# Relaxation of the Superconducting Order Parameter*

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We investigate the nonequilibrium behavior of a pure type-I superconductor and take into account the effects of the electron-phonon interaction. The formulation is based on Kadanoff-Baym transport equations suitably generalized to the case of superconductors. In order to investigate the relaxation behavior of the order parameter due to real phonon transitions, we solve the transport equations in the spatially homogeneous case when there is a slow time variation of the energy-gap function. We solve the transport equations treating the real phonon transitions in the weakest possible way, much in the spirit of the Bardeen-Rickayzen-Tewordt (BRT) transport equation used in the treatment of thermal conductivity. We discuss in detail, however, why the BRT equation fails in this case, and in general when there is a time variation of the order parameter. For weak-coupling materials, we find that for $0.9 \leq T/T_c \leq 0.99$, the gap relaxation rate ($10^8$–$10^9$ $\text{sec}^{-1}$) is about an order of magnitude slower than the quasiparticle decay rate, but that very close to $T_c$ it disappears as $T - T_c$.

## I. INTRODUCTION

RECENTLY there has been some interest in the nonequilibrium behavior of the order parameter or energy-gap function in superconductors[1–7] in conditions of slow time and space variation. In the references cited, the effect of lattice vibrations on the motion of the order parameter has been ignored. For example, in AT[4] it was assumed that the distribution functions of the superconductor have only a gentle variation in time and space about a state of local equilibrium in which the quasiparticle distribution function is of equilibrium form but with the instantaneous local value of the energy gap. For this to be true, it would be necessary for the quasiparticle-phonon interaction to be more rapid than the variation of the gap. In addition, the assumptions of the previous works make it impossible to treat the direct coupling of the gap to the lattice vibrations through processes which are known to be of some importance in transport properties such as the absorption of a phonon accompanied by the creation of two quasiparticles. It has been suggested by Lucas and Stephen[8] that near the transition temperature such processes give the largest contribution to the relaxation of the order parameter.

In this paper, we present a method which extends previous work to include the significant effects of the lattice vibrations on the motion of the order parameter.

We may remark here that the transport equation for the quasiparticle distribution function of Bardeen, Rickayzen, and Tewordt (BRT)[9] has been used to discuss similar problems.[8,10] We shall show that it is not valid for the present problem in which the energy gap is time-dependent and local equilibrium between the quasiparticles and energy gap does not obtain. It is because of this point that our results differ from those of Ref. 8.

We shall use the technique of thermodynamic Green's functions[11] and include the effect of lattice vibrations by means of the standard theory of strong-coupling superconductors.[12] The problem is essentially one of deriving transport equations for the superconductor in the presence of the electron-phonon interaction. There are several ways of doing this. For example, the methods of Refs. 2–4 and 7 consist, in effect, of expanding in wave number and frequency the susceptibilities for the linear response of the system to variations of the energy-gap function and external fields. A second method is that of Kadanoff and Baym[11]; it has been used for superconductors by several authors.[1,13] In this technique, the approximation of slow variation is made at the outset in the original equations of motion for the nonequilibrium Green's functions. We choose the second method here because in the derivation of the transport equations, the collision terms arising from the various scattering mechanisms are always clearly identifiable and separated from the

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[11] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962).
[12] J. R. Schrieffer, D. J. Scalapino, and J. Wilkins, Phys. Rev. Letters **10**, 336 (1963).
[13] James W. F. Woo, thesis, Cornell University, 1966 (unpublished).

driving terms. The latter include, in addition to external fields, the driving effect (on the distribution functions) of the deviations of the quasiparticle self-energies from equilibrium.[14]

The separation of collision and driving terms is useful in the present problem: Instead of carrying along the full machinery of strong-coupling superconductivity theory, we recognize that in the weak-coupling case, the important effects of phonons are contained entirely in the collision terms and all the self-energy (driving) terms may be treated in the usual weak-coupling approximation of Bardeen, Cooper, and Schrieffer[15] (BCS). Thus we shall define the energy-gap function as in the usual weak-coupling theory.[4] The essential feature of this approximation is then that we have a weak-coupling superconductor in which the excitations are independent quasiparticles which are scattered weakly by the lattice vibrations.

