Spin-Fluctuation Contributions to the Specific Heat

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The specific heat is calculated for weakly ferromagnetic and nearly ferromagnetic metals. These systems are characterized by an enhanced susceptibility indicating large spin fluctuations. An expression for the mass enhancement on both sides of the ferromagnetic instability is presented. The specific heat just above the Curie temperature is investigated. In this region, the terms which lead to the large low-temperature mass enhancements are greatly reduced, and a new term contributes a singular specific heat at the Curie temperature. Spin-fluctuation contributions to the low-temperature specific heat are calculated for both phases, and numerical results are presented. Very large magnetic fields are shown to be effective in reducing the strongly enhanced specific heat. Calculations of the temperature dependence of the specific heat due to spin fluctuations in the paramagnetic phase are compared with the experimental results on He^3.

1. INTRODUCTION AND RESUME

EARLY neutron-scattering measurements gave direct evidence for critical fluctuations in the magnetization of ferromagnets above the Curie temperature. Recently, it has been shown by Berk and Schrieffer[1] and by Doniach and Engelsberg[2] that fluctuations of this same type must also exist in materials that are not ferromagnetic but have an enhanced susceptibility. In both cases these fluctuations have a large effect on the specific heat of the metal. In a paramagnet, their contribution is such that the slope of the specific heat at low temperatures goes as the logarithm of the susceptibility. In a ferromagnet, they contribute a logarithmic singularity to the specific heat at the Curie temperature in addition to the very-low-temperature mass enhancement which persists from the paramagnetic phase into the ferromagnetic phase. The mass enhancement in the latter phase is predicted to go as the logarithm of the longitudinal susceptibility.[3] In this paper we present an investigation of the effects of spin fluctuations on the specific heat at low temperatures and near the Curie temperature for weakly ferromagnetic systems. We also consider the effect of a magnetic field, finite-range interaction, and higher-order corrections to the specific heat of almost ferromagnetic systems.

This section includes the motivation and justification for the approach that we use along with a brief description of the results. The detailed solution to each of the problems we consider is given in the sections following.

The model which we use is one in which there is a spatially uniform exchange enhancement of the susceptibility. As discussed in Refs. 1 and 2, the low-temperature enhancement of the specific heat in a paramagnet arises from the virtual scattering of electrons on the Fermi surface via spin fluctuations. This specific-heat change is thus viewed as an increase of the electron effective mass. The random-phase approximation (RPA) with an effective interaction is used to obtain the particle-hole correlation functions, which in turn give the dynamics of the spin fluctuations. Even though the excitations found are critically damped, they drastically alter the thermodynamic properties of a system. If one could completely determine the particle-hole scattering matrices as a function of coupling constant, the thermodynamic properties would be predictable. Recently, Schrieffer and Berk[4] and Doniach and Rice[5] have given arguments, applicable to low temperatures, as to why the effects which we might expect to alter the results of the RPA cancel out in the momentum and energy region where spin fluctuations are most important. These arguments are applied for systems in which the zero-temperature static spin susceptibility is greatly enhanced above the Pauli value for a noninteracting system. However, we know that the RPA leads to a result in disagreement with experiment for the static susceptibility in the region of the Curie temperature. It is also true that in the temperature range close to the critical temperature, the region of momenta and energies which are probed extend far beyond the region where the RPA is valid (see Sec. 3). Thus it is not surprising that the RPA predictions for the specific heat are in complete disagreement with experiment at the Curie temperature. However, the region of temperature about T_c over which the RPA breaks down is some fraction of T_c, so that in principle there will always be a region of validity at low temperatures. It is also true that there is currently no other approach whereby the present results can be readily obtained.

* Present address.
[1] N. Berk and J. R. Schrieffer, Phys. Rev. Letters 17, 433 (1966).
[2] S. Doniach and S. Engelsberg, Phys. Rev. Letters 17, 750 (1966).
[3] E. Bucher, W. F. Brinkman, J. Maita, and H. J. Williams, Phys. Rev. Letters 18, 1125 (1967).
[4] J. R. Schrieffer and N. F. Berk, Phys. Letters 24A, 604 (1967).
[5] S. Doniach and M. J. Rice, in Proceedings of Conference on Theoretical Physics for R. E. Pierels's Sixtieth Birthday (unpublished).

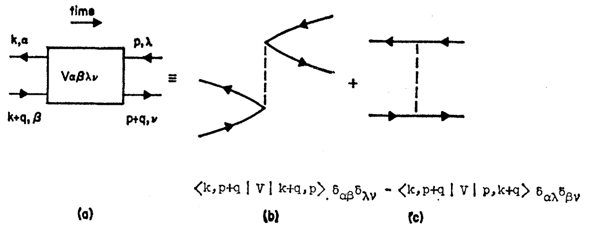

FIG. 1. (a) The basic particle-hole interaction; (b) and (c), constituent parts.

The effective particle-hole interaction discussed in Sec. 2 is taken to depend only on the total momentum of the particle-hole pair. The effective interaction is averaged over the other momenta involved. It is assumed that this approximation gives a solution that is qualitatively representative of the general equations in which the full momentum dependence is incorporated. There are several experimental results which indicate the importance of a momentum dependence for the interaction function. The first is that the compressibility or velocity of zero sound in He^3 is independent of the susceptibility enhancement. In the zero-range model, one parameter determines both. Second, for Coulomb forces we obtain with this model the plasma oscillations rather than finding only a zero-sound mode as in the zero-range model. Third, the large mass enhancements predicted by the zero-range model are strongly reduced by a small but finite range, in agreement with experiment.[3,6,7] Fourth, Clogston[8] has also given arguments for Pd which indicate the importance of the range of the interaction in determining the induced magnetic moment about an Fe impurity. The major disadvantage of the model is that the types of interactions which can occur are more complex than for zero range. The form of the interaction is that standard in Fermi-liquid theory[9] and is chosen to make transparent its spin symmetry. The infinite-wavelength limit of the two functions introduced are determined by the fermion contribution to the spin susceptibility and compressibility. For a paramagnet, the scattering matrices separate into spin-fluctuation and density-fluctuation parts. In a ferromagnet, the density fluctuations and the longitudinal spin fluctuations are coupled. This coupling is discussed in Sec. 4. With the solution obtained for the scattering matrices the shift of thermodynamic potential is calculated in Sec. 2. We believe that calculations of all the thermodynamic properties are most simply and correctly made by evaluating the thermodynamic potential as a function of temperature directly rather than evaluating fermion self-energy contributions. The direct approach has been given by Brenig, Mikeska, and Riedel.[10] However, a physical interpretation of the results is most easily achieved from the expression derived for the entropy shift in Sec. 2. It contains both a fermion contribution written in terms of a self-energy, plus a bosonlike contribution coming from the spin fluctuations.[11] The leading contribution to the specific heat of the latter term is proportional to $T^3 \ln T$, so that at low temperatures the specific heat is dominated by the fermion term. However, both contributions play an important role in the higher-order terms ($T^3 \ln T$, $T^3 \cdots$).

In Sec. 3 we examine the specific heat in the vicinity of the Curie temperature. As previously mentioned, we do not expect a correct prediction for two reasons. Even if we used an enhancement factor κ_0^2 which would give agreement with the measured static susceptibility, rather than the RPA result $\kappa_0^2 \propto T-T_c$, the RPA does not correctly account for the short-range correlations near the Curie temperature and the predicted specific heat would not be in agreement with experiment. Our reason for considering the RPA model for spin fluctuations is to find (a) the mechanism which is producing the singularity at the Curie temperature and (b) whether it is connected with the low-temperature mass enhancements. In Sec. 3 we show that the effective mass is not singular at the Curie temperature T_c but that another term in the self-energy leads to a $(T-T_c)^{-1/2}$ singularity in the specific heat. This term is essentially momentum- and energy-independent and does not appear in the bosonlike contribution to the entropy. The $(T-T_c)^{-1/2}$ type of singularity is characteristic of an Ornstein-Zernike or RPA and presumably would be replaced by a logarithm in an exact theory. However, from our analysis we would not expect this singularity to come from an effective-mass enhancement.

If one calculated the dynamic susceptibility for a ferromagnet in a fictitious paramagnetic phase, it would show an unstable singularity in the particle-hole continuum. The system adjusts its magnetization to remove this singularity. However, in adjusting, not all of the low-energy fluctuations that occur within the continuum of particle-hole states are removed. This is especially true for a system in which the Curie temperature is much less than the Fermi energy. We call such systems "weak itinerant ferromagnets." Rh-Ni

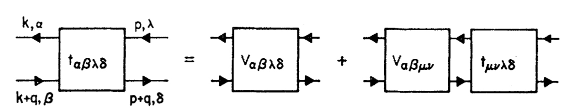

FIG. 2. Diagrammatic representation of the relation obeyed by the scattering matrices.

[6] In Sec. 5 we give the values of m^*/m predicted for He^3 in the zero-range RPA model and the range required to reduce the mass enhancement to its observed value.

[7] J. R. Schrieffer, Phys. Rev. Letters **19**, 644 (1967).

[8] A. M. Clogston, Phys. Rev. Letters **19**, 583 (1967).

[9] A. Abrikosov, L. Gor'kov, and I. Dzyaloshinski, in *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963), p. 75.

[10] W. Brenig, H. Mikeska, and E. Riedel, Z. Physik (to be published).

[11] E. Riedel, Z. Physik (to be published).

alloys[3] beyond the critical concentration, and possibly $ZrZn_2$,[12] are examples of weak ferromagnetism. The low-energy spin fluctuations are apparent for these systems from the measured enhancement of the longitudinal susceptibility.

In Sec. 4 we consider the effect of the low-lying spin fluctuations on the electron effective mass in weak itinerant ferromagnets. We demonstrate that the effective-mass enhancement shows the same effect in the ferromagnetic phase as in the paramagnetic phase, namely, that the mass enhancement goes to infinity symmetrically as the logarithm of the longitudinal susceptibility.

We also discuss the temperature variation of the specific heat for weak ferromagnets. The spin-wave collective modes give only a small contribution to the specific heat because they are valid excitations only in a small region of phase space in addition to the fact that their contribution to the specific heat goes as $T^{3/2}$. The electron interactions with the well-defined spin waves are also shown to lead to a negligible correction to the specific heat. The resulting temperature dependence is in agreement with the physically intuitive result that as one changes the system, bringing it closer to instability at zero temperature, the concurrent larger effective mass is removed at a lower temperature. Part of what one should see experimentally is shown in Fig. 13.

Application of large magnetic fields offers at least one way of testing the spin-fluctuation model. If the specific-heat enhancement is due to an ordering of local moments one would expect to quench the enhancement with fields $H \approx K_B/g\mu_B$ times the temperature at which enhancement effects are seen. However, in order to quench the spin-fluctuation enhancements we need $g\mu_B H$ of the order of the characteristic spin-fluctuation energy $\kappa_0^2\epsilon_F$. If one could find a system for which these energies are comparable, the specific heat would decrease rapidly with increasing field, as shown in Fig. 14.

In Sec. 5 the temperature dependence of the fermion contribution to the specific heat is discussed for the

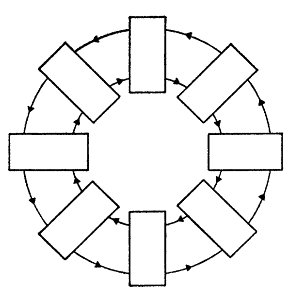

FIG. 3. General diagram included in the calculation of the shift in thermodymanic potential. The squares represent the basic particle-hole interactions shown in Fig. 1.

[12] S. Ogawa and N. Sakamoto, Phys. Letters **23**, 199 (1966). Recently, evidence has been presented that indicates that $ZrZn_2$ may not be ferromagnetic in the pure states; see S. Foner, E. J. McNiff, Jr., and V. Sudagopan, Phys. Rev. Letters **19**, 1233 (1967).

