Upper Bound to the Ground-State Energy of N-Body Systems and Conditions on the Two-Body Potentials Sufficient to Guarantee the Existence of Many-Body Bound States

F. CALOGERO* AND YU. A. SIMONOV
Institute of Theoretical and Experimental Physics, Moscow, USSR

(Received 13 November 1967)

We obtain simple conditions on the two-body potentials which guarantee that at least one bound state exists, when N particles interact pairwise with such potentials. The same approach also yields an upper bound to the energy of the N-body ground state.

1. INTRODUCTION

HERE we consider N quantum-mechanical nonrelativistic particles of masses m_i interacting pairwise with the spin-independent potentials $V_{ij}(r_{ij})$; r_{ij} indicates of course the interparticle distance, and $V_{ij}(r)$ is assumed to vanish as r diverges. We produce a simple upper bound to the ground-state energy of this N-body system, and we obtain simple conditions which, when satisfied by the potentials $V_{ij}(r)$, imply that the ground-state energy of the N-body system is negative.

Throughout this paper, by "N-body ground state" we mean the lowest-energy eigenstate of the N-body Hamiltonian. This state, even when its energy is negative, need not be bound (normalizable)[1]; for instance, in the ground state of an N-body system containing a particle which is repelled by all others, this particle sits (as a zero-energy free particle) a large distance away from all others. However, if the ground-state energy of the N-body system is negative, either this state is itself a bound state, or at least one of its (N-P)-body subsystems ($P>0$) can exist as a bound state.[1a] In any case, because the ground-state energy of an N-body system cannot exceed that of any of its subsystems, any upper bound to its value is of interest only if it falls below all known upper bounds to the ground-state energies of its subsystems. An analogous remark applies to conditions on the two-body forces sufficient to guarantee that the energy of the N-body ground state will be negative.

For simplicity in our discussion we focus attention on the $L=0$ case, $L(L+1)\hbar^2$ being the total angular momentum of the N-body system; but the same approach may be used for positive L, thereby obtaining an upper bound to the energy of the lowest-energy state with angular momentum L, and sufficient conditions for the existence of at least one N-body state with angular momentum L and negative energy. For the same reason we consider systems interacting only by two-body forces; the generalization to include many-body forces is trivial.

The extension to identical particles (bosons or fermions) and to spin and isospin-dependent interactions is straightforward.[2] Neither these topics, nor realistic applications of our results to nature, are discussed in this paper. We do, however, present some examples, which illustrate the power of this approach.

2. METHOD

Our approach is based on the treatment of the N-body problem by means of harmonic analysis in $3N-3$ dimensions. We report here the necessary results, referring to the literature[3] for all details.

The starting point is the expansion of the N-body wave function $\Psi(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)$ into a set of spherical harmonics $u_K{}^\nu$ (in $3N-3$ dimensions):

$$\Psi=\rho^{-\frac{1}{2}(3N-5)}\sum_{K\nu} u_K{}^\nu \chi_K{}^\nu(\rho). \tag{1}$$

The "radial" coordinate ρ is defined by

$$\rho=\Big[\sum_{i>j}^{N}\beta_{ij}|\mathbf{r}_i-\mathbf{r}_j|^2\Big]^{1/2} \tag{2a}$$

$$=\Big[\sum_{i=1}^{N} m_i R_i^2/m\Big]^{1/2}, \tag{2b}$$

* Permanent address: Physics Department, Rome University, Rome, Italy.

[1] For $N>2$.

[1a] *Note added in proof.* This very plausible statement has not yet been rigorously proved in the N-body case, although work in this direction by O. Yakubovski is in progress. In the 3-body case a rigorous proof can be found in Faddeev's work [L. D. Faddeev, Trudy Matem. Inst. V. A. Steklov, 1963, Vol. LXIX; English transl.: *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Israel Program for Scientific Translations, Jerusalem, 1965)]. One of us (F.C.) wishes to thank Dr. Sergeant of Saclay for raising this point; both of us wish to thank Prof. L. D. Faddeev for an illuminating discussion.

[2] Actually, all the results given in this paper hold without modification in the case of bosons.

