

High-Electric-Field Galvanomagnetic Effects in Piezoelectric Semiconductors*

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The Boltzmann transport equation is solved for the case of elastic and acoustic-phonon (deformation and piezoelectric interactions) scattering. The magnetoconductivity tensor is calculated for high magnetic and electric fields. The anisotropy of the piezoelectric scattering is treated for the zinc-blende and wurtzite structures.

I. INTRODUCTION

IN this paper we consider the transport properties of electrons in high electric and magnetic fields. We shall treat electron scattering by acoustic phonons (both deformation and piezoelectric potential interactions) and impurities. The anisotropy of the acoustic scattering is treated for the case of the piezoelectric potential.

The high-electric-field transport coefficients, assuming some, but not all, of the above electron-scattering mechanisms have been calculated by various authors.¹⁻⁶ Yamashita and Watanabe, and Pisarenko¹ considered the problem of acoustic phonons interacting via the deformation potential, whereas Laikhtman⁴ treated the piezoelectric interaction. Both acoustic-phonon (deformation potential) and ionized-impurity scattering were treated by Adawi.² Budd³ considered the same problem for high magnetic fields. Levinson⁵ obtained a general expression in the diffusion approximation.

The above treatments proceed by obtaining a solution of the Boltzmann transport equation in the diffusion approximation. A different approach is the energy balance equation method, originally used by Shockley.⁷ A Maxwellian distribution with an effective temperature is assumed. This approach has been successfully used by Matz and Garcia-Molinar⁸ to handle multiple scattering mechanisms in high magnetic fields. This approximation, while more tractable than solving the Boltzmann equation, can be justified only if the carrier density is high enough to permit electron-electron scattering to maintain a Maxwellian distribution.⁶ Because we are concerned with high-electric-field transport in

insulators and semiconductors at low temperatures, this approach does not suffice.

In Sec. II we obtain a solution to the Boltzmann transport equation for arbitrary magnetic and electric fields. This distribution function is used to calculate the conductivity mobility, Hall mobility, and magnetoconductivity tensor. In Sec. III we discuss the calculations for the particular case of CdS.

II. BOLTZMANN EQUATION

The time-independent Boltzmann equation can be written⁹

$$-\sum_i \left[E_i + \frac{1}{\hbar} \sum_{lr} \frac{\partial \mathcal{E}}{\partial K_l} B_{r\epsilon_{lri}} \right] \frac{\partial}{\partial K_i} f(\mathbf{K}) = \hat{C} f(\mathbf{K}), \quad (1)$$

where \mathbf{E} is the electric field, \mathbf{B} the magnetic field, \mathcal{E} the electron energy, \mathbf{K} the electron wave vector, and \hat{C} the collision integral operator. ϵ_{ilm} is the completely antisymmetric unit tensor of the third rank. [$\epsilon_{ilm} = 1$ if ilm is an even permutation of 123, -1 if an odd permutation, and zero if any two of these indices are the same.] To solve Eq. (1) we shall make the usual approximation¹⁰ of expanding the distribution function $f(\mathbf{K})$ in spherical harmonics, retaining only the first two terms.

$$f(\mathbf{K}) = f(\mathcal{E}) + \sum_i G_i(\mathcal{E}) K_i, \quad (2)$$

where $\mathbf{G}(\mathcal{E})$ is a vector to be determined later. When expansion (2) is substituted into Eq. (1) the Boltzmann equation reduces to two coupled equations. For the isotropic part

$$\begin{aligned} -\sum_i E_i \frac{\partial \mathcal{E}}{\partial K_i} \frac{\partial f(\mathcal{E})}{\partial \mathcal{E}} + e \sum_{ilmn} \frac{\partial \mathcal{E}}{\partial K_i} B_{l\epsilon_{ilm}} \frac{\partial}{\partial \mathcal{E}} (G_n(\mathcal{E}) K_n) \\ = \sum_i \hat{C} G_i(\mathcal{E}) K_i, \quad (3) \end{aligned}$$

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¹ J. Yamashita and M. Watanabe, *Progr. Theoret. Phys. (Kyoto)* **12**, 443 (1954); N. Pisarenko, *Izv. Akad. Nauk. SSSR, Fiz. Ser.* **3**, 631 (1938).

² I. Adawi, *Phys. Rev.* **112**, 1568 (1958).

³ H. Budd, *Phys. Rev.* **131**, 1520 (1963); **140**, A2170 (1965).

⁴ B. D. Laikhtman, *Fiz. Tverd. Tela* **6**, 3217 (1964) [English transl.: *Soviet Phys.—Solid State* **6**, 2573 (1965)].

⁵ I. B. Levinson, *Fiz. Tverd. Tela* **6**, 2113 (1964) [English transl.: *Soviet Phys.—Solid State* **6**, 1665 (1965)].

⁶ R. Stratton, *Proc. Roy. Soc. (London)* **A242**, 355 (1957).

⁷ W. Shockley, *Bell System Tech. J.* **30**, 990 (1951).

⁸ D. Matz and F. Garcia-Molinar, *Phys. Status Solidi* **5**, 495 (1964).

⁹ E. G. S. Paige, in *Progress in Semiconductors*, edited by A. F. Gibson and R. E. Burgess (John Wiley & Sons, Inc., New York, 1964), Vol. 8, p. 13.