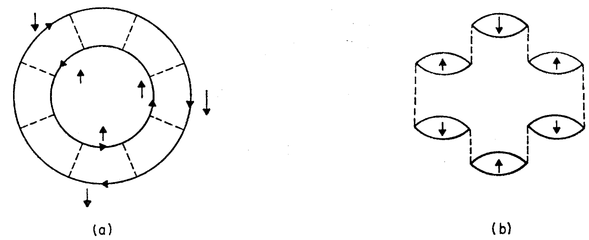

FIG. 4. Diagrams included in thermodynamic potential for the special case of a zero-range interaction.

paramagnetic phase. The inclusion of a finite-range interaction significantly reduces the total effective mass as was shown previously,[3,7] but in addition we find a reduction in the temperature variation of C_v/T versus T. These results are discussed and compared with experiments on He^3. We conclude that spin fluctuations are effective in a qualitative description of He^3. However, the parametrization using a range is an oversimplification for a quantitative description of He^3. It is also demonstrated that although $T^3 \ln T$ is the leading temperature dependence beyond the mass-enhancement term, this dependence is inadequate for describing the temperature dependence within the RPA model except at extremely low temperatures. In He^3 this region is essentially below the lowest temperatures for which measurements have been performed.

2. GENERAL FORMULATION

Our goal in this section is to give a somewhat generalized formulation of the RPA approximation from which the specific heat can be calculated in both the ferromagnetic and paramagnetic regime. The desire here is to obtain the total specific heat due to the large fluctuations in the magnetization that occur when the system is near a magnetic instability. Since the energy of these fluctuations goes to zero as such a system becomes unstable, virtual scattering of electrons by them causes large energy shifts at zero temperature. In the fermion system, the effect is a large change of the single-particle density of states in the vicinity of the Fermi surface.

In an itinerant system spin fluctuations are coherent particle-hole excitations. In order to describe these excitations one needs to consider the effective interaction between particle-hole pairs. This interaction is most easily discussed in a spin-invariant fashion similar to that outlined by Abrikosov, Gor'kov, and Dzyaloshinski.[9] The particle-hole interaction $V_{\alpha\beta\lambda\nu}(k, p; q)$ is the difference between the direct and exchange diagrams as shown in Fig. 1. Because of the invariance of the interaction with respect to rotations of the particle spins, the interaction can be written in terms of two functions;

$$V_{\alpha\beta\lambda\nu}(k, p; q) = V_s(k, p; q)\delta_{\alpha\beta}\delta_{\lambda\nu} + V_a(k, p; q)\, \boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{\sigma}_{\lambda\nu}. \quad (2.1)$$

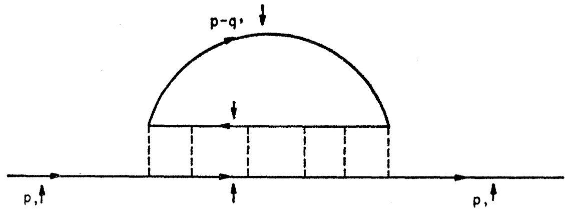

FIG. 5. Transverse-spin-fluctuation contribution to the self-energy.

Here $\boldsymbol{\sigma}_{\alpha\beta}$ are the matrix elements of the usual Pauli matrices. This interaction will be considered here as an effective scattering function which is the result of including, for example, particle-particle scattering and in the case of transition metals s-electron screening of the d-electron interaction.[13] When viewed this way, the form (2.1) must be considered as an assumption in a ferromagnet since in this case the spin invariance has been broken. There have been several papers discussing the form of this function for transition metals.[7,8] In a paramagnetic system, V_s describes the density oscillations. It characterizes zero-sound in liquid helium and in a charged system describes the plasma oscillations so that $V_s(k,p,q)\rightarrow 4\pi e^2/q^2$ as q goes to zero. For d electrons, however, if the s electrons effectively screen the Coulomb forces, V_s may remain finite as q goes to zero. For this to be valid the ratio of the d band mass to the s mass must be large. V_a characterizes the spin fluctuations of the type of interest here. However, since we are interested in the ferromagnetic phase we shall need to keep both types of fluctuations because the σ_z spin fluctuations couple to the density fluctuations in this case. Before we go on to solve this for the particle-hole scattering matrices, we note the reduction of the scattering functions to the zero-range model of Ref. 2;

$$V_s=-\tfrac{1}{2}I,\qquad V_a=2I. \tag{2.2}$$

We now consider the particle-hole scattering matrices which will enter our expressions for the shift in thermodynamic potential. The general equation obeyed by the scattering matrices is shown in Fig. 2. With the general form (2.1) these equations are quite impossible to solve but we note that the solution would always involve averages of V over $\mathbf{k}$ and $\mathbf{p}$. We thus make the approximation of replacing V_s and V_a by functions that are averaged over these two variables. This enables us to solve the equations and, we believe, retain most of the relevant physics. Our conventions for the scattering matrices are

$$\begin{aligned} &t_{\uparrow\uparrow\uparrow\uparrow}=t_{\uparrow\uparrow}^0, && t_{\downarrow\downarrow\downarrow\downarrow}=t_{\downarrow\downarrow}^0,\\ &t_{\downarrow\downarrow\uparrow\uparrow}=t_{\downarrow\uparrow}^0, && t_{\uparrow\uparrow\downarrow\downarrow}=t_{\uparrow\downarrow}^0,\\ &t_{\downarrow\uparrow\downarrow\uparrow}(\mathbf{q},\omega)=t^1(\mathbf{q},\omega), && t_{\uparrow\downarrow\uparrow\downarrow}(\mathbf{q},\omega)=t^1(\mathbf{q},-\omega). \end{aligned} \tag{2.3}$$

The equations shown diagrammatically in Fig. 2 are written as

$$\begin{aligned} t_{\uparrow\uparrow}^0(q)=&(V_s+\tfrac{1}{4}V_a)+(V_s+\tfrac{1}{4}V_a)\chi_\uparrow^0(q)t_{\uparrow\uparrow}^0(q)\\ &+(V_s-\tfrac{1}{4}V_a)\chi_\downarrow^0(q)t_{\downarrow\uparrow}^0(q),\\ t_{\downarrow\uparrow}^0(q)=&(V_s-\tfrac{1}{4}V_a)+(V_s-\tfrac{1}{4}V_a)\chi_\uparrow^0(q)t_{\uparrow\uparrow}^0(q)\\ &+(V_s+\tfrac{1}{4}V_a)\chi_\downarrow^0(q)t_{\downarrow\uparrow}^0(q), \end{aligned} \tag{2.4}$$

where the particle-hole propagators χ_σ^0 are defined by

$$\chi_\sigma^0(\mathbf{q},z)=\sum_{\mathbf{p}}(f_{\mathbf{p}+\mathbf{q},\sigma}-f_{\mathbf{p},\sigma})/(z+\epsilon_{\mathbf{p}\sigma}-\epsilon_{\mathbf{p}+\mathbf{q}\sigma}). \tag{2.5}$$

The Fermi functions $f_{\mathbf{p},\sigma}$ are given by

$$f_{\mathbf{p},\sigma}=[\exp(\beta\epsilon_{\mathbf{p}\sigma})+1]^{-1} \tag{2.6}$$

and

$$\epsilon_{\mathbf{p}\sigma}=\mathbf{p}^2/2m+V_s(0)(N_\sigma+N_{-\sigma})-\tfrac{1}{4}[V_a(0)](N_\sigma-N_{-\sigma})-\mu \tag{2.7}$$

are the Hartree single-particle energies measured relative to the chemical potential μ. N_σ denotes the number of particles with spin σ. This choice of single-particle energies is consistent with our averaged approximation for the t matrices since the Hartree-Fock energies are averages over k of the effective interaction.

Equation (2.4) may be solved algebraically for the zero-spin scattering matrix $t_{\uparrow\uparrow}^0(q)$;

$$t_{\uparrow\uparrow}^0(q)=\frac{V_s(q)+\frac{1}{4}V_a(q)-V_s(q)V_a(q)\chi_\downarrow^0(q)}{1-[V_s(q)+\frac{1}{4}V_a(q)][\chi_\downarrow^0(q)+\chi_\uparrow^0(q)]+V_s(q)V_a(q)\chi_\uparrow^0(q)\chi_\downarrow^0(q)}. \tag{2.8}$$

The equation for $t_{\downarrow\downarrow}^0$ is obtained by inverting the arrows above.

The spin-1 scattering matrix t^1 may also be obtained algebraically;

$$t^1(q)=\tfrac{1}{2}V_a(q)/[1-\tfrac{1}{2}V_a(q)\chi^0(q)]. \tag{2.9}$$

The spin-1 particle-hole propagator $\chi^0(q)$ is

$$\chi^0(\mathbf{q},z)=\sum_{\mathbf{p}}(f_{\mathbf{p}+\mathbf{q},\uparrow}-f_{\mathbf{p},\downarrow})/(z+\epsilon_{\mathbf{p}\downarrow}-\epsilon_{\mathbf{p}+\mathbf{q}\uparrow}). \tag{2.10}$$

The expression for the shift in thermodynamic potential is obtained from the general diagram shown in Fig. 3. The squares of Fig. 3 represent the interaction functions V. In the case of a zero-range interaction, this shift results from the two distinct types of diagrams

[13] C. Herring, in *Magnetism*, edited by G. T. Rado and H. Suhl (Academic Press Inc., New York, 1966), Vol. IV, Chap. 2.

shown in Fig. 4. The shift of the thermodynamic potential from its Hartree-Fock value is

$$\Delta\Omega=-(2\beta)^{-1}\sum_{\mathbf{q}}\sum_{m}\int_0^1\frac{d\lambda}{\lambda}\{t_{\uparrow\uparrow}{}^0(\mathbf{q},i\omega_m)\chi_\uparrow{}^0(\mathbf{q},i\omega_m)+t_{\downarrow\downarrow}{}^0(\mathbf{q},i\omega_m)\chi_\downarrow{}^0(\mathbf{q},i\omega_m)+2t^1(\mathbf{q},i\omega_m)\chi^0(\mathbf{q},i\omega_m)$$
$$-\lambda[V_s(\mathbf{q})+\tfrac{1}{4}V_a(\mathbf{q})][\chi_\uparrow{}^0(\mathbf{q},i\omega_m)+\chi_\downarrow{}^0(\mathbf{q},i\omega_m)]-\lambda V_a(\mathbf{q})\chi^0(\mathbf{q},i\omega_m)\}. \quad (2.11)$$

Here $\omega_m=(2m+1)\pi\beta$ and $\beta=1/K_BT$.

To obtain the shift in thermodynamic potential from our approximation to the scattering matrices, we have multiplied the interaction functions by a coupling constant λ and performed the indicated coupling-constant integration. This procedure again assumes that the detailed structure of the interaction functions are unimportant. Using (2.8) and (2.9) in (2.11) we obtain

$$\Delta\Omega=\beta^{-1}\sum_{\mathbf{q}}\sum_{m}\{\tfrac{1}{2}[\ln[1-(V_s+\tfrac{1}{4}V_a)(\chi_\uparrow{}^0+\chi_\downarrow{}^0)+V_sV_a\chi_\uparrow{}^0\chi_\downarrow{}^0]+(V_s+\tfrac{1}{4}V_a)(\chi_\uparrow{}^0+\chi_\downarrow{}^0)]$$
$$+\ln(1-\tfrac{1}{2}V_a\chi^0)+\tfrac{1}{2}V_a\chi^0\}. \quad (2.12)$$

It should be pointed out that in Eq. (2.12) the second-order diagram is counted twice. This diagram could be subtracted but since only infinite-order collective effects are of interest it will be neglected.