[3] Yu. A. Simonov, Yadern. Fiz. **3**, 630 (1966) [English transl.: Soviet J. Nucl. Phys. **3**, 461 (1966)]; Yu. A. Simonov and A. M. Badalyan, Yadern. Fiz. **3**, 1032 (1966): **5**, 88 (1967) [English transls.: Soviet J. Nucl. Phys. **3**, 755 (1966); **5**, 60 (1967)]; V. V. Pustovalov and Yu. A. Simonov, Zh. Eksperim. i Teor. Fiz. **51**, 345 (1966) [English transl.: Soviet Phys.—JETP **24**, 230 (1967)]; A. M. Badalyan, E. S. Galpern, V. N. Liakhovestskij, V. V. Pustovalov, Yu. A. Simonov, and E. L. Surkov, Yadern. Fiz. (to be published); Yu. A. Simonov, in Proceedings of the Symposium on Problems in Nuclear Physics, Tbilisi, 1967 (in Russian) (to be published). See, also, G. Morpurgo, Nuovo Cimento **9**, 461 (1952); F. T. Smith, Phys. Rev. **120**, 1058 (1960); J. Math. Phys. **3**, 735 (1962); A. J. Dragt, *ibid.* **6**, 533 (1965); J. M. Lévy-Leblond and F. Lurçat, *ibid.* **6**, 1564 (1965); J. M. Lévy-Leblond and M. Lévy-Nahas, *ibid.* **6**, 1571 (1965). Additional relevant references may be traced from these papers. Some results used in this paper (and in particular the N-body case with unequal masses) are not discussed in detail in the literature. They will be treated in detail in future publications.

where

$$\beta_{ij}=[m\sum_{k=1}^{N} m_k/(m_im_j)]^{-1}, \tag{3}$$

m is an arbitrary mass which enters as a scale constant, and R_i is the distance of the ith particle from the center of mass of the system. The bound-state problem for total angular momentum L is thereby reduced to the solution of the system of infinite coupled "radial" Schrödinger equations[4]:

$$\frac{d^2\chi_K^\nu}{d\rho^2}+\frac{1}{\rho}\frac{d\chi_K^\nu}{d\rho}+\left[2m\hbar^{-2}E-\frac{(K+\frac{1}{2}(3N-5))^2}{\rho^2}\right]\chi_K^\nu -2m\hbar^{-2}\sum_{K'\nu'}U_{KK'}{}^{\nu\nu'}\chi_{K'}{}^{\nu'}=0. \tag{4}$$

The index K takes either the values $L, L+2, L+4, \cdots$ or the values $L+1, L+3, L+5, \cdots$, depending on the parity of the state [the orbital parity is $(-1)^K$]. The index ν stands in general for a set of indices. In the three-body case ν is a single index, which takes the values $-K/2, -K/2+2, \cdots, K/2-2, K/2$; in the N-body case the set of indices ν spans the finite domain $D(K)$. In particular, for $K=0$ the set reduces to the single value $\nu=0$. From now on we consider only the $L=0$ case. Of course the ground state of the N-body system under consideration has $L=0$ and even orbital parity.

The potential matrix $U_{KK'}{}^{\nu\nu'}(\rho)$ is defined by the equation[4]

$$U_{KK'}{}^{\nu\nu'}(\rho)=\int d\Omega_{3N-3}u_K{}^\nu U u_{K'}{}^{\nu'}, \tag{5}$$

where the integration extends over the $3N-4$ angular "polar" coordinates, and

$$U=\sum_{i>j}^{N} V_{ij}(|\mathbf{r}_i-\mathbf{r}_j|). \tag{6}$$

Each matrix element $U_{KK'}{}^{\nu\nu'}(\rho)$ is the sum of $\frac{1}{2}N(N-1)$ terms, which behave in the origin ($\rho=0$) just as the corresponding two-body potentials $V_{ij}(r)$ do at $r=0$,[5] and asymptotically ($\rho\rightarrow\infty$) either as the two-body potentials themselves or as ρ^{-3}, whichever vanishes more slowly (in modulus). This long-range character of the matrix elements $U_{KK'}{}^{\nu\nu'}(\rho)$, even for short-range two-body forces, is a reflection of the more extended nature of the N-body system.[6]