¹⁰ E. G. S. Paige, in *Progress in Semiconductors*, edited by A. F. Gibson and R. E. Burgess (John Wiley & Sons, Inc., New York, 1964), Vol. 8, p. 182.

and for the anisotropic part

$$\frac{e}{\hbar} \sum_{in} E_i \frac{\partial}{\partial K_i} (G_n(\mathcal{E}) K_n) = \hat{C}f(\mathcal{E}). \quad (4)$$

It is convenient to define a relaxation time in the usual manner⁹ by

$$\frac{1}{\tau_i(\mathcal{E})} = -\sum_j \hat{C}G_j(\mathcal{E}) K_j / \sum_j G_j(\mathcal{E}) K_j, \quad (5)$$

where the direction i is along a principal crystal axis. The relaxation time tensor depends on energy and the crystallographic direction. We shall assume, however, that the tensor can be diagonalized. Of course, this tensor must be compatible with the crystal symmetry. In this derivation we shall assume a spherical parabolic conduction band. When the diagonal relaxation time tensor τ_i is substituted into Eq. (3), three simultaneous coupled equations result which can be readily solved for $\mathbf{G}(\mathcal{E})$.

$$G_i(\mathcal{E}) = -\frac{e}{m} \hbar \tau_i \frac{\partial f(\mathcal{E})}{\partial \mathcal{E}} \sum_{il} \left[E_i + \frac{e}{m} \tau_j E_j B_l \epsilon_{jli} / \right. \\ \left. (1 + \omega_0^2 \tau_0^2) + \left(\frac{e^2}{m^2} \right) \frac{\tau_1 \tau_2 \tau_3 E_j B_j B_i / \tau_i}{1 + \omega_0^2 \tau_0^2} \right]. \quad (6)$$

We use $\omega_0^2 \tau_0^2$ as a shorthand notation:

$$\omega_0^2 \tau_0^2 = \sum_i \frac{e^2}{m^2} \frac{\tau_1 \tau_2 \tau_3}{\tau_i} B_i^2. \quad (7)$$

Using the above expression for $\mathbf{G}(\mathcal{E})$ in Eq. (4), we find that

$$\hat{C}f(\mathcal{E}) = -\frac{2}{3} \frac{e^2}{m \mathcal{E}^{1/2}} \sum_{il} E_i \frac{\partial}{\partial \mathcal{E}} \\ \times \left(\mathcal{E}^{3/2} \tau_i \left[E_i + \frac{e^2}{m^2} \frac{\tau_1 \tau_2 \tau_3}{\tau_i} E_l B_l B_i \right] \frac{\partial f(\mathcal{E})}{\partial \mathcal{E}} / \right. \\ \left. (1 + \omega_0^2 \tau^2) \right). \quad (8)$$

If, on the other hand, the relaxation time is isotropic, Eqs. (6) and (8) reduce to Budd's³ results.¹¹

The next step is the evaluation of the collision integral terms. For nondegenerate statistics the linearized collision integral operator is⁸

$$\hat{C}f(\mathbf{K}) = \sum_{K'} (W(\mathbf{K}' \rightarrow \mathbf{K}) f(\mathbf{K}') - W(\mathbf{K} \rightarrow \mathbf{K}') f(\mathbf{K})), \quad (9)$$

where $W(\mathbf{K}' \rightarrow \mathbf{K})$ is the probability per unit time of an electron transition from the state \mathbf{K}' to the state \mathbf{K} . This transition probability can be calculated from first-order perturbation theory.¹²

We shall consider inelastic scattering by long-wavelength acoustic phonons and elastic scattering by impurities, the inelastic interaction being via both deformation and piezoelectric potentials. Optical-phonon and short-wavelength acoustic-phonon scattering will be neglected. Thus, the calculation will be valid only for electron energies less than the energy of an optical phonon.

For acoustic-phonon scattering, Eq. (9) is written¹³

$$\hat{C}f(\mathbf{K}) = \frac{V}{(2\pi)^2 \hbar^2} \int_{\text{zone}} \sum_{\lambda} d^3 K' M^2 \\ \times ([f(\mathbf{K}-\mathbf{q}) N_q - f(\mathbf{K})(N_q+1)] \delta(\mathcal{E}_{K'} - \mathcal{E}_K + \hbar\omega_{\lambda}) \\ + [f(\mathbf{K}+\mathbf{q})(N_q+1) - f(\mathbf{K}) N_q] \delta(\mathcal{E}_{K'} - \mathcal{E}_K - \hbar\omega_{\lambda})), \quad (10)$$

where \mathbf{q} is the phonon wave vector, $\mathbf{K}' = \mathbf{K} \pm \mathbf{q}$, V is the crystal volume, N_q is the phonon density, and M^2 is the square of the electron-phonon matrix element. M^2 can be taken as the sum of the matrix elements squared for each interaction. This simplification results because the matrix element for the deformation-potential interaction is pure imaginary, while that for the piezoelectric interaction is pure real. We shall assume that N_q is given by the Planck distribution. This assumption can be justified as long as the carrier density remains low to avoid appreciable phonon emission.¹⁴

It is convenient to formally expand the δ function to second order in the phonon frequencies.⁴

$$\delta(\mathcal{E}_{K'} - \mathcal{E}_K \pm \hbar\omega_{\lambda}) \simeq \left(1 \pm \hbar\omega_{\lambda} \frac{\partial}{\partial \mathcal{E}_{K'}} \right. \\ \left. + \frac{1}{2} (\hbar\omega_{\lambda})^2 \frac{\partial^2}{\partial \mathcal{E}_{K'}^2} \right) \delta(\mathcal{E}_{K'} - \mathcal{E}_K). \quad (11)$$

The differentials operate on the δ function. This expansion is only defined in terms of an integration over the δ function. The expansion of the δ function to second order is equivalent to expanding the symmetric part of the distribution function to second order in the phonon frequencies. After changing the integration over \mathbf{K}' to an integration over $\mathcal{E}_{K'}$, we remove the derivatives of the δ function by an integration by parts.