Equation (2.12) simplifies greatly in the paramagnetic state where

$$\chi_\uparrow{}^0=\chi_\uparrow{}^0=\chi^0, \quad (2.13)$$

leading to

$$\Delta\Omega=\beta^{-1}\sum_{\mathbf{q}}\sum_{m}\{\tfrac{3}{2}[\ln(1-\tfrac{1}{2}V_a\chi^0)+\tfrac{1}{2}V_a\chi^0]$$
$$+\tfrac{1}{2}[\ln(1-2V_s\chi^0)+2V_s\chi^0]\}. \quad (2.14)$$

The first term represents the total spin-fluctuation contribution. The longitudinal spin fluctuations were omitted in the original work,[1,2] but have subsequently been introduced.[3,10,14] The effect of including longitudinal spin fluctuations is seen to multiply the shift due to transverse spin fluctuations by the factor $\frac{3}{2}$. The second term of (2.14) is the density-fluctuation contribution.

Equation (2.14) represents the contribution of the simpler diagrams in Fig. 4, when we use the δ-function potential and the identities given in (2.2). For this case

$$\Delta\Omega=\beta^{-1}\sum_{\mathbf{q}}\sum_{m}\{\tfrac{3}{2}[\ln(1-I\chi^0)+I\chi^0]$$
$$+\tfrac{1}{2}[\ln(1+I\chi^0)-I\chi^0]\}. \quad (2.15)$$

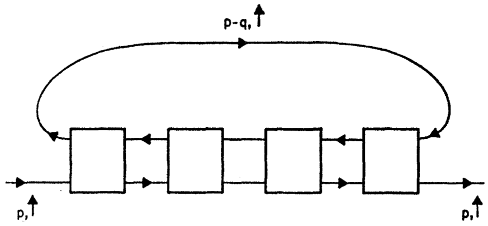

FIG. 6. Longitudinal contribution to the self-energy including both spin and density fluctuations.

[14] D. Penn, Phys. Letters **25A**, 269 (1967).

We break $\Delta\Omega$ into its longitudinal and transverse parts before deriving an expression for the shift in entropy. For the transverse part $\Delta\Omega_T$, the sum over integers may be rewritten as a contour integral;

$$\Delta\Omega_T=\sum_{\mathbf{q}}\int_c\frac{d\omega}{2\pi i}n(\omega)\{\ln[1-\tfrac{1}{2}V_a(\mathbf{q})\chi^0(\mathbf{q},\omega)]$$
$$+\tfrac{1}{2}V_a(\mathbf{q})\chi^0(\mathbf{q},\omega)\}, \quad (2.16)$$

where $n(\omega)$ is the Bose function

$$n(\omega)=[\exp(\beta\omega)-1]^{-1},$$

and the contour encircles the real axis from $-\infty$ to $+\infty$ in the clockwise direction, omitting the pole of $n(\omega)$ at $\omega=0$. Similarly, the longitudinal part $\Delta\Omega_L$ is written

$$\Delta\Omega_L=\tfrac{1}{2}\sum_{\mathbf{q}}\int_c\frac{d\omega}{2\pi i}n(\omega)$$
$$\times\{\ln[1-(V_s(\mathbf{q})+\tfrac{1}{4}V_a(\mathbf{q}))(\chi_\uparrow{}^0(\mathbf{q},\omega)+\chi_\downarrow{}^0(\mathbf{q},\omega))$$
$$+V_s(\mathbf{q})V_a(\mathbf{q})\chi_\uparrow{}^0(\mathbf{q},\omega)\chi_\downarrow{}^0(\mathbf{q},\omega)]$$
$$+[V_s(\mathbf{q})+\tfrac{1}{4}V_a(\mathbf{q})][\chi_\uparrow{}^0(\mathbf{q},\omega)+\chi_\downarrow{}^0(\mathbf{q},\omega)]\}. \quad (2.17)$$

The transverse contribution to the shift in entropy is

$$\Delta S_T=-\frac{\partial\Delta\Omega_T}{\partial T}$$
$$=-\sum_{\mathbf{q}}\int_c\frac{d\omega}{2\pi i}\frac{\partial n(\omega)}{\partial T}[\ln(1-\tfrac{1}{2}V_a\chi^0)+\tfrac{1}{2}V_a\chi^0]$$
$$-\sum_{\mathbf{q}}\int_c\frac{d\omega}{2\pi i}n(\omega)\frac{\partial}{\partial T}[\ln(1-\tfrac{1}{2}V_a\chi^0)+\tfrac{1}{2}V_a\chi^0]. \quad (2.18)$$

Carrying out the temperature differentiation in the second term and performing the contour integration, we obtain for this term

$$\sum_{\mathbf{q}} (\tfrac{1}{2}V_a)^2 \int_{-\infty}^{\infty} \frac{d\omega}{\pi} n(\omega)\, \mathrm{Im}\chi^{-+}(\mathbf{q},\omega) \times \sum_{\mathbf{p}} \left(\frac{\partial f_{\mathbf{p}\uparrow}}{\partial T} - \frac{\partial f_{\mathbf{p}-\mathbf{q}\downarrow}}{\partial T}\right) \frac{P}{(\omega - \epsilon_{\mathbf{p}\uparrow} + \epsilon_{\mathbf{p}-\mathbf{q}\downarrow})} - \sum_{\mathbf{p},\mathbf{q}} (\tfrac{1}{2}V_a)^2 n(\epsilon_{\mathbf{p}\uparrow} - \epsilon_{\mathbf{p}-\mathbf{q}\downarrow})\, \mathrm{Re}\chi^{-+} \times (q, \epsilon_{\mathbf{p}\uparrow} - \epsilon_{\mathbf{p}-\mathbf{q}\downarrow}) \left(\frac{\partial f_{\mathbf{p}\uparrow}}{\partial T} - \frac{\partial f_{\mathbf{p}-\mathbf{q}\downarrow}}{\partial T}\right), \tag{2.19}$$

where we have defined

$$\chi^{-+}(\mathbf{q},\omega) = \chi^0(\mathbf{q},\omega)/\{1 - \tfrac{1}{2}V_a(\mathbf{q})\chi^0(\mathbf{q},\omega)\} \tag{2.20}$$

and its real and imaginary parts are defined for $\omega + i0^+$.

We use the identity

$$n(\epsilon_{\mathbf{p}\uparrow} - \epsilon_{\mathbf{p}-\mathbf{q}\downarrow})(\partial f_{\mathbf{p}\uparrow}/\partial T - \partial f_{\mathbf{p}-\mathbf{q}\downarrow}/\partial T) = -(\partial/\partial T)[f_{\mathbf{p}\uparrow}(1 - f_{\mathbf{p}-\mathbf{q}\downarrow})] - (f_{\mathbf{p}\uparrow} - f_{\mathbf{p}-\mathbf{q}\downarrow})(\partial/\partial T) n(\epsilon_{\mathbf{p}\uparrow} - \epsilon_{\mathbf{p}-\mathbf{q}\downarrow}) \tag{2.21}$$

to rewrite (2.19). Doing the contour integration in the first term in (2.18) and combining Eq. (2.21), (2.19), and (2.18), we obtain

$$\Delta S_T = \sum_{\mathbf{q}} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\partial n(\omega)}{\partial T} \left[\tan^{-1} \frac{\tfrac{1}{2}V_a\chi_I^0(q,\omega)}{1 - \tfrac{1}{2}V_a\chi_R^0(q,\omega)} - \frac{\tfrac{1}{2}V_a\chi_I^0(q,\omega)[1 - \tfrac{1}{2}V_a\chi_R^0(q,\omega)]}{[1 - \tfrac{1}{2}V_a\chi_R^0(q,\omega)]^2 + [\tfrac{1}{2}V_a\chi_I^0(q,\omega)]^2}\right] - \sum_{\mathbf{p},\sigma} \frac{\partial f_{\mathbf{p},\sigma}}{\partial T} \mathrm{Re}\Sigma_\sigma^T(\mathbf{p}, \epsilon_{\mathbf{p},\sigma}). \tag{2.22}$$

χ_I^0 and χ_R^0 are the real and imaginary parts of χ^0 when $\omega = \omega + i0^+$. We have introduced the fermion self-energies

$$\Sigma_\uparrow^T(\mathbf{p}, \epsilon_{\mathbf{p}\uparrow}) = -\sum_{\mathbf{q}} (\tfrac{1}{2}V_a)^2 \times \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\mathrm{Im}\chi^{-+}(\mathbf{q},\omega)}{\omega - \epsilon_{\mathbf{p}\uparrow} + \epsilon_{\mathbf{p}-\mathbf{q}\downarrow}} [n(\omega) + 1 - f_{\mathbf{p}-\mathbf{q}\downarrow}], \tag{2.23}$$

$$\Sigma_\downarrow^T(\mathbf{p}, \epsilon_{\mathbf{p}\downarrow}) = \sum_{\mathbf{q}} (\tfrac{1}{2}V_a)^2 \times \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\mathrm{Im}\chi^{-+}(\mathbf{q},\omega)}{\omega - \epsilon_{\mathbf{p}-\mathbf{q}\uparrow} + \epsilon_{\mathbf{p}\downarrow}} [n(\omega) + f_{\mathbf{p}-\mathbf{q}\uparrow}]. \tag{2.24}$$

In the paramagnetic phase we can write

$$\mathrm{Im}\chi^{-+}(\mathbf{q},\omega) = -\mathrm{Im}\chi^{-+}(\mathbf{q},-\omega)$$

to obtain

$$\Sigma_\uparrow^T(\mathbf{p}, p_0) = \Sigma_\downarrow^T(\mathbf{p}, p_0) = \sum_{\mathbf{q}} (\tfrac{1}{2}V_a)^2 \int_0^{\infty} \frac{d\omega}{\pi} \mathrm{Im}\chi^{-+}(\mathbf{q},\omega) \times \left[\frac{f_{\mathbf{p}-\mathbf{q}} + n(\omega)}{p_0 - \epsilon_{\mathbf{p}-\mathbf{q}} + \omega} + \frac{1 - f_{\mathbf{p}-\mathbf{q}} + n(\omega)}{p_0 - \epsilon_{\mathbf{p}-\mathbf{q}} - \omega}\right]. \tag{2.25}$$

This is the self-energy contribution considered in Refs. 1 and 2 corresponding to the diagram shown in Fig. 5.

The total entropy in (2.22) has a bosonlike term in addition to the fermion self-energy contribution. For a paramagnetic system at low temperatures, the boson term has as its leading contribution a $T^3 \ln T$ temperature dependence. The bosonlike contribution may be incorporated into the $T^3 \ln T$ fermion contribution calculated by Doniach and Engelsberg by multiplying their result by a factor of $\frac{1}{3}$. Brenig *et al.*[10] have calculated both contributions by using the expansion for χ^0 directly in the RPA expression for $\Delta\Omega$ (2.16). The expression (2.22) shows that in this model the coherent particle-hole excitations give rise to a significant contribution to the entropy whenever the interactions cause an appreciable shift in the energy of a particle-hole excitation.

In the ferromagnetic case this boson term contains the magnon contribution to the entropy, which can be written as

$$\Delta S_{\mathrm{magnon}} = \sum_{\mathbf{q}} \int_{\omega_s(\mathbf{q})}^{\epsilon_l(\mathbf{q})} d\omega \frac{\partial n(\omega)}{\partial T}. \tag{2.26}$$

The magnon splits off from the lower side of the particle-hole continuum and $\epsilon_l(\mathbf{q})$ is the lower edge of this continuum.

We are primarily interested in the effective-mass enhancement and the qualitative temperature dependence of C_v/T. The bosonlike terms are at least of order $T^2 \ln T$ or T^2 in C_v/T and give quantitative corrections to the temperature dependence of the fermion-like contribution. We will ignore them in our subsequent calculations.