In particular we have

$$W_N(\rho)\equiv U_{00}{}^{00}(\rho)=\left\{\int_0^{\pi/2}d\theta\,\sin^2\theta(\cos\theta)^{3N-7}\right\}^{-1} \times\left\{\int_0^{\pi/2}d\theta\,\sin^2\theta(\cos\theta)^{3N-7}\sum_{i>j}^{N}V_{ij}(\rho\alpha_{ij}\sin\theta)\right\} \tag{7a}$$

$$=4\pi^{-1/2}[\Gamma(\tfrac{3}{2}N-\tfrac{3}{2})/\Gamma(\tfrac{3}{2}N-3)] \times\left\{\int_0^{\pi/2}d\theta\,\sin^2\theta(\cos\theta)^{3N-7}\sum_{i>j}^{N}V_{ij}(\rho\alpha_{ij}\sin\theta)\right\} \tag{7b}$$

$$=4\pi^{-1/2}[\Gamma(\tfrac{3}{2}N-\tfrac{3}{2})/\Gamma(\tfrac{3}{2}N-3)]\rho^{-3}\sum_{i>j}^{N}\alpha_{ij}{}^{-3} \times\int_0^{\rho\alpha_{ij}}dr r^2(1-\alpha_{ij}{}^{-2}r^2\rho^{-2})^{(3/2)N-4}V_{ij}(r), \tag{7c}$$

with $\alpha_{ij}=[m(m_i+m_j)/(m_im_j)]^{1/2}$. Note that, if all the two-body potentials $V_{ij}(r)$ are nowhere decreasing (corresponding classically to forces which are attractive everywhere),

$$W_N(\rho)\leq\sum_{i>j}^{N}V_{ij}(\rho\alpha_{ij}). \tag{8}$$

Also note that, as $\epsilon\rightarrow 0^+$, we get from Eqs. (7) $W_{2+\epsilon}(\rho)\rightarrow V_{12}(\rho\alpha_{12})$.

An N-body state with negative energy E corresponds to a solution of Eq. (4) such that each function $\chi_K{}^\nu(\rho)$ vanishes at the origin and at infinity but not all of them vanish identically. The corresponding wave function Ψ need not vanish asymptotically at large ρ, because the sum in Eq. (1) may diverge for special values of the angular coordinates; it does vanish asymptotically in the case of an N-body bound state.[7]

3. RESULTS

Our results are based on the following *Lemma: Let E_0 indicate the smallest eigenvalue associated with Eq. (4), and let $E(M)$ be an eigenvalue of the "truncated" problem which obtains setting, in Eq. (4),*

$$U_{KK'}{}^{\nu\nu'}(\rho)=0 \quad\text{for}\quad K, K'>M. \tag{9}$$

Then

$$E_0\leq E(M)\leq E(M-P),\quad P\geq 0. \tag{10}$$

Proof: Write down the Ritz variational principle for the energy of the ground state and insert as trial function

$$\Phi(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)=\rho^{-\frac{1}{2}(3N-5)}\sum_{K=0}^{M}\sum_{\nu\subset D(K)}\chi_K{}^\nu(\rho)u_K{}^\nu, \tag{11}$$

[4] Our $U_{KK'}{}^{\nu\nu'}(\rho)$ differs from that of Ref. 3 by a factor $-\hbar^2/(2m)$.

[5] Definition: two functions "behave" in the same way at a point if their ratio there is a nonvanishing constant (not necessarily unity). However, this statement applies only to the diagonal elements $U_{KK}{}^{\nu\nu}(\rho)$; the ratio of the nondiagonal elements to the diagonal ones is a finite constant or zero.

[6] It should be emphasized that while the divergence of any interparticle distance $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$ implies the divergence of ρ, the divergence of ρ implies only the divergence of at least one of the interparticle distances r_{ij}. Thus the total potential U of Eq. (6), considered as a function of ρ, does not vanish as ρ diverges (for special values of the "angular" coordinates). The compatibility of this asymptotic behavior with the asymptotic vanishing of each matrix element $U_{KK'}{}^{\nu\nu'}(\rho)$ is brought about by the nonuniformity in ρ of the convergence of the expansion of U into spherical harmonics.

[7] We do not present any proof of these statements, which should, therefore, be regarded as plausible conjectures. However, none of the following results depends on them.

where the functions $\chi_K^{\nu}(\rho)$ are the solutions of the "truncated" problem. This proves the first inequality in Eq. (10); the second inequality is proved by a similar argument (and in any case is not used below).