After some calculation we can write the collision integral terms as

$$\hat{C}f(\mathcal{E}) = \hat{C}_1 I_S \quad (12a)$$

¹² Reference 9, p. 58.

¹³ Reference 9, p. 64.

¹¹ The reviewer has brought to my attention the fact that Eqs. (6) and (8) have been derived by H. Budd, Phys. Rev. **134**, A1281 (1964).

¹⁴ E. Conwell, J. Phys. Chem. Solids **25**, 593 (1964); V. V. Paranjape, E. R. A. Technical Report No. L/T 384, 1959 (unpublished).

and

$$\frac{1}{\tau_i(\mathcal{E})} = -\sum_{\alpha} C_1 G_{\alpha}(\mathcal{E}) I_{A\alpha} / \sum_{\alpha} G_{\alpha}(\mathcal{E}) K_{\alpha}, \quad (12b)$$

where

$$\hat{C}_1 = \frac{(2m)^{3/2} V}{2h^4} \int \sum_{\lambda} d\Omega_{K'}. \quad (13)$$

The integrands over solid angle in K' space are

$$I_S = \frac{1}{\mathcal{E}^{1/2}} \frac{\partial}{\partial \mathcal{E}} \left(\mathcal{E} [M_{P^2} + M_{D^2}] \hbar \omega_{\lambda} \right. \\ \left. \times \left[f(\mathcal{E}) + \hbar \omega_{\lambda} \frac{1}{2} (2N_q + 1) \frac{\partial f(\mathcal{E})}{\partial \mathcal{E}} \right] \right) \quad (14)$$

and

$$I_{A_i} = \mathcal{E}^{1/2} (K'_i - K_i) ([M_{P^2} + M_{D^2}] \\ \times \frac{1}{2} (2N_q + 1) + M_e^2). \quad (15)$$

The squared matrix elements M_{P^2} and M_{D^2} are for the

electron-phonon interaction via the piezoelectric and deformation potentials, respectively. The matrix element for impurity scattering is M_e . It is apparent from Eq. (9) that for elastic scattering, \hat{C} operating on the symmetric part of the distribution function gives zero. With the above expressions at hand, we can proceed to calculate the symmetric part of the distribution function $f(\mathcal{E})$. We substitute expression (12a) into Eq. (8) and arrive at

$$\hat{C}_1 \mathcal{E} (M_{P^2} + M_{D^2}) \left(f(\mathcal{E}) + \frac{1}{2} \hbar \omega_{\lambda} (2N_q + 1) \frac{\partial f(\mathcal{E})}{\partial \mathcal{E}} \right) \hbar \omega_{\lambda} \\ + \frac{2}{3} \frac{e^2}{m} \sum_{i\lambda} E_i \left[E_i + \frac{e^2}{m^2} \frac{\tau_1 \tau_2 \tau_3 E_i B_i B_i}{\tau_i} \right] \\ \times \frac{\tau_i \mathcal{E}^{3/2}}{1 + \omega_0^2 \tau_0^2} \frac{\partial f(\mathcal{E})}{\partial \mathcal{E}} = 0, \quad (16)$$

which can be readily integrated to yield

$$f(\mathcal{E}) = N \exp \left(- \int_0^{\mathcal{E}} \frac{d\mathcal{E}_1 \hat{C}_1 \hbar \omega_{\lambda} (M_{P^2} + M_{D^2})}{C_1 (\hbar \omega_{\lambda})^{3/2} (2N_q + 1) (M_{P^2} + M_{D^2}) + \frac{2}{3} (e^2/m) \sum_i E_i^2 \tau_i (1 + Q_i) \mathcal{E}_1^{1/2}} \right), \quad (17)$$

where N is a normalization constant defined by

$$\int_0^{\infty} f(\mathcal{E}) \mathcal{E}^{1/2} d\mathcal{E} = 2\pi^2 \hbar^3 / (2m)^{3/2}, \quad (18)$$

and the quantity Q_i is defined by

$$Q_i = \frac{e^2}{m^2} \tau_1 \tau_2 \tau_3 \sum_l \left(\frac{E_l B_l B_l}{E_i \tau_i} - \frac{B_l^2}{\tau_l} \right) / (1 + \omega_0^2 \tau_0^2).$$

Equation (17) for $Q_i=0$ is a special case of the general expression for $f(\mathcal{E})$ derived by Levinson.⁵ The total relaxation time $\tau_i(\mathcal{E})$ is the sum of the reciprocals of the relaxation times for the various interactions.

The task remains to express the matrix elements M^2 and perform the integrations over the solid angle $d\Omega_{K'}$. Because we will be concerned with low-temperature phenomena, the electrons interact with long-wavelength acoustic phonons having a Debye spectrum. Thus, $\omega_{\lambda} = S_{\lambda} q$, where S_{λ} is the sound velocity.

For the deformation-potential interaction,¹⁵

$$M_{D^2} = \frac{\hbar q^2 D_{\lambda}^2}{2\rho V \omega_{\lambda}}, \quad (19)$$

where D_{λ} is the deformation potential and ρ is the crystal density. We shall consider the deformation-potential interaction to be isotropic.

