parameters were assigned the following values:

$$J_{0M}=20\hbar(J_{FM}=18\hbar)\,,$$
$$j_F^2=15\hbar^2 \text{ (i.e., } \alpha_F=1/10\hbar^2)\,.$$

The illustrated distribution can be compared with the observed strengths of ground-state band rotational transitions in distorted nuclei bombarded by 40-MeV α particles.[7,16] By subtracting the observed intensities of successive rotational lines, one obtains the intensities of the feeding of the rotational band from outside the band at each J of the band. It is found in Ref. 7, for example, that this spin distribution at entry into the ground-state band is shifted to a much lower average J than the one in Fig. 5. The median value of J of the compound nuclei produced in the 40-MeV α-particle bombardments in these experiments is about $14\hbar$. A statistical calculation, like that illustrated in Fig. 5, leads to an expected median J of about $13\hbar$ after the emission of three neutrons and several photons. The observed median J at entry into the ground-state band is, however, only about half this value. Thus instead of the expected negligible shift in the average nuclear angular momentum during the photon part of the cascade, one finds a very large one. Apparently, about as much angular momentum is "emitted" during the photon portion of the evaporation cascade which leads to the ground-state band as in the ground-state band itself. Similar discrepancies between experiment and expectations based on statistical theory are found in heavy-ion bombardments.[6]

These discrepancies probably stem mostly from two sources. One is a shortcoming of the statistical model and the other is a shortcoming of a ground-state rotational band as a fair sampler of the angular-momentum distribution at late stages of the evaporation cascade. The first difficulty is associated with the neglect, in the calculations, of the role of collective transitions in the photon cascades which lead into the ground-state band. This has been briefly discussed in Sec. IV. The second difficulty, first emphasized by Stephens,[18] is connected with the fact that the higher states in the ground-state band are not well separated in energy from states of comparable spin. They consequently share their probability for being excited with neighboring (generally unresolved) states. Both these reasons for the failure of statistical models to account for ground-state band intensities are discussed in detail elsewhere.[16–18]

VI. ANGULAR DISTRIBUTIONS OF PARTICLES AND RADIATION FROM RESIDUAL NUCLEI IN COMPOUND-NUCLEAR REACTIONS

A. Derivation of a General Formula for Angular Distributions

Let $\mathbf{J}_R$ be the angular momentum of some particular state which appears in the course of the evaporation cascade. We are interested in the angular distributions of particles or radiations emitted by this state. These can be determined once the distribution of the angular momentum associated with the emission is known. Let $\mathbf{j}_e$ represent this angular momentum. If now $w(\cos^2\psi)$ represents the angular distribution of the emission with respect to $\mathbf{j}_e$, the angular distribution of the emission with respect to $\mathbf{J}_R$ can be obtained by folding the function $w(\cos^2\psi)$ together with the distribution function for $\mathbf{j}_e$ about $\mathbf{J}_R$. We call the function for the latter

[16] H. Ejiri, M. Ishihara, M. Sakai, K. Katori, and T. Inamura, INS Report No. 116, Tokyo (to be published).

[17] C. F. Williamson, S. Ferguson, I. Halpern, and B. J. Shepherd (to be published).

[18] F. S. Stephens (private communication); see also J. Burde, R. M. Diamond, and F. S. Stephens, Nucl. Phys. A92, 306 (1967).

distribution $\mathfrak{D}(\cos\theta)$. The angles which appear in the functions w and $\mathfrak{D}$ have the following meanings: ψ is the angle between $\mathbf{j}_e$ and the direction of the emitted radiation, and θ is the angle between $\mathbf{j}_e$ and $\mathbf{J}_R$.