The following theorem is an immediate consequence of the preceding lemma (for $M=0$) and of the structure of Eq. (4).

Theorem: Let E indicate the ground-state energy of the N-body system, and let $E_l[W_N]$ be the energy of a bound state with angular momentum $l(l+1)\hbar^2$ of two particles of mass $2m$ (so that the reduced mass is m) interacting through the potential $W_N(r)$ of Eqs. (7). Then

$$E\leq E_{(3/2)N-3}[W_N]. \tag{12}$$

Note that corresponding to integral values of N the "equivalent" angular momentum $l=\frac{3}{2}N-3$ may take integral or half-integral values.[8] Of course, any upper bound to the energy $E_{(3/2)N-3}[W_N]$ of the two-body bound state provides *a fortiori* an upper bound to E. Such upper bounds may be obtained from the Ritz variational principle or from other formulas which exist in the literature.[9]

A corollary of this theorem, which obtains upon setting $E_{(3/2)N-3}[W_N]$ to zero, states:

Corollary 1: The existence of one bound state with angular momentum $l(l+1)\hbar^2$, $l=\frac{3}{2}N-3$, in the two-body problem (with reduced mass m) with the "equivalent" potential $W_N(r)$ of Eqs. (7), implies that the ground-state energy of the corresponding N-body system is negative.[10]

To ascertain whether a given two-body potential does or does not bind states with assigned angular momentum is an easy computing job. There also exists, in the literature, explicit formulas which provide sufficient conditions for the existence of two-body bound states.[9] They imply the following:

Corollary 2: If there exists either a positive constant R or a nondecreasing function $g(r)$, with the additional properties

$$0\leq g(r)\leq r^{3N-5}, \tag{13a}$$

$$\lim_{r\to 0}[r^{3N-3}W_N(r)/g(r)]=0, \tag{13b}$$

such that any one of the following inequalities hold:

$$2m\hbar^{-2}\int_0^\infty dr\, r^{6-3N}g^2(r)\,|W_N(r)|\geq(3N-5)g(\infty)$$

and

$$W_N(r)\leq 0, \tag{14a}$$

$$2m\hbar^{-2}\int_0^\infty dr\, R|W_N(r)|[(r/R)^{3N-6}+(r/R)^{6-3N}R^2 2m\hbar^{-2}|W_N(r)|]^{-1}\geq 1$$

and

$$W_N(r)\leq 0, \tag{14b}$$

$$\int_0^\infty dr\,\min[R^{-1}(r/R)^{3N-6},\, -R(r/R)^{6-3N}2m\hbar^{-2}W_N(r)]\geq\tfrac{1}{2}\pi, \tag{14c}$$

with $W_N(r)$ given, in terms of the two-body potentials, by Eqs. (7), then the energy of the ground state of the N-body system is negative.[10]

Note that the first two inequalities apply only in the case of nowhere repulsive "equivalent" potentials; obviously, a sufficient (but not necessary) condition for the potential $W_N(r)$ to be nowhere repulsive is that the original two-body potentials $V_{ij}(r)$ be themselves nowhere repulsive. Two choices of the arbitrary function $g(r)$ which may be convenient are

$$g(r)=r^{3N-5}[1+(r/R)^{3N-5}]^{-1}$$

or

$$g(r)=r^{3N-5}\theta(R-r)+R^{3N-5}\theta(r-R),$$

R being an arbitrary positive constant and $\theta(x)$ being the usual step function, $\theta(x)=(x+|x|)/(2x)$; alternatively, the choice of $g(r)$ may be adjusted to ease the integration in Eq. (14a). The symbol min in the third inequality is defined by $\min[A,B]=\frac{1}{2}(A+B-|A-B|)$.