For the piezoelectric interaction Zook¹⁶ has given for the square matrix element the expression

$$M_{P^2} = \frac{\hbar e^2 \omega_{\lambda} h_{3\lambda 3'}}{2V q^2 C_{\lambda 3\lambda 3'}}, \quad (20)$$

where the $h_{3\lambda 3'}$ are elements of the piezoelectric tensor and the $C_{\lambda 3\lambda 3'}$ are the elastic stiffness coefficients at constant displacement field.

For elastic scattering we shall be primarily concerned with ionized impurities. Nevertheless, we shall formulate the problem in a general form to include other elastic-scattering mechanisms such as dipoles and neutral impurities. We write the elastic-scattering time as

$$1/\tau_e(\mathcal{E}) = AL(\mathcal{E})\mathcal{E}^P. \quad (21)$$

Here, A and P are constants. $L(\mathcal{E})$ is a slowly varying function of the energy which depends on the choice of scattering potential. We consider the elastic scattering to be isotropic. For ionized impurity scattering,¹⁷ $P = -\frac{3}{2}$. Expressions for $L(\mathcal{E})$ have been given by various authors.¹⁷

To treat the anisotropy of the piezoelectric scattering, the Herring-Vogt¹⁸ procedure is useful in calculating the collision integral. Their prescription is to average the

¹⁶ J. David Zook, Phys. Rev. **136**, A869 (1964).

¹⁷ Reference 9, p. 75.

¹⁸ C. Herring and E. Vogt, Phys. Rev. **101**, 944 (1956).

¹⁵ Reference 9, p. 62.

collision integral over the solid angle $d\Omega_K$. Thus

$$\hat{C}_1 I_S \Rightarrow \frac{\int d\Omega_K \hat{C}_1 I_S}{\int d\Omega_K} \quad (22)$$

and

$$\frac{1}{\tau_i(\mathcal{E})} \Rightarrow \frac{-\int d\Omega_K K_i(\hat{C}_1 I_A)_i}{\int d\Omega_K K_i^2}. \quad (23)$$

The detailed calculations are carried out in the Appendix. The results of the various operations are

$$\begin{aligned} \hat{C}_1(\hbar\omega_\lambda)^{2\frac{1}{2}}(2N_q+1)[M_D^2+M_P^2] \\ = \frac{(2m)^{3/2}kT}{2\pi\hbar^2\rho} \left[\frac{2m\mathcal{E}}{\hbar^2} \sum_\lambda D_\lambda^2 + \frac{e^2}{4\pi} F_0 \right], \quad (24) \end{aligned}$$

where F_0 , a function of the piezoelectric coefficients, is expressed in the Appendix. In calculating expression (24) we assumed that the lattice temperature was high enough so that the phonon density could be approximated by $N_q \simeq kT/\hbar\omega_\lambda$. At low temperatures (a few degrees Kelvin) this approximation is not good for hot electrons. In this case $N_q \ll 1$ and phonon emission is the important electron-scattering process. In the region of dominant phonon emission we have

$$\begin{aligned} \hat{C}_{1\frac{1}{2}}(\hbar\omega_\lambda)^2[M_D^2+M_P^2] \\ = \frac{2m^2\mathcal{E}^{1/2}}{\pi\rho\hbar^2} \left(\frac{8m}{\hbar^2} \mathcal{E} \sum_\lambda D_\lambda^2 S_\lambda + e^2 F_0' \right). \quad (25) \end{aligned}$$

where

$$f(\mathcal{E}) = N \exp\left(-\int_0^{\mathcal{E}/kT} \frac{dZ(aZ+1)}{(aZ+1) + \sum_i P_i^2(1+Q_i)Z/(a_iZ+1+b_iZ^{P+1/2})}\right), \quad (30)$$

and

$$Z = \mathcal{E}/kT, \quad a = \frac{16\pi m k T}{\hbar^2 e^2 F_0} \sum_\lambda D_\lambda^2, \quad a_i = \frac{16\pi m k T}{3\hbar^2 e^2 F_i} \sum_\lambda D_\lambda^2/S_\lambda^2, \quad b_i = \frac{2^{7/2}(\pi\hbar)^2 A L(\mathcal{E})\rho}{3e^2 m^{1/2}(kT)^{-P+1/2} F_i}, \quad (31)$$

Because we have assumed that the deformation-potential interaction gives rise to an isotropic relaxation time, we assume that D_λ^2 is the same for each polarization direction, and define a deformation potential average by

$$D^2 = \frac{1}{3} \sum_\lambda D_\lambda^2.$$

The integration in Eq. (30) must be done numerically in most cases. We can, however, discuss some properties of the distribution function without performing the integration. The distribution function depends not only on the magnitude of the electric field but on its direction. We have this result because the term F_i depends on direction. The magnetic-field term Q_i occurs in conjunction with the electric field and thus is only influen-

For the relaxation times in the high-temperature approximation we find

$$\frac{1}{\tau^D(\mathcal{E})} = \frac{kT(2m)^{3/2}}{2\pi\rho\hbar^4} \mathcal{E}^{1/2} \sum_\lambda D_\lambda^2/S_\lambda^2 \quad (26)$$

for deformation-potential interaction and

$$\frac{1}{\tau_i^P(\mathcal{E})} = \frac{3kTe^2(2m)^{1/2}F_i}{(4\pi)^2\hbar^2\rho\mathcal{E}^{1/2}} \quad (27)$$

for the piezoelectric interaction. F_i is defined in the Appendix. In the low-temperature approximation (phonon emission only) we have similarly

$$\frac{1}{\tau^D(\mathcal{E})} = \frac{4m\mathcal{E}}{\pi\hbar^4\rho} \sum_\lambda D_\lambda^2/S_\lambda \quad (28)$$

and

$$\frac{1}{\tau_i^P(\mathcal{E})} = \frac{3e^2m}{4\pi\hbar^2\rho} F_i'. \quad (29)$$