The given form for w reflects two of its important properties. (a) There is no azimuthal dependence of the emission pattern about $\mathbf{j}_e$ and (b)

$$w(\pi-\psi)=w(\psi). \tag{42}$$

These properties hold for both emitted (heavy) particles and emitted radiations because the parity is presumably conserved in these emissions. Of course, the actual function w is not the same for different kinds of emissions. In particular, for particles w peaks at right angles to $\mathbf{j}_e$ but for dipole photons, say, w peaks in the direction (back and forth) along $\mathbf{j}_e$. It follows from (42) that it is permissible to express w as $w(\cos^2\psi)$.

The function $\mathfrak{D}$ cannot depend on the azimuth of $\mathbf{j}_e$ about $\mathbf{J}_R$. This is to say that angular-momentum directions at all azimuths about $\mathbf{J}_R$ must be equally likely. But in contrast to w, the function $\mathfrak{D}$ need not be symmetrical about an equatorial plane. In fact, since emissions tend to remove a share of a system's angular momentum, $\mathbf{j}_e$ tends to be distributed in a way that makes $\langle\mathbf{J}_R\cdot\mathbf{j}_e\rangle$ positive. The function $\mathfrak{D}$ may thus depend in an arbitrary way on the angle θ, and we write it $\mathfrak{D}(\cos\theta)$.

Clearly, the angular distribution of the emitted particles or radiation with respect to $\mathbf{J}_R$ may be written

$$W(\Theta)=\int w(\cos^2\psi)\,\mathfrak{D}(\cos\theta)d\Omega\,, \tag{43}$$

where the integration is over the distribution $\mathfrak{D}$ of the vectors $\mathbf{j}_e$ and Θ is the angle between $\mathbf{J}_R$ and the emission direction. The quantity $d\Omega$ is $\sin\theta d\theta d\phi$, where ϕ is the azimuth of $\mathbf{j}_e$ about $\mathbf{J}_R$. The connection between the various angles introduced so far is easily shown to be

$$\cos\psi=\cos\theta\cos\Theta+\cos\phi\sin\theta\sin\Theta. \tag{44}$$

From the form of $W(\Theta)$ it is possible to state some of its simple properties upon inspection. First, if either the function $\mathfrak{D}$ or the function w happens to be isotropic, then $W(\Theta)$ is isotropic. Second, if the function $\mathfrak{D}$ happens to be linear in $\cos\theta$, the function $W(\Theta)$ will still be isotropic. This follows from the fact that $\mathfrak{D}$ is independent of ϕ. Then, using (44), only even powers of $\cos\theta$ in the expansion of $w(\cos^2\psi)$ will lead to contributions to $W(\Theta)$. Consequently, there will then be no contributions from terms in $\mathfrak{D}$ which are odd in $\cos\theta$. Thus for there to be any anisotropy in the emissions, it is necessary that in the expansion of $\mathfrak{D}(\cos\theta)$,

$$\mathfrak{D}(\cos\theta)=\sum_i a_i\cos^i\theta\,,$$

there be at least one nonvanishing even coefficient ($i\geq 2$). These general features of angular distributions are well known and are illustrated in Fig. 6.

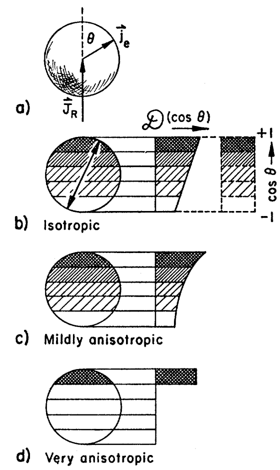

FIG. 6. Diagrams for understanding the angular distributions of radiations. The densities of shading on the spheres represent relative probabilities for different orientations of the vector $\mathbf{j}_e$ with respect to $\mathbf{J}_R$. The vector $\mathbf{j}_e$ represents the angular momentum of the emitted radiation and $\mathbf{J}_R$ is that of the radiating system. For each sphere, this density distribution per unit solid angle is also plotted at the right. If the density distribution is linear in $\cos\theta$ (i.e., in m_z), the angular distribution of radiation is isotropic (since the radiation pattern for m_z is the same as that for $-m_z$). Only when the density function has some curvature is there any anisotropy of radiation. In general, this curvature is relatively small for evaporations. For emissions in the ground-state rotational band of an aligned nucleus, the density function is a δ function at $m_z=|j_e|$. One can then expect appreciable anisotropy of the emitted radiation.