We may summarize the method as follows: We begin with Kadanoff-Baym transport equations written for a strong-coupling superconductor[16] and solve them for the motion of the order parameter keeping only the lowest-order electron-phonon terms, namely the electron-phonon scattering. The transport equations describe the evolution of the nonequilibrium state from an equilibrium one in the infinite past due to an adiabatically switched-on perturbation of the self-consistent pair potential or energy-gap function in exactly the same manner as that discussed in Sec. II of AT.[4] We shall restrict ourselves for simplicity to the spatially homogeneous case and shall find, for temperatures which are not too low, that the order parameter approaches equilibrium in a manner characterized by a temperature-dependent relaxation time which becomes infinite only in a very narrow temperature region near $T_c$.

In Sec. II of this paper, we write down the Kandanoff-Baym equations for the thermodynamic nonequilibrium real time Green's functions of the superconductor for slowly varying disturbances. We then show how they simplify for the case of a pure spatially homogeneous weak-coupling superconductor near equilibrium in the presence of weak phonon scattering. In Sec. III, we solve the equations and obtain the relaxation behavior of the order parameter near $T_c$. Section IV is devoted to a discussion of the results and comparison with previous work.

## II. TRANSPORT EQUATIONS

Kandanoff and Baym[11] have derived transport equations for the real time Green's function in a normal metal. Their equations are valid when the perturbations vary slowly with respect to the mean free path and lifetime (or $1/kT$ if it is smaller) of a quasiparticle. In an analogous fashion, we can obtain similar equations for a superconductor.[1,13] Instead of two equations as in a normal metal we now have eight, and the coherence length and $1/kT_c$ are the measure of slowness of space and time variations[4] if they are smaller than the mean free path and lifetime. The transport equations are for the elements of the matrix Green's function $G_{ij}(r_1, t_1, r_2, t_2)$, where

$$G_{ij}(1,2) = -i\langle T\Psi_i(1)\Psi_j^\dagger(2)\rangle, \qquad i,j=1,2$$

and

$$\Psi(1) = \begin{pmatrix} \psi_\uparrow(1) \\ \psi_\downarrow^\dagger(1) \end{pmatrix}$$

is the two-component field operator of Nambu.[17] In what follows the self-energy matrix will be denoted by $\Sigma_{ij}$, and the spectral weight function for $G_{ij}$ by $A_{ij}$. We shall also write $G_\mu$ for the $\tau_\mu$ ($\tau_i$ = Pauli matrices, $i\neq 0$; $\tau_0=1$) component of the matrix $G$. The elements $G_{12}$ and $G_{21}$ will be written as $F$ and $F^\dagger$, respectively. We define $G^>$ and $G^<$ by

$$G_{ij}(12) = G_{ij}^>(12), \qquad t_1 > t_2$$
$$= G_{ij}^<(12), \qquad t_1 < t_2.$$

We define the quantities $\Sigma_\mu$, $A_\mu$, and $\Sigma^{\gtrless}$ in similar fashion. Thus, in the BCS theory, for example, the self-energy is given by

$$\Sigma(1,2) = \begin{pmatrix} 0 & \Delta(1) \\ \Delta^\dagger(1) & 0 \end{pmatrix}\delta(1,2).$$

In the BCS theory we see that $\Sigma^{\gtrless}=0$, which is proper since they are related to scattering effects.

In equilibrium, all quantities are functions of the difference variables 1–2 only. In the following we shall Fourier-transform these; $\mathbf{k}$, $\omega$ are the corresponding wave vector and frequency (measured with respect to the chemical potential). The equilibrium self-energy in the general case of pairing due to virtual phonon exchange is given by

$$\Sigma(z_n) = \sum_{n',\mathbf{q}} D(\mathbf{q}, z_n - z_{n'})\tau_3 G(\mathbf{k}-\mathbf{q}, z_{n'})\tau_3,$$

$$\Sigma(z) = z[1-Z(z)]\tau_0 + Z(z)\Delta(z)\tau_1,$$

where $z_n$ is the imaginary discrete Matsubara frequency, $z$ is the complex frequency which is eventually continued to the real axis $z\to\omega\pm i\delta$, and $Z(z)$, $\Delta(z)$ are complex functions which are determined by the self-consistent Eliashberg equations.[12,18] We have not written the $\tau_3$ component of $\Sigma$ which is interpretable