4. EXAMPLES

(i) *Newton forces.* Let $V_{ij}(r)=-e^2/r$, $m_i=m$. We then find $W_N(r)=-\eta_N{}^2/r$, with

$$\eta_N{}^2=e^2\tfrac{1}{3}(2/\pi)^{1/2}N(N-1)(N-2)^{-1}\times\Gamma(\tfrac{3}{2}N-\tfrac{3}{2})/\Gamma(\tfrac{3}{2}N-3).$$

The preceding theorem then implies

$$(-E)^{1/2}\geq(2m)^{1/2}\hbar^{-1}e^2\tfrac{1}{3}(2/\pi)^{1/2}\times N(N-1)(N-2)^{-1}(3N-4)^{-1}\times\Gamma(\tfrac{3}{2}N-\tfrac{3}{2})/\Gamma(\tfrac{3}{2}N-3)\equiv\tfrac{1}{2}m^{1/2}\hbar^{-1}e^2q_N, \tag{15}$$

where E indicates the energy of the N-body ground state. In particular we find $q_3=32/(5\pi)\sim 2$ and $q_4=105/32\sim 3.3$, while the value $q_2=1$ corresponds to the exact value of the ground-state energy for $N=2$.[11] We may therefore assert that the binding energy of the ground state of such a three- (four-) body system is at least 4 (10) times that of the two-body system with the same forces.

---

[8] We assume every reader to understand what we mean when we refer to a bound state with nonintegral l. If in doubt consult, for instance, T. Regge, Nuovo Cimento **14**, 951 (1959).

[9] F. Calogero, J. Math. Phys. **6**, 161 (1965); **6**, 1105 (1965); Commun. Math. Phys. **1**, 80 (1965); *Variable Phase Approach to Potential Scattering* (Academic Press Inc., New York, 1967), Chap. 23.

[10] Let us emphasize again that if the ground-state energy of the N-body system is negative, either that system itself or at least one of its subsystems can form a bound state. See Ref. 1a.

[11] See the remark after Eq. (8), and note that the reduced mass is $\frac{1}{2}m$.

Note that $q_N \underset{N\to\infty}{\longrightarrow} \text{const}\times N^{3/2}$, so that the binding energy of an N-body system (of the type considered here), interacting by Newtonian forces, increases at least as N^3 at large N.[11a] This is a consequence of the singularity of the Newton potential at the origin, because when all interparticle potentials are bounded below by the constant $-|V|$, the binding energy of the N-body system satisfies the inequality

$$-E\leq\tfrac{1}{2}N(N-1)|V|. \qquad (16)$$

In fact, if we add to each Newtonian interparticle potential the "centrifugal" contribution g^2r^{-2}, Eq. (15) is modified by the substitution of the factor $\{1+[(3N-5)^2+2m\hbar^{-2}g^2N(N-1)(3N-5)]^{1/2}\}$ in place of $(3N-4)$ in the denominator, and, together with Eq. (16), it implies that $-E=(4\pi)^{-1}ce^4g^{-2}N^2$ at large N, with $1\leq c\leq\frac{1}{2}\pi$. The rather stringent determination obtained for c, together with the observation that the bound of Eq. (16) is certainly rather poor, because it is obtained neglecting completely the kinetic-energy contribution and taking the maximum contribution for the potential energy disregarding any geometrical constraint, imply that our bound must be, in this case, quite close to the exact value.

(ii) *Coulomb forces.* Let $V_{i1}(r)=-(N-1)e^2/r$, $m_1=\infty$, $V_{ij}(r)=e^2/r$, $m_j=m$, $j\neq 1$. This corresponds to $N-1$ particles of negative charge $-e$ ("electrons") and one particle of positive charge $(N-1)e$ ("nucleus"), so that the system is neutral. In the same manner as in the preceding example we obtain in this case[12]

$$\begin{aligned}(-E)^{1/2}\geq{}&\hbar^{-1}(2m)^{1/2}e^2\tfrac{1}{3}(2/\pi)^{1/2}2\\ &\times\max_{1\leq n\leq N}\{(\sqrt{2}N-\tfrac{1}{2}n+1-\sqrt{2})(n-1)(n-2)^{-1}\\ &\times(3n-4)^{-1}\Gamma(\tfrac{3}{2}n-\tfrac{3}{2})/\Gamma(\tfrac{3}{2}N-3)\}. \qquad (17)\end{aligned}$$

For large N, the value of n which corresponds to the maximum is $n=(2^{3/2}/3)N$, so that

$$(-E)^{1/2}\geq(2m)^{1/2}\hbar^{-1}e^2(4/9)(2^{1/4}/\sqrt{\pi})N^{3/2},\quad (N\gg 1). \qquad (18)$$

On the other hand, we may also assert that

$$(-E)^{1/2}\leq(2m)^{1/2}\hbar^{-1}e^2\tfrac{1}{2}(N-1)^{3/2}, \qquad (19)$$

because the right-hand side of this equation is the exact result with all repulsive interactions switched off. Thus in this case we may conclude that, at large N,

$$-E=2m\hbar^{-2}e^4 2^{9/2}3^{-4}\pi^{-1}cN^3,$$

with $1\leq c\leq 81\pi 2^{-13/2}\sim 3$.