B. Angular Distributions of Statistically Emitted Photons

The character of the function $\mathfrak{D}$ that is relevant for evaporations is given by the spin-cutoff or warping factor $\exp[-\beta(\mathbf{J}_R-\mathbf{j}_e)^2]$ that was introduced in Eq. (7). Expanding the angle-dependent part of this exponential,

$$\begin{aligned}\mathfrak{D}(\cos\theta)&\simeq\exp[-\beta J_R^2-\beta j_e^2+2\beta J_R j_e\cos\theta]\\&\simeq\{1+2\beta J_R j_e\cos\theta+2\beta^2 J_R^2 j_e^2\cos^2\theta+\cdots\}\\&\qquad\times\exp[-\beta J_R^2-\beta j_e^2],\end{aligned} \tag{45}$$

it is seen that the earliest term which can be responsible for any anisotropy is the third term in the expansion. The actual anisotropy of the distribution $W(\Theta)$ will, of course, depend on the particular form of $w(\cos^2\psi)$, which appears in (43), but in general the coefficient which measures the magnitude of the anisotropy will not exceed $2\beta^2 J_R^2 j_e^2$,[11] the coefficient of $\cos^2\theta$ in (45). Now βJ_R^2 may be a number as large as unity for 40-MeV α-particle bombardments of large nuclei. It can be somewhat larger for bombardments with energetic heavy ions and it will be smaller for proton bombardments. The other factor βj_e^2 is for, say, dipole photon emissions of the order of a few percent. Therefore, even in the most favorable circumstances one can hardly expect anisotropies of statistically emitted dipole photons

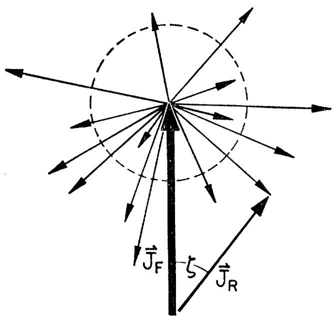

FIG. 7. Diagram shows the orientation in space of angular-momentum vectors $\mathbf{J}_R$ in a residual system after a sequence of evaporations. $\mathbf{J}_F$ is parallel to the initial angular momentum $\mathbf{J}_0$ and is slightly smaller in magnitude. If the average angle ζ is small, the evaporations did not significantly disorient the original nuclear alignment. When ζ is very large, the system has been isotropized, and consequently any radiations emitted by the system will also be rather isotropic.

greater than about 5%.[13] That is, despite the considerable amount of angular momentum which is contained in some evaporating systems, the angular distributions of evaporated photons (the bulk of which are presumably electric dipole) will remain rather isotropic. Anisotropies of photons of higher multipolarity or of evaporating particles will be somewhat larger than those of dipole photons because j_e^2 is larger.

C. Angular Distributions of Rotational Radiations in Distorted Nuclei

It is instructive to contrast the (nearly isotropic) angular distribution expected for normally evaporated photons with the distributions expected for ground-state band rotational radiations which are observed in the same bombardments. As has been emphasized in Sec. IV C, the quadrupole photons emitted from a rotational state with spin J_R necessarily go to a state with spin J_R-2 instead of being allowed to reach states with spins from J_R-2 to J_R+2 as they would be for more nearly "statistical" quadrupole photons. Thus the function $\mathfrak{D}$ for radiations in a ground-state rotational band is sharply peaked instead of being spread smoothly over all directions with respect to the spin of the emitting nucleus. This is illustrated in Fig. 6 and accounts for the large anisotropy that one expects, and observes, for such radiations.