[14] For an example of how the two approaches treat the same problem, compare Ref. 11, Chap. 11 and J. M. Luttinger and P. Nozières, Phys. Rev. **127**, 1423 (1962); **127**, 1431 (1962).
[15] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957), henceforth referred to as BCS.
[16] These were first set down by J. W. F. Woo, Ref. 13.
[17] Y. Nambu, Phys. Rev. **117**, 648 (1960).
[18] G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **38**, 960 (1960) [English transl.: Soviet Phys.—JETP **9**, 1385 (1959)].

as a renormalization of the chemical potential.[18] It can be shown[18] that the self-energy is, to good approximation, a function of frequency only. We shall assume that this is true also in the nonequilibrium case for slowly varying disturbances.[19] The equilibrium Nambu matrix Green's function in the absence of superfluid flow is then given by

$$G(k,z)=\frac{zZ(z)+\epsilon(k)\tau_3+Z(z)\Delta(z)\tau_1}{z^2Z^2(z)-\epsilon^2(k)-Z^2(z)\Delta^2(z)},$$

where $\epsilon(k)$ is the kinetic energy measured with respect to the chemical potential. We shall write

$$Z(z)=Z_1(z)+iZ_2(z),$$
$$\Delta(z)=\Delta_1(z)+i\Delta_2(z),$$

where all subscripted quantities are real. The quasiparticle lifetime $\Gamma^{-1}$ for excitation frequency $\omega$ is then given by[20]

$$\omega Z_1\Gamma=2Z_2(\omega^2-\Delta_1^2)-2\Delta_1\Delta_2Z_1, \tag{1}$$

where $Z$, $\Delta$ are evaluated at $z=\omega+i\delta$. For weak-coupling materials, it is a good approximation to set $Z_1=1$ and $\Delta_2=0$.

The nonequilibrium matrix Green's function $G_{ij}(1,2)$ satisfies the Nambu[17]-Eliashberg[18] equation of motion. We follow Kadanoff and Baym[11] and transform to center-of-mass (c.m.) variables $(\mathbf{r},t)=\frac{1}{2}(1+2)$ and difference variables $(\mathbf{R},T)=1-2$. We assume a slow variation in the c.m. variables and expand with respect to it keeping only the first-order terms. After a Fourier transformation of the difference variables we find the equation of motion satisfied by $G_{ij}(\mathbf{k},\omega;\mathbf{r},t)$:

$$\begin{aligned}&i\partial(G_0\tau_0+G_1\tau_1+G_2\tau_2+G_3\tau_3)/\partial t-2\epsilon(G_1\tau_2+G_2\tau_1)\\&\quad+(i\mathbf{k}\cdot\nabla r/m)(G_0\tau_3+G_3\tau_0)-\sum_{\mu,\nu}(\tau_\mu\tau_\nu-\tau_\nu\tau_\mu)\\&\times[(\mathrm{Re}\Sigma_\mu)G_\nu^<+\Sigma_\mu^<\mathrm{Re}G_\nu]+i\tfrac{1}{2}\sum_{\mu,\nu}\{([\mathrm{Re}\Sigma_\mu,G_\nu^<]\\&\quad-[\mathrm{Re}G_\mu,\Sigma_\nu^<])(\tau_\mu\tau_\nu+\tau_\nu\tau_\mu)+\tfrac{1}{2}([\Sigma_\mu^>,G_\nu^<]\\&-[\Sigma_\mu^<,G_\nu^>])(\tau_\mu\tau_\nu-\tau_\nu\tau_\mu)\}=\tfrac{1}{2}\sum_{\mu,\nu}(\Sigma_\mu^>G_\nu^<-\Sigma_\mu^<G_\nu^>)\\&\qquad\times(\tau_\mu\tau_\nu+\tau_\nu\tau_\mu),\end{aligned} \tag{2}$$