The entropy contribution from $\Delta\Omega_L$ given in (2.17)

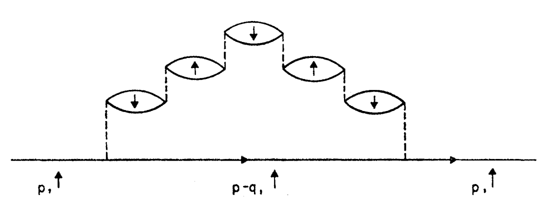

FIG. 7. Longitudinal contribution to the self-energy in the special case of a zero-range model.

can be derived in exactly the same way as ΔS_T. If we define

$$\chi_\sigma(\mathbf{q},\omega)=\frac{[V_s(\mathbf{q})+\frac{1}{4}V_a(\mathbf{q})]-V_s(\mathbf{q})V_a(\mathbf{q})\chi_\sigma{}^0(\mathbf{q},\omega)}{1-[V_s(\mathbf{q})+\frac{1}{4}V_a(\mathbf{q})][\chi_\uparrow{}^0(\mathbf{q},\omega)+\chi_\downarrow{}^0(\mathbf{q},\omega)]+V_s(\mathbf{q})V_a(\mathbf{q})\chi_\uparrow{}^0(\mathbf{q},\omega)\chi_\downarrow{}^0(\mathbf{q},\omega)}-[V_s(\mathbf{q})+\tfrac{1}{4}V_a(\mathbf{q})], \quad (2.27)$$

we obtain

$$\Delta S_L=\tfrac{1}{2}\sum_{\mathbf{q}}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\partial n(\omega)}{\partial T}\left\{\tan^{-1}\frac{(V_s+\frac{1}{4}V_a)(\chi_{\uparrow I}{}^0+\chi_{\downarrow I}{}^0)-V_sV_a(\chi_{\uparrow I}{}^0\chi_{\downarrow R}{}^0+\chi_{\uparrow R}{}^0\chi_{\downarrow I}{}^0)}{1-(V_s+\frac{1}{4}V_a)(\chi_{\downarrow R}{}^0+\chi_{\uparrow R}{}^0)+V_sV_a(\chi_{\downarrow R}{}^0\chi_{\uparrow R}{}^0-\chi_{\downarrow I}{}^0\chi_{\uparrow I}{}^0)}-\sum_\sigma[\mathrm{Re}\chi_{-\sigma}+(V_s+\tfrac{1}{4}V_a)]\chi_{\sigma I}{}^0\right\}-\sum_\sigma\sum_{\mathbf{p}}\frac{\partial f_{\mathbf{p},\sigma}}{\partial T}\mathrm{Re}\Sigma_\sigma{}^L(\mathbf{p},\epsilon_{\mathbf{p},\sigma}), \quad (2.28)$$

where

$$\Sigma_\sigma{}^L(\mathbf{p},\epsilon_{\mathbf{p},\sigma})=\tfrac{1}{2}\sum_{\mathbf{q}}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\,\mathrm{Im}\chi_{-\sigma}(\mathbf{q},\omega)\left[\frac{f_{\mathbf{p}-\mathbf{q},\sigma}+n(\omega)}{\epsilon_{\mathbf{p},\sigma}-\epsilon_{\mathbf{p}-\mathbf{q},\sigma}+\omega}+\frac{1-f_{\mathbf{p}-\mathbf{q},\sigma}+n(\omega)}{\epsilon_{\mathbf{p},\sigma}-\epsilon_{\mathbf{p}-\mathbf{q},\sigma}-\omega}\right]. \quad (2.29)$$

This self-energy is shown diagrammatically in Fig. 6. The squares of Fig. 6 are the zero-spin interaction functions of (2.1). In the zero-range model, Fig. 6 reduces to Fig. 7. In the paramagnetic case the coefficient of $\partial n/\partial T$ of (2.28) splits into two contributions, one that is identical to the first part of (2.22) but multiplied by $\frac{1}{2}$ and another contribution of the form

$$\tfrac{1}{2}\sum_{\mathbf{q}}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\partial n(\omega)}{\partial T}\left[\tan^{-1}\left(\frac{2V_s\chi_I{}^0}{1-2V_s\chi_R{}^0}\right)-\frac{2V_s\chi_I{}^0(1-2V_s\chi_R{}^0)}{(1-2V_s\chi_R{}^0)^2+(2V_s\chi_I{}^0)^2}\right]. \quad (2.30)$$

Contributions from any collective density fluctuations such as zero sound or plasma oscillations are contained in (2.30). Again, since we are primarily interested in lowest-order effects in temperature, we ignore these contributions.

The shift in entropy will be calculated using the fermionlike contributions given in Eqs. (2.22) and (2.28).

In this section we have developed a relation for the shift in thermodynamic potential from terms most important in a fermion system close to a ferromagnetic instability. From this relation we have derived an expression for the shift in entropy due to collective excitations and the single-particle self-energy. In this form the effect of interactions on the single-particle spectrum and density of states is apparent and its contribution to equilibrium properties is manifest.

3. SPECIFIC HEAT NEAR THE CURIE TEMPERATURE

It is known that in a nearly ferromagnetic system, spin fluctuations lead to large enhancements of the effective mass measured by the slope of the low-temperature specific heat.[1,2] It has also been shown[2] that the low-temperature behavior is changed qualitatively from the expected linearity. The question to which we address ourselves at this point is: For the itinerant model we consider, does the effective-mass enhancement which appeared at zero temperature for the nearly ferromagnetic system become tied to the Curie temperature, or do those enhancement terms remain at zero temperature and a new type of singularity appear from another mechanism? We will see that it is the latter behavior which prevails.

For the ferromagnetic system I of Eq. (2.2) is such that[15]

$$\bar{I}\equiv IN(0)>1, \quad (3.1)$$

where $N(0)$ is the density of states for a single spin at the Fermi energy for the unstable paramagnetic phase. The condition for ferromagnetism is that the enhanced susceptibility becomes infinite. Under this condition the Curie temperature is determined by

$$1=-I\int d\epsilon N(\epsilon)\left.\frac{\partial f}{\partial\epsilon}\right|\equiv\bar{I}(T_c). \quad (3.2)$$

We consider $T_c\ll T_F$, that is, weak itinerant ferromagnets, in which case

$$\bar{I}(T)\approx\left\{1+\frac{\pi^2}{12}\left(\frac{T_c}{T_F}\right)^2\left[1-\left(\frac{T}{T_c}\right)^2\right]\right\}+O\left(\frac{T_c}{T_F}\right)^4 \quad (3.3)$$

for parabolic bands. The quantity $\kappa_0{}^2=1-\bar{I}(T)$ may be written

$$\kappa_0{}^2\approx\tfrac{1}{6}\pi^2\bar{I}(0)(T_c/T_F{}^2)(T-T_c),\quad\text{for}\quad T\gtrsim T_c. \quad (3.4)$$

In the paramagnetic phase we consider only the spin-fluctuation contribution of (2.14). We neglect the effect of density fluctuations appearing in $\Delta\Omega_L$, since they will not be affected as we approach the Curie temperature from the paramagnetic phase. The self-energy (2.25) is broken into two parts for this calcula-

[15] Since the spin fluctuations decouple from the density fluctuations in the paramagnetic region, we will characterize the relevant interaction $\frac{1}{2}V_a(q)$ by the constant I in this section.

tion: a fermion term

$$\Sigma_f(\mathbf{p},p_0)=\tfrac{3}{2}I^2\sum_{\mathbf{q}}\int_0^\infty\frac{d\omega}{\pi}\,\mathrm{Im}\chi^{-+}(\mathbf{q},\omega+i\delta)\times\left[\frac{f_{\mathbf{p}-\mathbf{q}}}{p_0-\epsilon(\mathbf{p}-\mathbf{q})+\omega}+\frac{1-f_{\mathbf{p}-\mathbf{q}}}{p_0-\epsilon(\mathbf{p}-\mathbf{q})-\omega}\right] \quad (3.5)$$

and a boson term

$$\Sigma_n(\mathbf{p},p_0)=\tfrac{3}{2}I^2\sum_{\mathbf{q}}\int_0^\infty\frac{d\omega}{\pi}\,\mathrm{Im}\chi^{-+}(\mathbf{q},\omega+i\delta)n(\omega)\times\frac{2[p_0-\epsilon(\mathbf{p}-\mathbf{q})]}{[p_0-\epsilon(\mathbf{p}-\mathbf{q})]^2-\omega^2}. \quad (3.6)$$

The reason for making this separation is twofold. In a system where the collective excitation is an independent entity, as phonons in a crystal, Σ_n depends explicitly on the boson occupation numbers $n(\omega)$. In the present model the collective excitations arise from coherent fermion excitations. However, the major reason for the separation is that in a nearly ferromagnetic system all of the mass enhancement at low temperatures comes from the fermion part Σ_f. The boson part Σ_n contributes a leading term to the entropy which is no more important than T^3. We will find the situation reversed near the Curie temperature. In this region the low-frequency sharply defined spin-fluctuation modes no longer contribute a singular term to the specific-heat mass enhancement arising from Σ_f, but a new singular term going as $(T-T_c)^{-1/2}$ is contributed by Σ_n.[16]

The term in Σ_f not including a fermi function has no structure about $p_0=0$. Thus this term corresponds essentially to a shift in chemical potential. We write the other two terms of (3.5) as

$$\Sigma_f[\mathbf{p},\epsilon(\mathbf{p})]=\tfrac{3}{2}I^2\sum_{\mathbf{q}}\chi^{-+}[\mathbf{q},\epsilon(\mathbf{p}-\mathbf{q})-\epsilon(\mathbf{p})]f_{\mathbf{p}-\mathbf{q}} \quad (3.7)$$

using the spectral relation for χ^{-+}. We adopt the analytic approximation for χ^{-+} given in Ref. 2.

$$\mathrm{Re}\chi^{-+}(\mathbf{q},\omega)=\frac{N(0)(\kappa_0^2+\alpha\bar{q}^2)}{(\kappa_0^2+\alpha\bar{q}^2)^2+(\gamma\bar{\omega}/\bar{q})^2}, \quad (3.8)$$

where

$$\alpha=\tfrac{1}{12}\bar{I},\qquad \gamma=\tfrac{1}{4}\pi\bar{I},$$
$$\bar{q}=q/p_F,\qquad \bar{\omega}=\omega/\epsilon_F.$$

This form for χ^{-+} retains the large enhancements of the susceptibility at $\bar{\omega}\approx\kappa_0^2\bar{q}$ for $\kappa_0^2\to0$. Integrating (3.7) by parts and using (3.8), we obtain

$$\Sigma_f[\mathbf{p},\epsilon(\mathbf{p})]=-\frac{3}{2}\frac{N(0)I^2\epsilon_F}{\gamma}\sum_{\mathbf{q}}\bar{q}\frac{\partial f_{\mathbf{p}-\mathbf{q}}}{\partial\epsilon(\mathbf{p}-\mathbf{q})}\times\tan^{-1}\frac{[\epsilon(\mathbf{p}-\mathbf{q})-\epsilon(\mathbf{p})]\gamma}{\epsilon_F(\kappa_0^2\bar{q}+\alpha\bar{q}^3)}. \quad (3.9)$$

[16] T. Izuyama and R. Kubo, J. Appl. Phys. **35**, 1074 (1964).