If a classical quadrupole is rotating about its axis, it is readily shown that the angular distribution of emitted radiation has a very anisotropic form with respect to the axis;

$$W(\cos^2\Theta)=\sin^2\Theta(1+\cos^2\Theta). \quad (46)$$

In a typical bombardment the angular-momentum axes of the compound nuclei originally formed will lie in a plane perpendicular to the beam. If we overlook disorientations of the directions of these axes that may be brought about by the evaporations of neutrons and photons or by initial spins, then the vectors $\mathbf{J}_R$ for the rotational states will lie in the plane perpendicular to the beam. To obtain the angular distribution of the rotational radiation with respect to the beam one must average (46) over a set of axes uniformly distributed around the beam. The result is easily found to be

$$W(\Theta_B)=\tfrac{5}{32}[5+6\cos^2\Theta_B-3\cos^4\Theta_B]. \quad (47)$$

Here Θ_B is the angle between the beam and the observed rotational radiation. This expression can alternatively be derived[17] by computing the $E2$ radiation pattern for the transition $J_R\rightarrow J_R-2$, with the value of m in the initial state set equal to zero (in order to reflect the fact that there is no component of $\mathbf{J}_R$ in the beam direction). One must then let $\mathbf{J}_R$ become infinite. If one proceeds in this way, he finds that as $\mathbf{J}_R$ is increased from its lowest possible value, 2, the angular patterns very quickly approach the classical ($J=\infty$) limit given by (47).

Thus neglecting the angular-momentum reorientations due to neutron and photon emissions which occur before the cascade reaches the ground-state rotational band, one would expect all of the rotational lines to show an angular distribution very close to that given by (47). It is necessary at this point to make an estimate of the reorientation or rocking effect associated with the evaporations which occur on the way to the rotational band.

For this purpose let us consider that the initially excited nuclei are formed with angular momentum $\mathbf{J}_0$. After the evaporation of a certain number of neutrons and photons, the system has reached the state with spin $\mathbf{J}_R$ in the ground-state rotational band. We are interested in the distribution of the possible $\mathbf{J}_R$'s in direction about the original $\mathbf{J}_0$ direction. One rotational state, i.e., one value of $|\mathbf{J}_R|$, is considered at a time. The distribution function for $\mathbf{J}_R$ is given by Eq. (33). The geometrical relations between the vectors in this equation are illustrated in Fig. 7. It is seen that (33) can be written

$$d^2N/J_R^2dJ_Rd\Omega_R=\exp[-\alpha_F(J_R^2+J_F^2-2J_RJ_F\cos\zeta)], \quad (33')$$

where ζ is the angle between $\mathbf{J}_R$ and $\mathbf{J}_F$ (or $\mathbf{J}_0$).

By rewriting this in the form

$$d^2N/J_R^2dJ_Rd\Omega_R=\exp[-\alpha_F(J_R-J_F)^2]\times\exp[-\alpha_F(4J_RJ_F\sin^2\zeta/2)], \quad (33'')$$

it is apparent that for a given value of $|\mathbf{J}_R|$ the distribution falls to $1/e$ of its ($\zeta=0°$) peak value when $\zeta=\zeta_c$, where