where the bracket

$$[x,y]=(\partial x/\partial\omega)(\partial y/\partial t)-(\partial x/\partial t)(\partial y/\partial\omega)$$
$$-\nabla_{\mathrm{k}}x\cdot\nabla_{\mathrm{r}}y+\nabla_{\mathrm{r}}x\cdot\nabla_{\mathrm{k}}y$$

and the sums on Greek indices run from 0 to 3. The left-hand side of Eq. (2) contains the driving terms which include the usual BCS terms as well as correction terms arising because the quasiparticles do not propagate as BCS quasiparticles between collisions. These corrections involve either mass renormalization effects ($\mathrm{Re}\Sigma_\mu$) or lifetime effects ($\Sigma^{\gtrless}$). For weak-coupling materials, mass renormalization effects are negligible. In the quasiparticle limit, the lifetime effects are also small; they arise mainly from off-energy-shell contributions as in $[\mathrm{Re}G_\mu,\Sigma_\nu^<]$ or else are of the form $[\Sigma^>,G^<]$ and are smaller than the scattering terms of the right-hand side of Eq. (2) by a factor $\Omega/kT$, where $\Omega$ is a characteristic c.m. frequency. We shall see that near the transition temperature $T_c$, $\Omega/kT_c\ll1$. The scattering terms are on the right-hand side; they lead, in Born approximation, to the usual Golden rule result for transition rates. They are, of course, zero in equilibrium.

We now discuss the approximations we shall make to simplify and solve the transport equations, Eq. (2). We shall apply the transport equations to the case of a pure weak-coupling superconductor with the assumption that the phonons are always in equilibrium. We therefore take $\Delta$ to be frequency-independent and given by the BCS self-consistency condition

$$\Delta=-i\lambda\sum_{\mathrm{k}}\int\frac{d\omega}{2\pi}F^<(\mathbf{k},\omega),$$

where $\lambda$ is the BCS-Gor'kov[21] coupling constant. We also neglect the imaginary part of $\Delta$ which is related to relaxation processes. This term is very small even for strong-coupling materials near $T_c$,[22] and should be completely negligible in the weak-coupling case. In accordance with our remarks following Eq. (2), we make the quasiparticle approximation for the driving terms and retain only the BCS contributions. In fact, we shall go to an extreme quasiparticle limit and evaluate all quantities at $\omega=E_k$, where $E_k$ is the location of the peak of the BCS spectral function for wave number $k$. Thus lifetime effects will appear only in the scattering terms and here, as previously remarked in connection with Eq. (1), we take $Z_1=1$ and $\Delta_2=0$. It is evident that our approach amounts to separating the part of the electron-phonon interaction giving rise to real transitions from that part responsible for superconductive pairing. We treat the latter as in BCS and the former appears only in the collision terms. Thus the effects of the real transition part are treated in the weakest possible way. These approximations lead to the ordinary Boltzmann equation in the case of a normal metal.[11] The approach is in the spirit of BRT[9] in their treatment of thermal conductivity. When local equilibrium obtains, our final equations are equivalent to the BRT transport equations. As we shall explain in Sec. IV, the BRT equations are not valid in the general time-dependent case.

Before reducing the transport equations on the basis of the approximations we have discussed, we mention

[19] The same approximation for the normal state is made by R. E. Prange and L. P. Kadanoff, Phys. Rev. **134**, A566 (1964).
[20] L. Tewordt, Phys. Rev. **128**, 12 (1962).
[21] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958) [English transl.: Soviet Phys.—JETP **7**, 505 (1958)].
[22] J. Swihart, D. J. Scalapino, and Y. Wada, Phys. Rev. Letters **14**, 106 (1965).

several further simplifications. Since we want to isolate that part of the relaxation of the energy-gap function which is caused by real phonon transitions we shall restrict ourselves to the case of spatial homogeneity in the c.m. variable. This eliminates the diffusion behavior previously discussed by AT near $T_c$.[4] We shall neglect scattering-in terms in the calculation of the relaxation and postpone a discussion of their effect to Sec. IV. We shall, furthermore, only consider small deviations of all quantities from thermodynamic equilibrium and we shall linearize the transport equations with respect to these deviations.