On the other hand, for $N=n=3$ (helium atom)[13] we obtain

$$(-E)^{1/2}\geq b(-E_H)^{1/2}\equiv b(2m)^{1/2}\hbar^{-1}e^2/2,$$

with $b=128(1-\frac{1}{8}\sqrt{2})/(15\pi)\sim 2.25$, while the corresponding values of b obtained by the standard perturbative treatment of the repulsive-electronic interaction are $b=8^{1/2}\sim 2.84$ (in zeroth order) and $b=(11/2)^{1/2}\sim 2.35$ (in first order). The corresponding result obtained by a simple variational improvement of the first-order perturbative calculation yields $b=27/(8\sqrt{2})\sim 2.39$. The exact result is $b\sim 2.405$.[14]

(iii) *Long-range attraction with short-range repulsion.* Let $m_i=m$ and

$$V_{ij}(r)=e^2r^{-2-2p}-fr^{-2-p},\quad 0<p<\tfrac{1}{2},\ f>0, \qquad (20)$$

which, when inserted in Eqs. (7), obviously yields

$$W_N(r)=\eta_N{}^2r^{-2-2p}-\varphi_Nr^{-2-p}, \qquad (21)$$

the constants η_N and φ_N being easily evaluated. The main reason for considering this potential is that the exact condition for the existence of at least one bound state with angular momentum l in the two-body problem with reduced mass m is given by the simple formula[15]

$$|f/e|(2m)^{1/2}\hbar^{-1}\equiv a\geq 2l+3p+1. \qquad (22)$$

Thus, from Eqs. (20–22) and Corollary 1, we obtain as a sufficient condition for the existence of at least one bound state in the N-body case the inequality

$$\begin{aligned}a\geq{}&2(3p+3N-5)[N(N-1)(3N-5)]^{-1/2}\\ &\times\Gamma(\tfrac{3}{2}N-\tfrac{5}{2}-p/2)[\Gamma(\tfrac{3}{2}N-\tfrac{5}{2})\Gamma(\tfrac{3}{2}N-\tfrac{5}{2}-p)]^{-1/2}\\ &\times[\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}-p)]^{1/2}/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}p)\equiv\sqrt{2}(3p+1)\alpha_N. \qquad (23)\end{aligned}$$

Again $\alpha_2=1$ yields the exact limiting value for $N=2$.[11] On the other hand we find, for $p=\frac{1}{3}$, $\alpha_3=0.82$ and $\alpha_N<1$ for $N>2$, and similar results for other values of p, so that there are cases when the potential is not enough attractive to sustain a two-body bound state but we may be able to assert that it does bind three or more particles. For instance if, say, $a=2.5$, and $p=\frac{1}{3}$, we may assert that the three-body ground state is bound, while the two-body ground state is not. Note finally that $\alpha_N \underset{N\to\infty}{\longrightarrow} \text{const}\times N^{-1/2}$, so that, in spite of its repulsive

---

[11a] *Note added in proof.* It can be easily proved that the binding energy in this case increases indeed as N^3 at large N, and that the bound of Eq. (15) is a very stringent one. See, H. R. Post, Proc. Phys. Soc. (London) **A79**, 819 (1962); F. Calogero and C. Marchioro (unpublished).

[12] This result obtains upon considering the subsystem with only $n-1$ "electrons" and using the remark of the second paragraph of Sec. 1. Strictly speaking, the maximum should be taken only over integral values of n, but presumably the result holds true even without this restriction. Incidentally, the possibility of exploiting the approach described in this paper to perform an analytic continuation in the particle number N is very appealing.

[13] Note that in the ground state of the helium atom the Pauli principle is inoperative, so that our results are indeed applicable in this case.