$$\sin\tfrac{1}{2}\zeta_c=\tfrac{1}{2}(\alpha_FJ_RJ_F)^{-1/2}. \quad (48)$$

To determine whether the nuclear reorientations due to the evaporation cascade will significantly wash out the angular distribution of the radiation, one must ask whether an angular spread in the orientations of J_R of this amount will seriously distort the distribution (46). If $2l$ gives the highest power of $\cos\Theta$ in a distribution, one does not generally expect significant smearing until the smearing angle is about $(2l)^{-1/2}$. For the distribution (46) this would suggest that for values of ζ_c less than $\frac{1}{2}$ rad the rocking should not smear the observed radiation distribution very much. As an example, let us estimate the value of ζ_c for the α, $3n$ reaction for which the parameters were given in Sec. V B: $J_{FM}=18\hbar$ (making a typical J_F equal to about $12\hbar$), and $\alpha_F=0.1\hbar^{-2}$. Let us say that we are interested in the 8^+ level in the rotational band. Then $J_R=8\hbar$. Using (48), we learn that ζ_c is about $\frac{1}{3}$ rad. According to our criterion, this should not produce serious smearing. Suppose, however, that one is interested in the angular distribution of the $2^+\rightarrow 0^+$ transition. Then J_R is only $2\hbar$ and perhaps the appropriate value of J_F is now less than $12\hbar$ since the 2^+ level probably is produced most often from the lowest angular-momentum depositions in the bombardment. The application of (48) in this case would lead one to expect some washing out of the distribution.

At this point it is necessary to call attention to an important feature of angular distributions of rotational radiations. If the evaporation cascade enters a rotational band at, say, the 8^+ level, the radiations $8^+\rightarrow 6^+$, $6^+\rightarrow 4^+$, and so on down the chain, will all have exactly the same angular distributions.[19] The rocking effect estimated above for the 2^+ level had to do only with events where the 2^+ level is reached from a level outside the band. However, the 2^+ populations are generally mainly fed from earlier entries into the band. Therefore, they inherit the angular distributions of the rotational radiations associated with these higher spin states. As we have indicated, these distributions are very little smeared by evaporation rocking. It follows, therefore, that one can expect but little washing out of the classical quadrupole radiation pattern, Eq. (47), for any of the rotational radiations.

This is indeed what one observes. The spins of low-lying nuclear levels which are formed in bombardments where considerable amounts of angular momentum are deposited tend very much to retain the orientations of the angular momenta of the compound states initially produced in the bombardment.[5–8]

VII. CONCLUDING REMARKS

Some of the main results of this paper can be stated in qualitative terms. At nuclear excitations which are high enough so that a reasonable variety of residual state spins are available after particle or photon emissions, it is useful to divide the angular momentum removed by the emissions into two distinct parts. In classical language, one part can be associated with the random thermal motions of the nucleons in the nucleus. The average value of this angular momentum is zero and the average of the square of this angular momentum is proportional to the nuclear temperature. The second part of the removed angular momentum is a fixed fraction of that of the emitting system. It was shown that one can automatically identify these two angular-momentum components if one represents the emission spectrum of angular momentum by a Gaussian function which is warped by a Boltzmann factor to depress the relative probabilities for residual states of high spin.

As long as there are plenty of available residual states with higher spin than the emitting state, it is found that the average angular momentum of a decaying nucleus changes very little with each emission. In higher-energy bombardments, particle evaporation generally takes place before photon emission and the excitation energies are such that spin availability criteria are met. One may therefore say in rough approximation that in the decay of a hot nucleus the particles leave first, taking almost all of the excitation energy and almost none of the angular momentum. At the very end of the decay, when the particles have removed all of the energy that they can, the photons are emitted and it is left for them to remove the nuclear angular momentum. As long as the excitation energies are still reasonably high in this photon stage of decay, the general statistical descriptions of this paper should apply. There must come a point, however, where a decrease of residual angular momentum in a photon emission is on the average much more probable than an increase. Although emissions under such circumstances are in a sense still statistical, they cannot easily be treated by the methods discussed here or by equivalent methods. The rate of removal of angular momentum in the final stages of decay apparently depends on both the angular-momentum content of the nucleus and on its detailed level structure.