Of particular note is the change in the energy dependence of the relaxation times.¹⁹

With the above expressions at hand we may write the symmetric part of the distribution function in the high-temperature limit as

tial at high electric fields. If $\mathbf{E} \parallel \mathbf{B}$, then $Q_i = 0$. In all other cases Q_i is negative and of absolute value less than unity. Thus, the effect of the magnetic field is to reduce the strength of the electric field. It is convenient to define an effective electric field

$$(\bar{E}_i)^2 = E_i^2(1+Q_i), \quad (32)$$

which is less than or equal to the total electric field. The total electric field includes the Hall field. This decrease of the effectiveness of the electric field is referred to as magnetic cooling.^{3,8}

If there is no elastic scattering ($b_i = 0$) and $Q_i \ll 1$, the integration in Eq. (36) can be readily performed to

¹⁹ E. M. Conwell and A. L. Brown, J. Phys. Chem. Solids **15**, 208 (1960); E. M. Conwell, *ibid.* **8**, 234 (1959).

yield

$$f(\mathcal{E}) = N[\exp(-Z)][Z+Z_+]^t[Z+Z_-]^{-\beta}, \quad (33)$$

where

$$Z_{\pm} = \frac{P_i^2 + a + a_i}{2aa_i} \left[1 \pm \left(1 - \frac{4aa_i}{P_i^2 + a + a_i} \right)^{1/2} \right],$$

and

$$t = (aa_i Z_+^2 - Z_-(a+a_i) + 1) / (aa_i [Z_- - Z_+]),$$

$$\beta = (aa_i Z_-^2 - Z_-(a+a_i) + 1) / (aa_i [Z_- - Z_+]).$$

Here, a , a_i , and P_i are defined by expression (31). The above distribution function represents acoustic-phonon

scattering via deformation and piezoelectric potentials. In the limit of only the deformation-potential interaction, $1/a$, $1/a_i$, and $1/P_i \rightarrow 0$. Thus, Eq. (33) becomes identical with the result obtained by Yamashita and Watanabe.¹ In the opposite approximation of only the piezoelectric interaction, a and $a_i \rightarrow 0$. Then Eq. (33) reduces to the result obtained by Laikhtman.⁴ [Note added in proof. Equation (33) has also been obtained by Saitoh [M. Saitoh, J. Phys. Soc. Japan **21**, 2540 (1966)].

For the low-temperature approximation, where phonon emission is the dominant process, the distribution function $f(\mathcal{E})$ is

$$f(\mathcal{E}) = N \exp\left(-\frac{F_0'}{2^{3/2} m^{1/2}} \int_0^{\mathcal{E}} \frac{d\mathcal{E}_i [a_0 \mathcal{E}_i + 1]}{\mathcal{E}_i^{1/2} [a_2 \mathcal{E}_i + 1 + \sum_i T_i^2 (1 + Q_i) / (1 + b_0 \mathcal{E}_i^P + a_3 \mathcal{E}_i)]}\right), \quad (34)$$

where

$$a_0 = a/kT, \quad a_2 = \frac{16m\pi}{3\hbar^2 e^2 F_0'} \sum_{\lambda} D_{\lambda}^2 / S_{\lambda}, \quad a_3 = \frac{a_2 F_0'}{F_i'}, \quad b_0 = \frac{8\pi^2 A L(\mathcal{E}) \hbar^2 \rho}{3F_i'}$$

and

$$T_i^2 = \frac{1}{2} \left(\frac{8\pi^2 \hbar^2 \rho}{3m^2 e} \right)^2 \frac{E_i^2}{F_0' F_i'}.$$

All the scattering mechanisms are included in Eq. (34).

We should remark that this distribution function is only valid for large electric fields, since for low electric fields the expansion of the δ function, Eq. (11), must be carried to higher order. Since the emitted phonon energy is the order of the electron energy, expansion (11) is not rapidly convergent.

To make a comparison between theory and experiment we must evaluate the transport coefficients which are the readily available experimental quantities. Measurements of the distribution function can only be made in very special cases.²⁰

The transport coefficients can be readily evaluated from the expression for the current:

$$J_i = \frac{2e}{\hbar V} \sum_{\mathbf{K}} \frac{\partial \mathcal{E}}{\partial K_i} f(\mathbf{K})$$

$$= \frac{2e}{(2\pi)^3 \hbar} \int_{\text{zone}} d^3 K \frac{\partial \mathcal{E}}{\partial K_i} \sum_i K_i G_i(\mathcal{E})$$

$$= \sum_e \sigma_{il} E_l, \quad (35)$$

where

$$\sigma_{il} = -\frac{e^2 (2m)^{3/2}}{3\pi^2 m \hbar^3} \int_0^{\infty} \mathcal{E}^{3/2} d\mathcal{E} \Gamma_{il} \frac{\partial f(\mathcal{E})}{\partial \mathcal{E}}$$

$$= -\frac{e^2}{m} \langle \Gamma_{il} \rangle. \quad (36)$$

²⁰ W. E. Pinson and R. Bray, Phys. Rev. **136**, A1449 (1964).