If we use this form in (2.22), we obtain the shift in entropy due to the fermion part of the self-energy.

$$\Delta S_f=\frac{3\bar{I}^2N(0)\epsilon_F}{4T\gamma}\int_{-\infty}^{\infty}d\epsilon\frac{\partial f}{\partial\epsilon}\int_{-\infty}^{\infty}d\epsilon'\frac{\partial f'}{\partial\epsilon'}(\epsilon-\epsilon')\times\int_0^{\bar{p}_1}d\bar{q}\bar{q}^2\tan^{-1}\frac{(\epsilon-\epsilon')\gamma}{\epsilon_F(\kappa_0^2\bar{q}+\alpha\bar{q}^3)}. \quad (3.10)$$

The low-temperature expansion given by Doniach and Engelsberg[2] may be obtained from (3.10) for temperatures much less than the characteristic spin-fluctuation temperature T_s.

$$T_s=\kappa_0^2\epsilon_F/K_B\equiv\kappa_0^2T_F.$$

To show the relevant variations of the functions in the integrand, we plot $\tan^{-1}\epsilon/T_s$ and $-\partial f/\partial\epsilon$ for $T\ll T_s$ in Fig. 8. We see that for $T\ll T_s$ the arc tangent is slowly varying and very small in the region where $\partial f/\partial\epsilon$ has all its weight. Thus by using this fact and integrating on $\bar{q}$ by parts the low-temperature expansion is achieved.

For temperature approaching the critical temperature, $\kappa_0^2\to0$ and we find the opposite extreme, $T_s\ll T$. The behavior of the important functions in this limit is shown in Fig. 9. In this case $\partial f/\partial\epsilon$ is almost constant in the region where the arc tangent goes through its entire variation from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$. This feature of the functions involved in (3.10) suggests an approximation which allows an analytic evaluation. The approximation we use is

$$\partial f/\partial\epsilon=-1/6K_BT,\qquad -3K_BT<\epsilon<3K_BT$$
$$=0,\qquad \text{otherwise.} \quad (3.11)$$

Equation (3.10) may then be written as

$$\Delta S_f=\frac{9}{16}\frac{K_BN(0)\epsilon_F\bar{I}^2}{\gamma}\int_0^{\bar{p}_1}d\bar{q}\bar{q}^2\int_{-1}^{1}dx\times\int_{-1}^{1}dx'(x-x')\tan^{-1}\frac{x-x'}{\Delta}, \quad (3.12)$$

where

$$\Delta=(\epsilon_F/\gamma)(\kappa_0^2\bar{q}+\alpha\bar{q}^3/3K_BT).$$

The x and x' integrals may now be performed by changing variables of integration. We neglect the $\bar{q}^3$ terms in

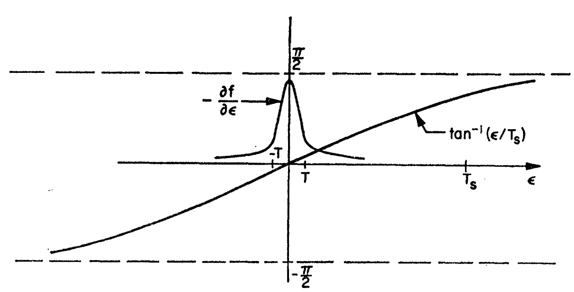

FIG. 8. Illustration of the variation of the functions appearing in the integrand of 3.10 for $T\ll T_s$.

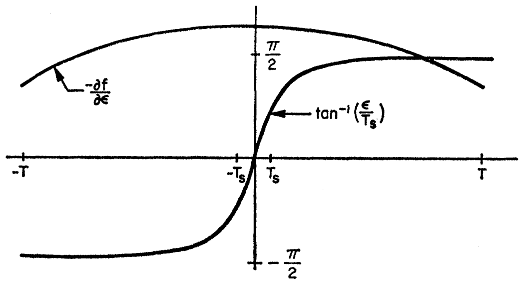

FIG. 9. Illustration of the variation of the functions appearing in the integrand of 3.10 for $T \gg T_s$.

Δ to obtain the leading contribution to the specific heat for $T_s \ll T$;

$$\Delta C_f = \frac{3K_B N(0)\epsilon_F \bar{p}_1^4}{\pi^2} \frac{T_c}{T} + O\left(\frac{T_s}{T}\right). \quad (3.13)$$

In taking a temperature derivative to obtain this result, we include the temperature dependence of κ_0^2. Equation (3.13) shows that the mass enhancement has not gone away but at the same time it is not singular at T_c.

We next consider the contribution of Σ_n, (3.6), to the specific heat. This term was not included in the low-temperature expansion of Doniach and Engelsberg.[2] We will indicate briefly why its contribution is negligible compared with the terms previously calculated for $T_s \gg T$. In the opposite regime $T_s \ll T$, near the ferromagnetic Curie point, it is just this term which contributes a singular term to the specific heat. Using the same analytic approximation which gave (3.8), we may write Σ_n as

$$\Sigma_n[\mathbf{p}, \epsilon(\mathbf{p})] = 3I^2 N(0)\gamma \sum_{\mathbf{q}} \int_0^\infty \frac{d\omega}{\pi} n(\omega)$$

$$\times \frac{\bar{\omega}\bar{q}}{(\kappa_0^2\bar{q} + \alpha\bar{q}^3)^2 + (\gamma\bar{\omega})^2} \frac{\epsilon(\mathbf{p}) - \epsilon(\mathbf{p}-\mathbf{q})}{[\epsilon(\mathbf{p}) - \epsilon(\mathbf{p}-\mathbf{q})]^2 - \omega^2}. \quad (3.14)$$

Whether $T \ll T_s$ or $T \gg T_s$, the major contribution of the integral above comes from the region where $\omega \ll K_B T$,

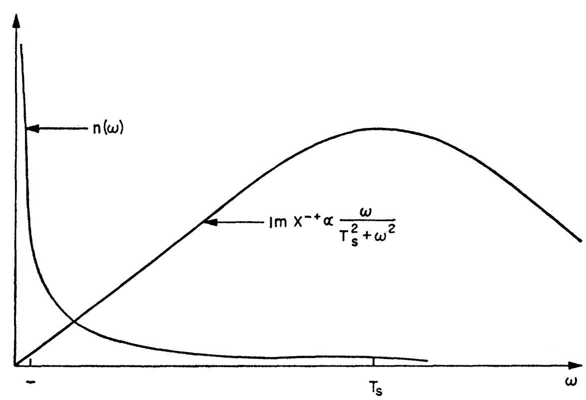

FIG. 10. Illustration of the variation of the functions appearing in the integrand of 3.16 for $T \ll T_s$.

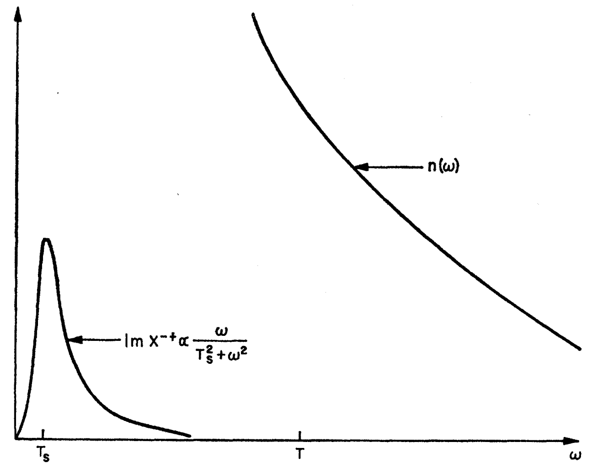

FIG. 11. Illustration of the variation of the functions appearing in the integrand of 3.16 for $T \gg T_s$.

due to the factor $n(\omega)$. Since $|\epsilon(\mathbf{p}) - \epsilon(\mathbf{p}-\mathbf{q})|$ is essentially unbounded, we neglect ω^2 compared to $[\epsilon(\mathbf{p}) - \epsilon(\mathbf{p}-\mathbf{q})]^2$ and obtain

$$\Sigma_n[\mathbf{p}, \epsilon(\mathbf{p})] = 3I^2 N(0)\gamma \sum_{\mathbf{q}} \int_0^\infty \frac{d\omega}{\pi} n(\omega)$$

$$\times \frac{\bar{\omega}\bar{q}}{(\kappa_0^2\bar{q} + \alpha\bar{q}^3) + (\gamma\bar{\omega})^2} [\epsilon(\mathbf{p}) - \epsilon(\mathbf{p}-\mathbf{q})]^{-1}. \quad (3.15)$$

The error in this approximation is of order $(T/T_F)^2$.

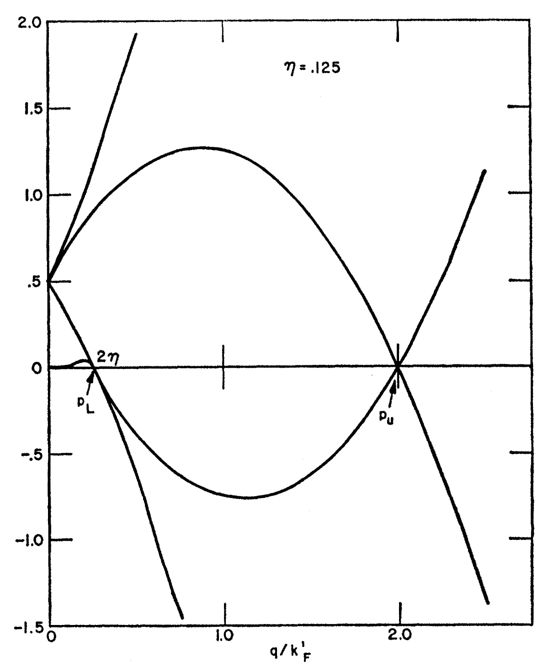

FIG. 12. The boundaries of the continuum for the transverse susceptibility $\chi^0(q, \omega)$. The p_l and p_u mark the limits of the integration in Eq. (4.3). At long wavelengths the low-frequency spin wave splits off from the continuum.

The major contribution of the q integration arises for momenta $q \ll p_f$. If we perform the angular integration in (3.15) and expand for $q \ll p_f$, there results

$$\Sigma_n[\mathbf{p}, \epsilon(\mathbf{p})] = -\frac{3I^2N(0)\gamma p_F^3}{8\pi^2(\epsilon+\epsilon_F)\epsilon_F}\int_0^{\bar{p}_1} d\bar{q}\,\bar{q}^3 \int_0^\infty \frac{d\omega}{\pi} \times \frac{n(\omega)\omega}{(\kappa_0^2\bar{q}+\alpha\bar{q}^3)^2+(\gamma\bar{\omega})^2}. \quad (3.16)$$

At this stage we must consider a specific temperature regime to obtain the leading contributions of (3.16). The physical situation is illustrated in Fig. 10 for $T_s \gg T$. In this temperature regime we may expand the denominator of $\mathrm{Im}\chi^{-+}$ as a power series in ω^2/T_s^2. If we keep only the leading term, we find

$$\Sigma_n[\mathbf{p}, \epsilon(\mathbf{p})] = \frac{3I^2N(0)\gamma p_F^3(K_BT)^2}{16\pi^3(\epsilon+\epsilon_F)\alpha\epsilon_F} \times \left(\frac{1}{\kappa_0^2+\alpha\bar{p}_1^2}-\frac{1}{\kappa_0^2}\right)\int_0^\infty dZ\,Zn(Z), \quad (3.17)$$

where $Z=\omega/K_BT$.

The shift in entropy due to $\Sigma_n(\epsilon)$ is then obtained using (2.22) and expanding $\Sigma_n(\epsilon)$ about $\epsilon=0$. For $\kappa_0^2 \to 0$, this term gives as a dominant contribution

$$\Delta C_n = -[3\pi^4N(0)\bar{I}^2/4\kappa_0^2](T/T_F)^2K_BT. \quad (3.18)$$

Note that we find no $T^3\ln T$ terms and the T^3 term has as its coefficient $1/\kappa_0^2$, in contrast to the leading contribution from Σ_f. As shown in Ref. 2, Σ_f contributes a large mass enhancement, a $T^3\ln T$ term and a T^3 term with coefficient $1/\kappa_0^6$. The ratio of the T^3 term from Σ_n to that from Σ_f is approximately $(T_s/T_f)^2$. Thus the term Σ_n may be neglected for $T \ll T_s$.