Although these effects of level structure make it difficult to predict yields of low-lying states of various spins that would be observed in moderate energy bombardments, they are less troublesome as regards orientation effects. It is shown that the statistical emissions of particles and photons do not tend to reorient the direction of angular momentum significantly. This is basically because (a) the evaporating particles and photons remove only their share, a small fraction of the ngular momentum $J_{\text{in}}\sim(2M_{\text{inc}}E_{\text{inc}}R^2)^{1/2}$; (b) the random or thermal angular momentum contributed by an evaporation $J_{\text{out}}\sim(2M_{\text{out}}(2T)R^2)^{1/2}$ is small compared with J_{in}; and (c) for a sequence of emissions, the various thermal angular momenta must be added incoherently or randomly. Thus if the average angular momentum of the initial compound system is sizeable, the system will tend to maintain its original orientation distribution after many evaporations. If some of these evaporations happen to consist of photon emissions between states in rotational bands, this will not tend to introduce reorientations. Thus, although the presence of

[19] S. R. deGroot, H. A. Tolhoek, and W. J. Huiskamp, in *Alpha-, Beta-, and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (North-Holland Publishing Co., Amsterdam, 1965).

transitions in higher rotational bands may make it difficult to predict the yields of states of different J, they will not upset the conclusion based on a conventional statistical treatment that nuclei of high initial spin retain their orientations as they cool.

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Unified Theory of Alpha Decay*

KICHINOSUKE HARADA
Japan Atomic Energy Research Institute, Tokai-Mura, Japan

AND

ELIZABETH A. RAUSCHER
University of California, Lawrence Radiation Laboratory, Livermore, California

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A new theory of α decay is presented in which the nuclear-radius parameter is not used explicitly. An expression is derived—similar to the one in Feshbach's unified reaction theory, and based on the time-dependent perturbation method—which overcomes the inadequacies of previous α-decay theories. It is pointed out that the new decay formula makes it possible to calculate even the absolute value of the decay rate. In this theory, the α-decay process is interpreted as the transition which is caused by the difference between four times the nucleon-nucleus and α-nucleus potentials. The method of numerical calculation is illustrated for the case of a simple α decay.

I. INTRODUCTION

MANY previous α-decay-rate calculations based on the nuclear shell model have been performed, and they reproduce the experimental fine structures of decay rates fairly well, so far as the relative intensities are concerned.[1,2] Therefore, the basic formula used in the decay-rate calculations, which is the well-known expression for the resonance level width in R-matrix theory, seems to be useful.

The applicability of the one-level formula in R-matrix theory to the α-decay problem was shown by Thomas,[3] who suggested the possibility of using the shell-model microscopic calculations of the α-decay rate. Mang[4,5] was the first to perform practical microscopic calculations for the α-decay rates. He derived the basic formula by generalizing Casimir's time-dependent α-decay theory,[6] and then used it in his calculations. If the boundary condition which assures the continuity of the internal and the external α-particle wave functions, is introduced into Mang's formula, it becomes identical with Thomas's.

However, there are two shortcomings in these microscopic calculations based on the one-level formula. (a) We must introduce the nuclear radius, which divides the space sharply into internal and external regions. The barrier penetrability, relevant to α decay, is very sensitive to the radius parameter. For that reason many theoretical calculations have been performed only for the determination of relative intensities, but not for the calculation of absolute decay rates. (b) The shell-model wave function used for the parent nucleus does not satisfy the boundary condition, which we normally impose upon a compound state in R-matrix theory. Fortunately, the energies of the emitted α particle are much lower than the height of the Coulomb barrier, and these energies lie in a narrow range of 5–10 MeV, so that the error due to this defect would be nearly constant. Strictly speaking, however, this defect certainly brings some errors even into the theoretical predictions of the relative intensities.

Recently, we[7] studied the α decay of Po^{212m}, taking into account the effects of collective octupole coupling, and found that the conventional microscopic theory does not explain the large α-decay group to the first excited state of Pb^{208}. A major cause of discrepancy may be attributed to the lack of octupole core polarization in the wave function for Po^{212m}. However, it is also the case that the conventional theory is not satisfactory in describing this isomeric state decay because the

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