The Γ tensor is

$$\Gamma_{il} = \tau_i \delta_{il} - \left(\tau_i \omega_0^2 \tau_0^2 \delta_{il} + \sum_j \frac{e}{m} \tau_i \tau_j B_j \epsilon_{ilj} - \frac{e^2}{m^2} \tau_1 \tau_2 \tau_3 B_i B_l \right) / (1 + \omega_0^2 \tau_0^2). \quad (37)$$

The first term gives the conductivity, the second term the magnetoconductivity. The term containing ϵ_{ilj} is responsible for the Hall effect.

$$\mu_i^e = -\frac{e}{m} \langle \Gamma_{ii} \rangle, \quad (38)$$

and the Hall mobility is

$$\mu_i^H = \frac{1}{B_k} \frac{\langle \Gamma_{ji} \rangle}{\langle \Gamma_{jj} \rangle} = \frac{e}{m} \frac{\langle \tau_i \tau_j / (1 + \omega_0^2 \tau_0^2) \rangle}{\langle \tau_j / (1 + \omega_0^2 \tau_0^2) \rangle}. \quad (39)$$

We must remember that the distribution function depends on the Hall voltage. The electric field appearing in the expression for $f(\mathcal{E})$ is the total electric field. Similar expressions can be generated for the elements of the magnetoconductivity tensor.

III. VALIDITY AND LIMITATIONS

We carried out the calculation assuming a spherical conduction band because there is little evidence for large mass anisotropies in the important wurtzite and zinc-blende structure materials. There is about a 10% mass anisotropy in CdS, but this contributes only about

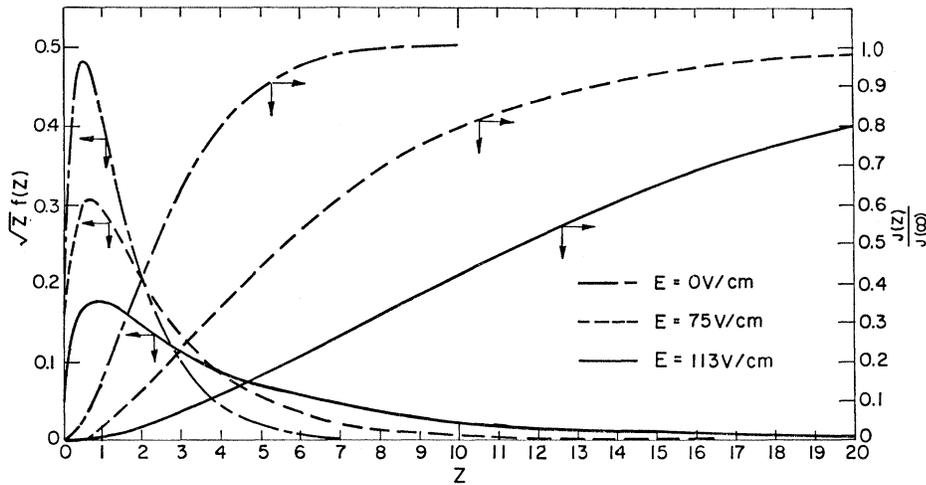


FIG. 1. The number density $Z^{1/2}f(Z)$ versus Z and the fractional current $J(Z)/J(\infty)$ versus Z for CdS at 20.5°K. $Z = \mathcal{E}/kT$. The area under the $Z^{1/2}f(Z)$ curves is unity.

5% to the relaxation time anisotropy.¹⁶ Considering the other errors involved in the calculation, we do not feel this added refinement is necessary.

If, on the other hand, nonspherical bands were considered, the effects in a magnetic field would possess a new feature. A longitudinal magnetic field ($\mathbf{E} \parallel \mathbf{B}$) would cool the electrons. This result is analogous to the result that nonspherical bands give rise to a longitudinal magnetoresistance.

A serious limitation of this calculation is the neglect of polar-optical-phonon interactions. If optical-phonon scattering had been included, we could not have solved the Boltzmann equation by expanding the distribution function in a power series in the phonon frequencies. The large energy of the optical phonon makes this expansion useless except for high-energy electrons. Only for the case of covalent optical phonons has the Boltzmann equation been solved²¹ without the electron-temperature approximation. This solution, by the path integral method, is a difficult numerical problem.

The present calculation is thus restricted to temperatures well below the optical-phonon energy. As we show in the next section, the present theory can be used to indicate at what electric field the optical-phonon interaction becomes important.

For high electron density the theory breaks down because of the neglect of electron-electron scattering. If, however, electron-electron scattering is predominant, the electron-temperature model⁶ can be used to solve the Boltzmann equation. The present theory is intended for the low-carrier regime where the electron-temperature model is not justified.

We have assumed that the phonon population is not changed. This assumption implies low-carrier densities. In piezoelectric semiconductors large phonon population

changes result from the acoustoelectric effect.²² Thus, our calculation can not be applied if there is an acoustoelectric effect.

A final concern is the choice of the relaxation time for ionized-impurity scattering.¹⁷ Neither the Brooks-Herring nor the Conwell-Weisskopf expressions are well justified at low temperatures. Since the expressions for ionized-impurity scattering are more reliable for high-energy electrons, this objection is not as stringent if we consider the hot-electron situation. Non-Ohmic transport is due mainly to the electrons in the high-energy tail of the distribution. Since the expressions for ionized-impurity scattering are more reliable for high-energy electrons, the present theory is better justified for high fields than consideration of low-field transport would indicate.