[14] See any quantum mechanics textbook, for instance, L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Moscow, 1948) [English transl.: Pergamon Press, Inc., New York, 1958].

[15] The number of bound states of angular momentum l in the two-body problem with reduced mass m due to this potential is the integral part of $(2p)^{-1}(a-2l-p-1)$; F. Calogero and G. Cosenza, Nuovo Cimento **45**, 867 (1966).

core, this potential, no matter how small the parameter a measuring its attractive strength, yields a negative energy for the ground state of all N-body systems with $N>\bar{N}$, $\bar{N}$ finite. Presumably the binding energies of these ground states also increase (in modulus) with N at large N. This is a consequence of the long-range attraction of the two-body forces (compare with previous examples, and with the following one).

(iv) *Repulsion both at small and at large distance.* Let $m_i=m$ and

$$V_{ij}(r)=e^2r^{-2-2p}-fr^{-2-p}+g^2r^{-2},\quad 0<p<\tfrac{1}{2},\qquad f>0,\ f^2-4e^2g^2\equiv F^2>0. \quad (24)$$

This potential, which obtains upon adding a "centrifugal" contribution to that of the preceding example, is attractive[16] only in the finite interval $r_-<r<r_+$, with $r_\pm=(2e^2)^{1/p}(f\mp F)^{-1/p}$. We find, in this case, as a sufficient condition for the N-body ground-state energy to be negative,

$$\begin{aligned} a\geq 2\{3p+[(3N-5)^2+N(N-1)(3N-5)g^2 2m\hbar^{-2}]^{1/2}\} \\ \times[N(N-1)(3N-5)]^{-1/2}\Gamma(\tfrac{3}{2}N-\tfrac{5}{2}-\tfrac{1}{2}p) \\ \times[\Gamma(\tfrac{3}{2}N-\tfrac{5}{2})\Gamma(\tfrac{3}{2}N-\tfrac{5}{2}-p)]^{-1/2}[\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}-p)]^{1/2}/ \\ \Gamma(\tfrac{1}{2}-\tfrac{1}{2}p)\equiv\sqrt{2}\{3p+[1+4g^2m\hbar^{-2}]^{1/2}\}\alpha_N', \quad (25)\end{aligned}$$

with a defined as in Eq. (22). Again $\alpha_2'=1$ yields the exact limiting value for a two-body bound state to exist; but now α_N' tends, as N diverges, to the finite limit

$$\begin{aligned}\alpha_\infty'=2^{1/2}\pi^{1/4}(1-p)(1-2p)^{-1/2}[\Gamma(\tfrac{3}{2}-p)]^{1/2} \\ \times[\Gamma(\tfrac{3}{2}-\tfrac{1}{2}p)]^{-1}\{3p+[1+4g^2m\hbar^{-2}]^{1/2}\}gm^{1/2}\hbar^{-1}. \quad (26)\end{aligned}$$

[16] As usual, we term "attractive" a negative potential, independently from the sign of its slope.

This limiting value may be larger or smaller than unity, depending on the values of p and g.

5. FINAL REMARKS

In future publications we hope to treat the topics which have been mentioned but not dealt with in this paper. The application of this approach to bound states other than the ground state is another interesting possibility. Finally, we mention that the derivation of *lower* bounds to the ground-state energy of N-body systems, and the determination of *necessary* conditions to be satisfied by the two-body interactions in order for N-body bound states to exist, is now in progress. Such results, besides being interesting *per se*,[17] may be useful in conjunction with those given in this paper, to yield simultaneous upper and lower bounds, asymptotic estimates which are exact up to a constant factor (as in some of the examples given above), and unambiguous conclusions concerning the possibility of a N-body system to exist as a bound state.[18]

ACKNOWLEDGMENTS

One of us (F.C.) wishes to acknowledge the grant of a Traveling Fellowship by CERN, which made his visit to Moscow possible, and the kind hospitality of the Theoretical Group of ITEP, which made his visit there worthwhile and pleasant.

[17] Especially in connection with the question of stability of various kinds of matter.

[18] If, by using the results of this paper it is proved that the ground state of an N-body system has negative energy, and at the same time (using results such as those mentioned here) it is shown that none of its subsystems can exist as a bound state (or equivalently, that the ground states of all its subsystems have positive energy), then it may be concluded that the N-body ground state is bound.