We now treat the opposite extreme, $T_s \ll T$. Equation (3.16) is still valid, but now the situation is that shown in Fig. 11. In this situation, $\mathrm{Im}\chi^{-+}$ has all of its significant variation in the frequency region where $n(\omega) \approx K_BT/\omega$. This approximation is adequate only for the resulting singular terms, since it is a high-temperature expansion, $T \gg T_s$. After this replacement for $n(\omega)$, (3.16) becomes

$$\Sigma_n[\mathbf{p}, \epsilon(\mathbf{p})] = -\frac{3I^2N(0)K_BTp_F^3}{16\pi^2(\epsilon+\epsilon_F)}\int_0^{\bar{p}_1} d\bar{q}\,\bar{q}^3\int_{-\infty}^\infty \frac{dy}{\pi} \times \frac{1}{(\kappa_0^2\bar{q}+\alpha\bar{q}^3)^2+y^2}$$
$$= -\frac{3I^2N(0)K_BTp_F^3}{16\pi^2(\epsilon+\epsilon_F)}\int_0^{\bar{p}_1} d\bar{q}\,\frac{\bar{q}^2}{\kappa_0^2+\alpha\bar{q}^2}. \quad (3.19)$$

We perform the momentum integration and neglect a term independent of κ_0^2, since it results in a negligible effective-mass shift. Use of (2.22) now gives the term in the entropy which will lead to a singular specific heat;

$$\Delta S_n = \tfrac{1}{8}(\bar{I}^2K_Bp_F^3)\left(\frac{T}{T_F}\right)^2\frac{\kappa_0}{\alpha^{3/2}}\tan^{-1}\left(\frac{\bar{p}_1\alpha^{1/2}}{\kappa_0}\right)$$
$$\approx \tfrac{1}{16}(\pi\bar{I}^2K_Bp_F^3)(T/T_F)^2(\kappa_0/\alpha^{3/2}). \quad (3.20)$$

Using (3.4) the specific heat can be written as

$$\Delta C_n/C_0 = \tfrac{9}{4}\sqrt{2}\,\pi^2\bar{I}(T_c/T_F)^2[T_c/(T-T_c)]^{1/2}, \quad (3.21)$$

where $C_0=\frac{2}{3}\pi^2N(0)K_B^2T$ is the bare fermion specific heat.

The bosonlike contribution to the entropy (2.22) and (2.28) can be shown to be nonsingular by making use of the same approximation for χ as we used in evaluating Σ. The point we wish to emphasize is that the Σ_n term giving the singular specific heat is actually going to zero as T goes to T_c.

As mentioned before, the singularity in C_v has been obtained previously by Izuyama and Kubo.[16] It is a rather general result of going one approximation stage beyond the molecular-field approximation and is incorrect. Experimentally the specific heat of itinerant magnetic systems are described by a logarithmically divergent temperature dependence.[17] The point we wish to emphasize is that the low-temperature mass enhancement and the singularity in the specific heat at T_c come from different contributions to the entropy. This aspect of the theory may carry over to a more exact theory.

4. SPECIFIC HEAT FAR BELOW THE CURIE TEMPERATURE

In studying the specific heat for temperatures much less than the Curie temperature, we will assume again the weak itinerant-ferromagnetic condition that T_c be much less than the Fermi energy. The Hartree-Fock single-particle energies are given by Eq. (2.7).

In a ferromagnet there are two Fermi surfaces with characteristic momenta, $p_{F\uparrow}$ and $p_{F\downarrow}$. It is convenient to introduce the two alternative parameters η and p_F' defined so that

$$p_{F\uparrow} = p_F'(1-\eta)$$

and

$$p_{F\downarrow} = p_F'(1+\eta).$$

The equilibrium conditions are then

$$p_F' = p_F(1+3\eta^2)^{-1/3}$$

and

$$(1+3\eta^2) = [\tfrac{1}{2}\bar{V}_a(0)]^3(1+\tfrac{1}{3}\eta^2)^3,$$

where p_F and $[\frac{1}{2}\bar{V}_a(0)]$ are the paramagnetic Fermi momentum and dimensionless interaction, respectively. In terms of η the magnetization per particle is

$$\tfrac{1}{2}\mu_B\,\eta(3+\eta^2)/(1+3\eta^2) \approx \tfrac{1}{2}\mu_B\,3\eta.$$

[17] Y. A. Kraftmakher and T. Y. Romashima, Fiz. Tverd. Tela **8**, 1562 (1966) [English transl.: Soviet Phys.—Solid State **7**, 2040 (1966); Y. A. Kraftmakher, *ibid.* **8**, 1306 (1966) [English transl.: *ibid.* **8**, 1048 (1966).

Since $T_c/E_F \approx \eta$, we will be interested in the region in which $T/E_F \ll \eta \ll 1$.

In a ferromagnet, the transverse and longitudinal fluctuations are quite different and will be discussed separately. However, as in Sec. 3, the main contribution to the specific heat comes from the terms in the self-energy involving the Fermi function. Their contribution can be discussed independently of the response function involved. All such terms can be written in the form

$$\Sigma_{f\sigma}(\mathbf{p}, \epsilon) = \sum_{\mathbf{q}} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\mathrm{Im}F(q, \omega+i\delta)}{\epsilon - \epsilon_{\mathbf{p}-\mathbf{q},\sigma'} + \omega} f(\epsilon_{\mathbf{p}-\mathbf{q},\sigma'}), \qquad (4.1)$$

where $F(q, \omega+i\delta)$ is the appropriate t matrix. For example, $F(q, \omega) = [\frac{1}{2}V_a(q)]^2 \chi^{-+}(q, \omega)$ for transverse fluctuations. The σ' spin variable is dependent on which type of self-energy and which response function is under consideration.

Performing the azimuthal angular integration and changing variables in the polar angular variable, we have

$$\Sigma_{f\sigma}(\mathbf{p}, \epsilon) = \frac{m}{\hbar^2 p(2\pi)^2} \int_0^{\infty} q\, dq \int_{\epsilon_{p-q,\sigma'}}^{\epsilon_{p+q,\sigma'}} d\epsilon' \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \times \frac{\mathrm{Im}F(q, \omega+i\delta)}{\epsilon - \epsilon' + \omega} f(\epsilon'). \qquad (4.2)$$

We make the standard approximation of the electron-phonon problem.[18] The Fermi function in the integrand restricts the q integral to the region where $\epsilon_{\mathbf{p}-\mathbf{q},\sigma'} < 0$. Thus the q integral effectively extends from p_l to p_u as illustrated in Fig. 12 for the transverse-response case. p_l is nonzero because of the splitting of the Fermi surfaces by the self-consistent field. In almost all this region $\epsilon_{p-q,\sigma} \ll -\omega_s(q)$ and $\epsilon_{p+q,\sigma} \gg \omega_s(q)$. The main variation of $\Sigma_{f\sigma}$ comes from the region where $|\epsilon'| \lesssim \omega_s(q)$. Therefore, we may integrate by parts with respect to ϵ' and neglect the endpoint contribution. The ϵ' limits on the remaining term may then be set equal to $\pm\infty$ to give

$$\Sigma_{f\sigma}(\mathbf{p}, \epsilon) = \frac{m}{(2\pi)^2 p} \int_{p_l}^{p_u} q\, dq \int_{-\infty}^{\infty} d\epsilon' \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \mathrm{Im}F(q, \omega+i\delta) \times \frac{\partial f(\epsilon')}{\partial \epsilon'} \ln(\epsilon - \epsilon' + \omega), \qquad (4.3)$$

$$\mathrm{Re}\Sigma_{f\sigma}(\mathbf{p}, \epsilon) = -\int_{-\infty}^{\infty} d\epsilon' \frac{\partial f(\epsilon')}{\partial \epsilon'} \Sigma_{f\sigma}^0(\epsilon - \epsilon').$$

In (4.3) we have introduced the zero-temperature limit of the self-energy $\Sigma_{f\sigma}^0$;

$$\Sigma_{f\sigma}^0(\epsilon) = -\frac{m}{(2\pi)^2 p_{F\sigma}} \int_{p_l}^{p_u} dq\, q \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \mathrm{Im}F(q, \omega) \ln(\epsilon + \omega). \qquad (4.4)$$

[18] J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, Inc., New York, 1964), p. 154.

We have set the momentum p equal to the appropriate Fermi momentum $p_{F\sigma}$. This approximation, justified above, is equivalent to ignoring the momentum dependence of the self-energy. Inserting this expression into the formula for the shift in entropy ΔS, we obtain

$$\Delta S = -N_\sigma(0) K_B \int_{-\infty}^{\infty} dx \frac{\partial f(x)}{\partial x} x \int_{-\infty}^{\infty} dx' \times \frac{\partial f(x')}{\partial x'} \Sigma_{f\sigma}^0[K_B T(x-x')], \qquad (4.5)$$

where $N_\sigma(0)$ is the density of states for spin σ at $\epsilon_\sigma = 0$. The resulting shift in specific heat is

$$\Delta C_v = T \frac{\partial \Delta S}{\partial T} = -K_B T N_\sigma(0) \int_{-\infty}^{\infty} dx \frac{\partial f(x)}{\partial x} x \times \int_{-\infty}^{\infty} dx' \frac{\partial f(x')}{\partial x'} (x-x') \Sigma_{f\sigma}^{0\prime}[K_B T(x-x')]. \qquad (4.6)$$

Here

$$\Sigma_{f\sigma}^{0\prime}(\epsilon) = -\frac{m}{\hbar^2 p_{F\sigma}} \int_{p_l}^{p_u} \frac{dq}{(2\pi)^2} qP \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{F(q, \omega+i\delta)}{\epsilon+\omega} = -\frac{m}{\hbar^2 p_{F\sigma}} \int_{p_l}^{p_u} \frac{dq}{(2\pi)^2} q\, \mathrm{Re}F(q, -\epsilon). \qquad (4.7)$$

By changing variables in (4.6), we can perform one of the integrations and obtain the formula

$$\Delta C_v = -K_B^2 N_\sigma(0) T \int_{-\infty}^{\infty} du\, \tfrac{1}{2} u^2 J(u) \Sigma_{f\sigma}^{0\prime}(K_B T u), \qquad (4.8)$$

where

$$J(u) = 2\int_0^{\infty} dx\, f'(x+\tfrac{1}{2}u) f'(x-\tfrac{1}{2}u) = \frac{T^3}{\omega} \frac{\partial^2}{\partial T^2} n(\omega) \bigg|_{\omega/T=u}. \qquad (4.9)$$

From (4.8) we obtain the zero-temperature limit of the specific heat;

$$\Delta C_v / [\pi^2/3 N_\sigma(0) K_B^2 T] = -\Sigma_{f\sigma}^{0\prime}(0). \qquad (4.10)$$

The temperature dependence of the susceptibility has been neglected in calculating the specific heat from the entropy. We use formula (4.8) to obtain a qualitative idea of how the specific heat will vary with temperature. The errors made in deriving this expression for the fermion contribution are of the order $(T/T_F)^2$ but the bosonlike contributions in formulas (2.22) and (2.28) will tend to make C_v/T a less rapidly decreasing function of temperature as in the paramagnetic case.