IV. APPLICATION TO CdS

Because measurements of hot-electron effects of low temperature were made recently in semi-insulating CdS,²³ we shall discuss the distribution function for this material. The comparison between experimental and theoretical mobility values is made in the preceding paper.²³

In Fig. 1 the number density $n(Z) = Z^{1/2}f(Z)$ is plotted versus a dimensionless energy $Z = \mathcal{E}/kT$. The area under each curve is unity. The high-temperature approximation is used in calculating the distribution function at 20.5°K. The ionized-impurity, piezoelectric, and deformation-potential limited zero-field mobilities are in the ratio 1:1.8:15, respectively. As the electric field increases, the high-energy tail of the distribution increases, but the maximum of the distribution shifts only slightly to high energy. It is the strong low-energy

²¹ L. V. Keldysh, Zh. Eksperim. i Teor. Fiz. **48**, 1692 (1965) [English transl.: Soviet Phys.—JETP **21**, 1135 (1965)]; H. Budd, J. Phys. Soc. Japan Suppl. **21**, 1420 (1966).

²² A. R. Hutson and D. L. White, J. Appl. Phys. **33**, 40 (1962); D. L. White, *ibid.* **33**, 2547 (1962); R. W. Smith, Phys. Rev. Letters **9**, 527 (1962).

²³ R. S. Crandall, preceding paper, Phys. Rev. **169**, 577 (1968).

ionized-impurity scattering that keeps the bulk of the distribution from shifting to high energy. When ionized-impurity scattering is omitted, the bulk of the distribution shifts to high energy.

By plotting the fractional current versus Z , we show in Fig. 1 how the distribution affects the current. The fractional current $J(Z)/J(\infty)$ at an energy Z is the current $J(Z)$ due to all the electrons in the energy interval 0 to Z divided by the total current $J(\infty)$. At 75 V/cm, one-half the current is due to the few electrons in the region $Z > 5.8$. Less than 8% of the electrons are in this region. At 113 V/cm, one-half the current is due to the few electrons in the region $Z > 11.5$. In this domain 20% of the current comes from electrons whose energy is greater than an optical-phonon energy ($Z = 21$). Since optical-phonon emission will strongly scatter these electrons to low energy, we do not expect them to contribute as much to the current as we have calculated.

A crude estimate of the current could be made by including only the electrons for $Z < 21$. This approximation, however, overestimates the effect of optical-phonon emission. The electrons are not removed from the distribution, but scattered to low energy where they still contribute to the current. This contribution, however, is much less than the contribution for $Z > 21$.

The distribution function $f(\mathcal{E})$ is not a Maxwellian at high fields; thus the electron-temperature model can not describe the non-Ohmic effects.

Because of the effects not included in the calculation, non-Ohmic transport in CdS is not the ideal situation to test the theory. In a future work we shall show a more convincing comparison of measurements in GaAs. We do feel, however, that in the absence of the acoustoelectric effect the theory has some utility in the low-field region where optical-phonon emission is unimportant. Here comparison between experiment and theory can be justified. Because of the sensitivity of the distribution function to the energy-loss mechanism, the acoustic scattering can be determined, even though the zero-field mobility is due to ionized-impurity scattering.

V. SUMMARY

We have derived expressions for the electrical conductivity which are valid at low temperatures in large electric and magnetic fields. Because of the inclusion of elastic scattering from impurities as well as inelastic scattering by acoustic phonons interacting via both deformation and piezoelectric potentials, these expressions are particularly well suited for analyzing non-Ohmic transport in piezoelectric semiconductors. The sensitivity of the transport coefficients to the energy-loss mechanism (acoustic phonons) permits a determination of the acoustic-phonon interaction even though the low-field mobility is determined by elastic scattering.

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APPENDIX

In performing the integrations over the solid angle in Eqs. (22) and (23) we shall follow the procedure of Zook.¹⁶ In a coordinate system whose 3' axis is in the direction of phonon propagation, the matrix element M_p^2 is that given by Eq. (20). This matrix element can be transformed from the phonon coordinate system (the primed system) to the crystal coordinate system (the unprimed system) by the transformation matrix given by Zook.¹⁶ The 3' axis is defined to be along the phonon axis \mathbf{q} where the phonon polar coordinates are (θ, β, φ) . The 2' axis is chosen in the (1,2) plane. The piezoelectric tensor in the primed system can thus be expressed by its components in the unprimed system. We remove the transformed stiffness coefficients from the integrations over solid angle by replacing these quantities by their spherical averages and then expressing them in terms of the sound velocities. For wurtzite symmetry and [0001]-type valleys,

$$h_{333'}^2 = h_x^2 \left[\left(1 - \frac{h_{33}}{h_x} \right)^2 \cos^2 \theta + 2 \left(\frac{h_{33}}{h_x} - 1 \right) \cos^4 \theta + \cos^6 \theta \right], \quad (\text{A1})$$

$$h_{323'}^2 = 0,$$

and

$$h_{313'}^2 = h_{15}^2 \left[1 + \left(\frac{2h_x}{h_{15}} - 1 \right) \cos^2 \theta + \left(\frac{h_x^2}{h_{15}^2} - \frac{2h_x}{h_{15}} \right) \times \cos^4 \theta - \frac{h_x^2}{h_{15}^2} \cos^6 \theta \right],$$

where the h_{ij} are elements of the piezoelectric tensor

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 & h_{15} & 0 \\ 0 & 0 & 0 & h_{15} & 0 & 0 \\ h_{31} & h_{31} & h_{33} & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A2})$$

The 3 axis is the symmetry axis and

$$h_x = h_{33} - h_{31} - 2h_{15}. \quad (\text{A3})$$

The spherical averages of the stiffness coefficients are

$$C_3 = \langle C_{3333'} \rangle = \frac{1}{3}(2C_{11} + C_{33}) - \frac{2}{15}C_x = \rho S_3^2$$

and

$$C_1 = \langle C_{1313'} \rangle = C_{44} + \frac{2}{15}C_x = \rho S_1^2 = C_2 \quad (\text{A4})$$

in terms of the sound velocities. Here, $C_x = C_{11} + C_{33}$

$-2C_{13}-4C_{44}$. The quantities F_0 and F_i are defined by

$$F_0 = \sum_{\lambda} \int d\Omega_K h_{3\lambda 3'}^2 \quad (\text{A5})$$

and

$$F_i = \sum_{\lambda} \int d\Omega_K \frac{h_{3\lambda 3'}^2}{S_{\lambda}^2} \cos^2 \phi, \quad (\text{A6})$$

where ϕ is the angle between \mathbf{E} and \mathbf{K} . Thus,

$$F_0 = 4\pi h_{33}^2 \left[\frac{1}{3} - \frac{4}{15} \frac{h_x}{h_{33}} \right] + h_{15}^2 \left[\frac{2}{3} + \frac{4}{15} \frac{h_x}{h_{15}} + \frac{2}{15} \frac{h_x^2}{h_{15}^2} \right], \quad (\text{A7})$$

$$F_3 = \left(\frac{1}{S_3} \right)^2 F_{3,3} + \left(\frac{1}{S_1} \right)^2 F_{3,1}, \quad (\text{A8})$$

and

$$F_1 = \left(\frac{1}{S_3} \right)^2 F_{1,3} + \left(\frac{1}{S_1} \right)^2 F_{1,1} = F_2. \quad (\text{A9})$$

Here

$$F_{3,3} = 4\pi h_{33}^2 \left[\frac{1}{5} - \frac{4}{35} \frac{h_x}{h_{33}} + \frac{8}{315} \left(\frac{h_x}{h_{33}} \right)^2 \right], \quad (\text{A10})$$

$$F_{3,1} = 4\pi h_{15}^2 \left[\frac{2}{15} + \frac{4}{35} \frac{h_x}{h_{15}} + \frac{2}{63} \left(\frac{h_x}{h_{15}} \right)^2 \right], \quad (\text{A11})$$

$$F_{1,3} = 2\pi h_{33}^2 \left[\frac{1}{15} - \frac{8}{105} \frac{h_x}{h_{33}} + \frac{4}{315} \frac{h_x^2}{h_{33}^2} \right], \quad (\text{A12})$$

and

$$F_{1,1} = 2\pi h_{15}^2 \left[\frac{4}{15} + \frac{8}{105} \frac{h_x}{h_{15}} + \frac{23}{315} \left(\frac{h_x}{h_{33}} \right)^2 \right]. \quad (\text{A13})$$

The quantities F_0' , F_i' for the low-temperature approximation are

$$F_0' = S_3 h_x^2 \left[\frac{1}{3} \left(1 - \frac{h_{33}}{h_x} \right)^2 + \frac{2}{5} \left(\frac{h_{33}}{h_x} - 1 \right) + \frac{1}{7} \right] + S_{1,2} h_{15}^2 \left[1 + \frac{1}{3} \left(\frac{2h_x}{h_{15}} - 1 \right) + \frac{1}{5} \left(\frac{h_x^2}{h_{15}^2} - 2 \frac{h_x}{h_{15}} \right) - \frac{1}{7} \frac{h_x^2}{h_{15}^2} \right], \quad (\text{A14})$$

$$F_3' = \frac{1}{S_3} F_{3,3} + \frac{1}{S_1} F_{3,1}, \quad (\text{A15})$$

and

$$F_1' = \frac{1}{S_3} F_{1,3} + \frac{1}{S_1} F_{1,1}. \quad (\text{A16})$$

For zinc-blende symmetry and (000)-type valleys,

$$h_{333'}^2 = 36h_{14}^2 (\sin^4 \theta \cos^2 \theta \sin^2 \beta \cos^2 \beta) \quad (\text{A17})$$

and

$$h_{313'}^2 + h_{323'}^2 = 4h_{14}^2 \sin^2 \theta (\cos^2 \theta + \sin^2 \theta \cos^2 \beta \sin^2 \beta), \quad (\text{A18})$$

where h_{14} is an element of the piezoelectric tensor

$$h = \begin{bmatrix} 0 & 0 & 0 & h_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & h_{14} \end{bmatrix}. \quad (\text{A19})$$

The spherical averages of the stiffness coefficients are

$$C_3 = \langle C_{3333'} \rangle = \frac{2}{3} C_{11} + \frac{4}{5} C_{44} + \frac{2}{5} C_{12} = \rho S_3^2$$

and

$$C_1 = \frac{1}{2} \langle C_{1313'} + C_{2323'} \rangle = \frac{3}{5} C_{44} - \frac{1}{5} C_{12} + \frac{1}{5} C_{11} = \rho S_1^2 = C_2. \quad (\text{A20})$$

The quantities F are

$$F_0 = \frac{16\pi}{5} h_{14}^2 \quad (\text{A21})$$

and

$$F_3 = F_1 = \frac{32\pi}{35} h_{14}^2 \left[\frac{1}{S_3^2} + \frac{4}{3S_1^2} \right] \quad (\text{A22})$$

in the high-temperature approximation. In the low-temperature approximation

$$F_0' = \frac{h_{14}^2}{34} [54S_3 - 22S_1] 4\pi \quad (\text{A23})$$

and

$$F_3' = F_1' = \frac{8}{35} h_{14}^2 \left[\frac{1}{S_3} + \frac{4}{3S_1} \right] 4\pi. \quad (\text{A24})$$