We now discuss the form of the susceptibilities, ignoring for the present the momentum dependence of the interaction. We introduce the parameters $I = \frac{1}{2}[V_a(0)]$ and $I_s = -2V_s(0)$. For transverse fluctuations,

$$F(q, \omega+i\delta) = I^2 \chi^{-+}(q, \omega+i\delta).$$

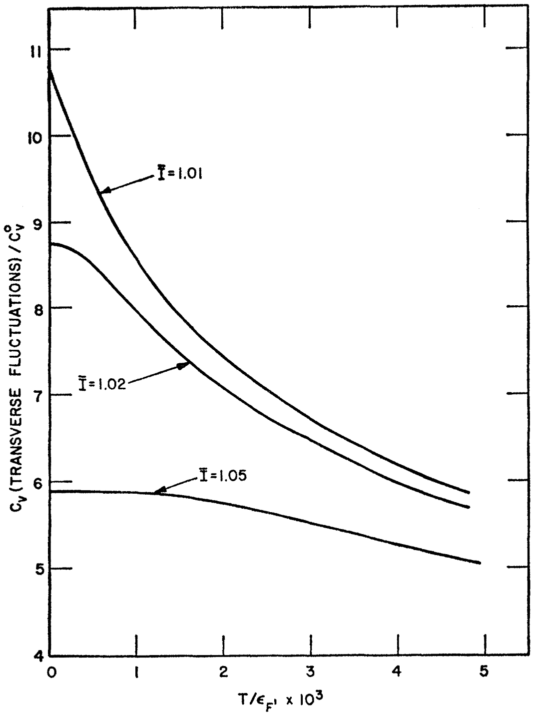

FIG. 13. The transverse fluctuation contribution to C_v/C_v^0 as a function of temperature for three different values of the paramagnetic $\bar{I}$.

The response function $\chi^{-+}(q,\omega)$ has been discussed extensively in the literature.[16,19] It exhibits a real spin-wave pole outside a continuum of single-particle excitations as illustrated in Fig. 12. For the transverse case,

$$p_l = p_{F\downarrow} - p_{F\uparrow} = 2p_F'\eta$$

and

$$p_u = p_{F\downarrow} + p_{F\uparrow} = 2p_F'.$$

The region of integration in $\Sigma_{f\sigma}'$ is restricted to be within the continuum. This region does not go to $q=0$ since it involves the virtual excitation of a particle with spin σ to a state with spin $-\sigma$ with zero energy change. The analytic structure of χ^{-+} is somewhat complicated for both ω and q small. However, we can obtain an expression for the effective-mass enhancement by determining $\chi^{-+}(q,0)$.

Expanding $\chi^0(q,0)$ to second order in q, we obtain

$$\chi^0(q,0) \approx N'(0)\left[(1+\tfrac{1}{3}\eta^2) - \tfrac{1}{12}\bar{q}'^2(1-\tfrac{1}{5}\eta^2)\right]. \quad (4.11)$$

Inserting this into the form of χ^{-+} and using the equilibrium condition gives

$$\chi^{-+}(q,0) \approx N'(0)12(1+\tfrac{1}{3}\eta^2)/(\bar{I}'(1-\tfrac{1}{5}\eta^2)\bar{q}'^2). \quad (4.12)$$

Here $\bar{q}' = q/p_F'$ and $N'(0)$ is the density of states at the energy corresponding to the momentum p_F'. The effective-mass enhancement is then

$$(m^*/m-1)_\sigma = -\Sigma_{f\sigma}^{0\prime} = -6\,\frac{(1+\frac{1}{3}\eta^2)\bar{I}'}{(1-\frac{1}{5}\eta^2)(1-\sigma\eta)}\ln\eta. \quad (4.13)$$

The mass enhancement of the specific heat from the above terms is

$$(m^*/m)_T = 1 - 6\bar{I}'(1+\tfrac{1}{3}\eta^2)/(1-\tfrac{1}{5}\eta^2)\ln\eta. \quad (4.14)$$

If we write η in terms of the paramagnetic $\bar{I}$, we obtain to lowest order

$$\eta^2 = \tfrac{3}{2}(\bar{I}-1)$$

and in the limit as $\eta \to 0$, the singular term in (4.14) reduces to

$$(m^*/m)_T \approx -3\ln|1-\bar{I}|.$$

The calculation of $C_v(T)$ is complicated in this case because it requires a detailed knowledge of the behavior of $\chi^{+-}(q,\omega)$ for $q \approx 2p_F'\eta$ and ω small. The real spin-wave excitations come into the continuum in this region so that the structure is nonanalytic. However, one can surmise what the temperature dependence will be. For most values of q, $\mathrm{Re}\chi^{-+}(q,\omega)$ will be large at $\omega=0$ and drop off symmetrically whenever $|\omega| \simeq \omega_s(q)$. For q near $2\eta k_F$ however, $\mathrm{Re}\chi^{-+}(q,\omega)$ is asymmetric because of the buildup of the real spin-wave mode. In this region, $\mathrm{Re}\chi^{-+}(q,\omega)$ peaks for positive ω at approximately $\omega_s(q)$ while for ω negative $\mathrm{Re}\chi^{-+}(q,\omega)$ decreases rapidly. This is reflected in the self-energy. For positive frequencies $|\Sigma'(\epsilon)|$ increases until ϵ is equal to the frequency at which the real spin waves

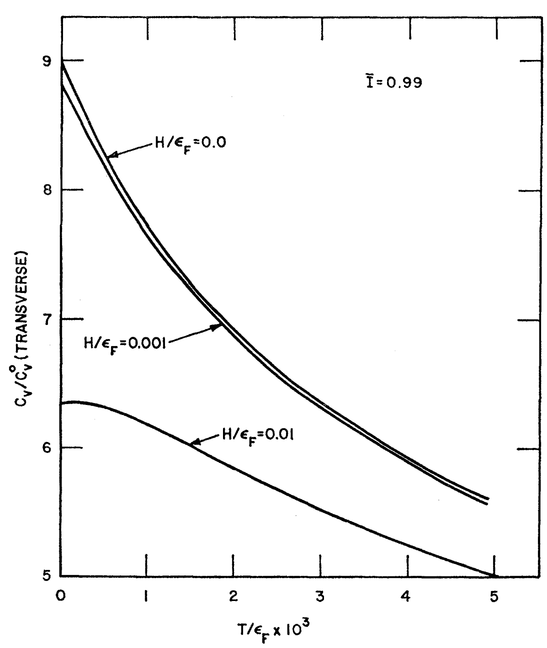

FIG. 14. The effect of an external field on the transverse contribution to the specific-heat enhancement. To see such an effect one needs a material with a low Fermi energy.

[19] T. Izuyama, D. J. Kim, and R. Kubo, J. Phys. Soc. Japan **18**, 1025 (1963).

enter the continuum $\approx(\frac{1}{6}\eta^3)\epsilon_{p'F}$. For negative ϵ, $|\Sigma'(\epsilon)|$ decreases continuously. In C_v/T the increase of Σ' for positive frequencies at first cancels the decrease for negative frequencies. The effect is to give C_v/T a width at $T=0$ of about $\Delta T\simeq 0.2\,\frac{1}{6}\eta^3 T_F$. The 0.2 is due to the fact that $J(u)$ in (4.8) peaks at $u=5$. These properties are borne out by numerical calculations. In Fig. 13 we plot C_v^T/C_v^0 versus temperature for $\bar{I}=1.01$, 1.02, and 1.05. For $\bar{I}=1.01$, $\eta=0.124$, so that on the scale plotted one cannot see a width to the peak at low temperatures. However, for $\bar{I}=1.02$, $\eta=0.178$ and $\frac{1}{6}\eta^3 T_F=0.003T_F$ and we see a width to the peak at $T=0$ of about $\frac{1}{5}$ of this temperature.

Before we go on to the longitudinal spin-fluctuation terms, there are two other contributions from $\Delta\Omega^T$ to the specific heat that should be mentioned. The first is the spin-wave contribution. It is easily seen that the total energy given in Eq. (2.26) is always at least of the order η^3 because of the cutoff in phase space $2\eta p_F'$ where the real spin-wave branch enters the continuum. Thus this term is completely negligible compared to the effective-mass enhancement.

The second term of interest is the contribution of the spin wave-electron interactions coming from Σ_n. Using the same approximation that led to (3.15) and picking up only the spin-wave poles in $\chi^{-+}(q,\omega)$, this term is

$$\Sigma_{n\sigma}(\text{spin waves})\approx I^2\sum_q\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\times\frac{(N_\downarrow-N_\uparrow)\delta[\omega-\omega_s(q)]}{I(N_\downarrow-N_\uparrow)}n(\omega)=\sigma I\sum_q n[\omega_s(q)].$$

By itself this term would give an erroneous $T^{3/2}$ dependence of the self-energy. However it is exactly cancelled by a term coming from the spin-wave renormalization of the Hartree contribution,

$$\tfrac{1}{2}I(N_{-\sigma}-N_\sigma)\equiv\sigma IM(T)=\sigma I\big(M(0)-\sum_q n[\omega_s(q)]\big).$$

This is a different cancellation from that obtained by Izuyama and Kubo,[16] who claim that Σ_n itself starts with $T^{5/2}$. Our result agrees with the decoupling approximations of Cornwall.[20] As far as our calculations are concerned however, these terms are not important, because they give small contributions to the specific heat.

We now turn to the longitudinal spin-fluctuation contribution. In this case, $F(q,\omega)$ is given by Eq. (2.27). The interesting feature is that the spin fluctuations and the density fluctuations do not separate as in the paramagnetic case. The actual enhancement of the effective mass will depend only weakly on this coupling because the low-frequency excitations are still primarily of the spin-fluctuation type. For ω/q small and $q\ll p_F$,

$$\chi_\sigma^0\approx N_\sigma(0)[1-\tfrac{1}{12}\bar{q}'^2+\tfrac{1}{4}(i\pi)\bar{\omega}'/\bar{q}']=N'(0)[(1-\sigma\eta)-\tfrac{1}{12}\bar{q}'^2+\tfrac{1}{4}(i\pi)\bar{\omega}'/\bar{q}'].\quad(4.15)$$

Here $p_{F\sigma}$ is replaced by p_F' in the $\bar{q}^2$ term. This replacement neglects a term of order η^2 in the mass enhancement and leaves unchanged the most singular temperature-dependent terms. Inserting (4.15) into our expression for $\chi_\sigma(\mathbf{q},\omega)$, (2.27), we find that there is a low-lying pole at

$$\bar{\omega}'/\bar{q}'=-\tfrac{1}{4}i\pi(1/\bar{I}')[(1-\bar{I}')+\eta^2\bar{I}'^2/(1+\bar{I}'/\bar{I}_s')+\tfrac{1}{12}\bar{I}'\bar{q}'^2]+O(\eta^4).\quad(4.16)$$

We define

$$\kappa_0'^2=[(1-\bar{I}')+\eta^2\bar{I}'^2/(1+\bar{I}'/\bar{I}_s')].$$

The frequency dependence of the t matrix is essentially described by keeping only the pole contribution of (4.16);

$$t_{\uparrow\uparrow}\approx\frac{\frac{1}{2}I_s}{(1+\bar{I}_s)}+\frac{1}{2}\frac{I}{(\kappa_0'^2+\frac{1}{12}\bar{I}'\bar{q}'^2)+\frac{1}{4}i\pi\bar{I}'\bar{\omega}'/\bar{q}'}-\tfrac{1}{2}I_s-\tfrac{1}{2}I.\quad(4.17)$$

The quantity entering the self-energy is

$$\mathrm{Im}t_{\uparrow\uparrow}\approx\frac{1}{2}\frac{I^2N'(0)\frac{1}{4}\pi\bar{\omega}'/\bar{q}'}{(\kappa_0'^2+\frac{1}{12}\bar{I}'q'^2)^2+(\frac{1}{4}\bar{I}'\pi\bar{\omega}'/\bar{q}')^2}.\quad(4.18)$$

This is exactly the form used by Doniach and Engelsberg[2] in calculating these effects in the paramagnetic case. The specific heat for the present case is simply obtained from their Eq. (13) by using the new value of κ_0^2 and multiplying the total contribution by $\frac{1}{2}$.

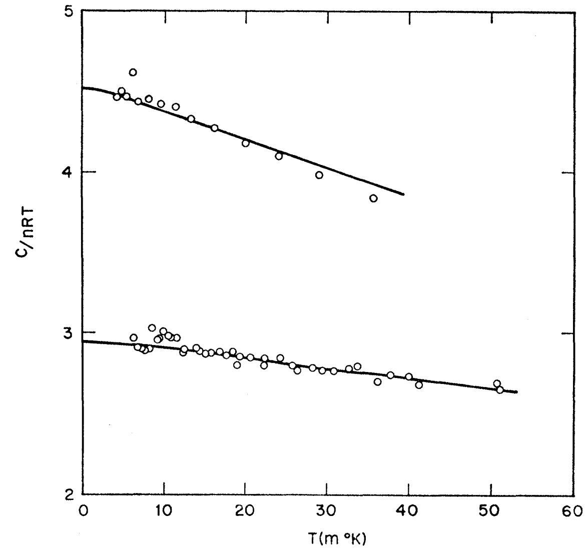

FIG. 15. Temperature dependence of the specific heat from (4.8) for spin fluctuations. The two parameters which determine the curves completely (κ_0^2 and b) were obtained from the measured susceptibility (Ref. 21) and the zero-temperature limit of C/T. Experimental points are those given in Ref. 24 for He^3 under high and low pressures.

[20] J. F. Cornwell, Proc. Roy. Soc. (London) **A284**, 423 (1965).

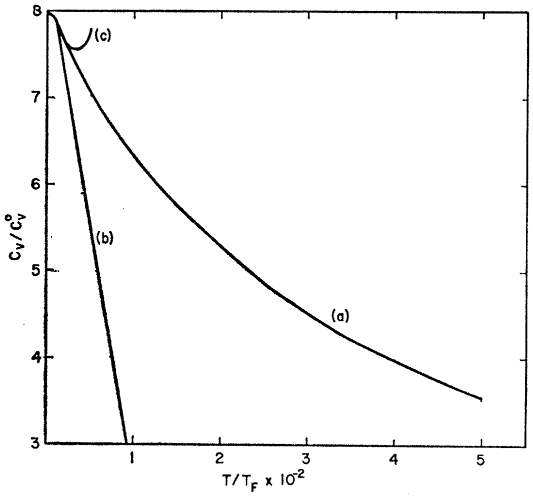

FIG. 16. The temperature dependence of C_v/C_v^0 for $\kappa_0^2=0.05$. (a) Computer calculation, (b) analytic formula Eq. (5.3), and (c) curve obtained by changing coefficient of T^3 to fit the computer results at low temperatures.

If we again write $\eta^2=\frac{3}{2}(\bar{I}-1)$, we find that

$$\kappa_0'^2 \approx \tfrac{1}{2}(\bar{I}-1)[1+3/(1+\bar{I}/\bar{I}_s)].$$

For $\bar{I}-1$ very small, the total effective mass from both types of spin fluctuation is

$$m^*/m \approx -\tfrac{9}{2}\ln|1-\bar{I}|.$$

Thus the singular term in m^*/m is symmetric approaching the ferromagnetic instability from either the paramagnetic or ferromagnetic phase. On the ferromagnetic side, $1-\bar{I}$ is proportional to the zero-temperature magnetization, so that the mass enhancement is proportional to the log of the magnetization. This was previously reported in Ref. 3. In the experiments on Rh-Ni alloys a singularity in the γ value was observed which goes as the log of χ^{-+} for concentrations for which the alloy is paramagnetic. On the ferromagnetic side the behavior is more complicated. This may be due to the fact that the bare density of states is varying rapidly in this region. The above result contains only the singular terms. The nonsingular terms depend on the equilibrium properties determined by the density of states as a function of energy and on the various effective interactions. For example, the longitudinal spin fluctuations will be damped more or less rapidly, depending on their coupling to the density fluctuations, which in turn depend on $V_s(q)$. If $V_s(q)$ goes as $4\pi e^2/q^2$, the spin fluctuations are essentially restricted by the condition that they not be accompanied by density fluctuations. The calculations of the mass enhancement can then be performed using a longitudinal susceptibility restricted by this condition. If the s electrons screen the d-electron interaction, $V_s(q)$ will not diverge. In this case there is the possibility of acoustic plasmons. In the ferromagnet it is possible to couple to these modes with a magnetic field parallel to the magnetization.

The inclusion of the momentum dependence of the interaction reduces the total effective-mass enhancement. Its effect on the temperature dependence of C_v in the paramagnetic phase is discussed in Sec. 5. The effect is similar for the longitudinal fluctuations in the ferromagnetic phase. For the transverse fluctuations, however, the range of the interaction will increase the spin-wave energy for a given value of q so that the width of the peak at $T=0$ of C_v^T/T will extend to somewhat higher temperatures.

The effect of an external field on C_v can be included by shifting the single particle energies by $\frac{1}{2}g\mu_B H$. The new equilibrium condition is

$$4\eta(1-\bar{I})=\bar{H}(1+3\eta^2)+\tfrac{4}{3}\bar{I}\eta^3+4\eta[1-(1+3\eta^2)^{1/3}].$$

In the paramagnetic regime the effective mass is reduced in the presence of a field because the cutoff at long wavelengths $p_l=2\eta_{p'F}$ due to the Zeeman splitting. This cutoff eliminates the very-low-energy spin fluctuations. This is readily included in the program to calculate the transverse contribution to the specific heat. In Fig. 14 we plot this contribution to the specific heat for $\bar{I}=0.99$ and for three values of the field. As can be seen, the field will suppress the mass enhancement if $\bar{H}=g\mu_B H/\epsilon_F$ is of the order of $1-\bar{I}$. This is also true for the longitudinal fluctuations. It would be interesting if one could find a system in which this condition is achieved.

5. EFFECT OF FINITE RANGE, COMPARISON WITH He^3

It has been found experimentally that the RPA model with zero-range interactions overestimates the effective-mass enhancement for a given value of the susceptibility enhancement κ_0^2.[3] It has been suggested that this can be corrected by introducing a finite-range interaction[3,7]

$$\bar{I}(q)=\bar{I}(0)-bq^2. \qquad (5.1)$$

We will consider the effect of a finite range on the temperature dependence of the specific heat given in (4.8) and compare our results with the experimental results for He^3.

First we maintain a δ-function potential to describe the two particle interactions. There are then no undetermined parameters. The susceptibility measurements give us values for κ_0^2 as a function of pressure. According to Wheatley,[21] at a pressure of 0.28 atm, $T_F=5.0$°K and $\kappa_0^2=0.11$ and at high pressures $p=27$ atm, $T_F=6.2$°K and $\kappa_0^2=0.05$. Use of these values gives a mass enhancement which is too large at both pres-

[21] J. C. Wheatley, in *Quantum Fluids*, edited by D. R. Brewer (North-Holland Publishing Co., Amsterdam, 1966), Table III, p. 198 and Table IV, p. 205; $\kappa_0^2=\frac{3}{2}K_BT^*/(p_0^2/2m)$.

sures. We find

$$(m^*/m)_{p=0.28\text{ atm}}=5.1,$$

$$(m^*/m)_{p=27\text{ atm}}=7.9.$$

The values of b were chosen for Fig. 15 by demanding that the experimental values[22] of m^*/m be given completely by the spin-fluctuation contribution. The values, at 0.28 and 27 atm, respectively, are

$$m^*/m=2.98,\qquad b=0.127,$$

$$m^*/m=5.65,\qquad b=0.06.$$

To give some feeling for these values of b, we compare the term in $\bar{q}^2$ which we have added to χ^{-+} to the term which appears in the analytic expansion (3.8), namely, $\alpha\bar{q}^2$.

$$\alpha=\tfrac{1}{12}\bar{I}=0.074,\qquad \text{for } p=0.28\text{ atm}$$

$$=0.079,\qquad \text{for } p=27\text{ atm}.$$

It is important to note that the range b also has an appreciable effect on the temperature dependence of C_v/T. For example, in the high-pressure case, where $\kappa_0^2=0.05$,

$$m^*|_{b=0}/m^*|_{b=0.06}=1.4,$$

$$\frac{[C_v(T_1)/T_1-C_v(T_2)/T_2]|_{b=0}}{[C_v(T_1)/T_1-C_v(T_2)/T_2]|_{b=0.06}}=1.5.$$

We have used the temperature range from $T_1=1\times10^{-3}\epsilon_F$ to $T_2=4\times10^{-3}\epsilon_F$, where C_v/T is a fairly linear function of temperature. The comparison with experiment shown in Fig. 15 confirms the fact that spin fluctuations in He^3 are effective in leading to the observed decrease in C_v/T with increasing temperature. There are several factors which indicate that our model is too simplified to be the whole story. For one, the experimental value of m^*/m is certainly due in part to other mechanisms, e.g., Hartree-Fock corrections as calculated by Brueckner[23] and fermion-phonon interactions. Thus the range used for spin fluctuations will have to be increased further. As indicated above, an increase in the range decreases the falloff in C/T. Furthermore, we have not included the bosonlike contribution to C/T. As was pointed out earlier, explicitly for the $T^3\ln T$ terms, the bosonlike term also decreases the falloff in C/T. Thus the real experimental situation in He^3 appears more complicated than the simple parametrization allows.

The computer calculations have given us an important fact about the low-temperature expansions which include only $T^3\ln T$ terms exactly. We find the low-temperature expansion to be valid only for a very small temperature range: The falloff of $T^3\ln T$ is much more rapid than the correct temperature dependence, especially for very small κ_0^2. In Figs. 16 we try to illustrate this fact by including the computer calculation of C_v/C_v^0 for $\kappa_0^2=0.05$ in the zero-range model [Fig. 16(a)] along with the $T^3\ln T$ and T^3 terms which arise from the analytic approximation to the susceptibility. The analytic approximation to Eq. (13) of Ref. 2 leads to

$$\frac{C_v}{C_v^0}=\frac{m^*}{m}+\frac{3}{2}\left[\tfrac{6}{5}\pi^2\frac{\bar{I}^2}{\kappa_0^2}\left(\frac{T}{\bar{T}_F}\right)^2\left\{\ln\left(\frac{T}{\bar{p}_1\bar{T}_F}\right)+1.78\right\}\right],\qquad(5.2)$$

where

$$\bar{T}_F=(4\kappa_0^2/\pi\bar{I})T_F.$$

We have included the factor $\frac{3}{2}$ to account for longitudinal spin fluctuations. The last T^3 term in the above expression is the result of a numerical integration not included in Ref. 1. We choose the cutoff $\bar{p}_1=1.6$ so that for $\kappa_0^2=0.05$, the mass enhancement given by the analytic form, Eq. (10) of Ref. 1, agrees with the computer calculation, namely, $m^*/m=7.9$. When the values $\kappa_0^2=0.05$, $\bar{p}_1=1.6$ are used, Eq. (5.2) is evaluated to be

$$\frac{C_v}{C_v^0}=7.9-7.2\times10^4\left(\frac{T}{T_F}\right)^2\ln\left|\frac{0.0202}{T/T_F}\right|.\qquad(5.3)$$

This equation is plotted in Fig. 16(b). By changing the scale of temperature in the logarithm, that is, replacing 0.0202 by a parameter λ, we change the T^3 terms of the specific heat. If we try to vary λ in order to force a $T^3\ln T$ fit to the low-temperature part of the calculation in Fig. 16(a) we get the curve shown in Fig. 16(c) corresponding to $\lambda=0.00546$.

These curves show why Doniach and Engelsberg[2] had to apply large renormalizations (roughly 100% at high pressure) to κ_0^2 in order to obtain a reasonable agreement with experiment. The fact that the experiments have not yet reached temperatures where $(T/T_s)^3\ln(T/T_s)$ is a good approximation to the calculated temperature dependence leads one to question attempts at obtaining the observed specific heat from a Fermi-liquid calculation of the coefficients of the $T^3\ln T$ term.[24]

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