We now reduce the transport equations in terms of the following notation: Capital letters henceforth denote equilibrium quantities and lower case letters denote (small) deviations from equilibrium. We shall separate each of the matrices $g^>$ and $g^<$ into two parts. Let

$$g^>=(g^>-g^<)(1-f_0)+g^{>(2)}\equiv g^{>(1)}+g^{>(2)},$$
$$g^<=-(g^>-g^<)f_0+g^{<(2)}\equiv g^{<(1)}+g^{<(2)},$$

where $f_0$ is the Fermi function of $\omega$. It follows then that $g^{>(2)}=g^{<(2)}\equiv g^{(2)}$ which is interpretable as the change in distribution function. Since $G^>-G^<=-iA$, we see that $g^{(1)}$ is related to $a$, the deviation of the matrix spectral function from equilibrium. Thus

$$g^{<(1)}=iaf_0, \qquad g^{>(1)}=-ia(1-f_0).$$

Since $f_0$ is the equilibrium Fermi function, we see that $g^{(1)}$ does not contribute to the linearized scattering terms. It will be recalled that the lower case quantities are functions of $k$, $\omega$; $t$ in the spatially homogeneous case. We Fourier-transform Eq. (2) with respect to the c.m. time $t$, expand to the first order in the c.m. frequency $\Omega$, and equate $-i\Omega$ with $\partial/\partial t$. Finally we only consider changes in the magnitude of $\Delta$. With this notation, and making all the approximations we have discussed, the transport equations, Eq. (2), are reduced to the following set (with the scattering-in terms omitted, cf. Sec. IV):

$$\Omega g_3^{<(1)}+\Delta[f^{<(1)}-f^{\dagger<(1)}]=0,$$
$$\Omega g_3^{(2)}+\Delta[f^{(2)}-f^{\dagger(2)}]=-2iZ_2\omega g_3^{(2)},$$
$$(\Omega-2\epsilon)f^{<(1)}+2G_3^<\delta+2g_3^{<(1)}\Delta+i\Omega\delta f_0\partial A_0/\partial\omega=0,$$
$$(\Omega+2\epsilon)f^{\dagger<(1)}-2G_3^<\delta-2g_3^{<(1)}\Delta+i\Omega\delta f_0\partial A_0/\partial\omega=0,$$
$$(\Omega-2\epsilon)f^{(2)}+2g_3^{(2)}\Delta+i\Omega\delta A_0\partial f_0/\partial\omega=-2i[Z_2\omega f^{(2)}-\Delta Z_2 g_0^{(2)}],$$
$$(\Omega+2\epsilon)f^{\dagger(2)}-2g_3^{(2)}\Delta+i\Omega\delta A_0\partial f_0/\partial\omega=-2i[Z_2\omega f^{\dagger(2)}-\Delta Z_2 g_0^{(2)}],$$
$$\Omega g_0^{(2)}+i\Omega\delta A_1\partial f_0/\partial\omega=-2i[Z_2\omega g_0^{(2)}-\Delta Z_2\tfrac{1}{2}(f^{(2)}+f^{\dagger(2)})]. \quad (3)$$

## III. SOLUTION OF TRANSPORT EQUATIONS

The solution of Eqs. (3) is straightforward and we give only the results for the off-diagonal components of the matrix $g^<$.

$$f^{<(1)}=\{(\Omega-2\epsilon+i\eta)^{-1}[-2G_3^<\epsilon^2/E^2-iA_0'f_0(\Omega-2\Delta^2\epsilon/E^2)]\}\delta,$$
$$f^{(2)}=\{\gamma(D_+-\alpha)/(D_+D_--\alpha^2)\}\delta,$$
$$f^{\dagger(2)}=\{\gamma(D_--\alpha)/(D_+D_--\alpha^2)\}\delta, \quad (4)$$

where

$$D_\pm=\Omega\pm2\epsilon+2iZ_2\omega-2\Delta^2(1-Z_2^2)/(\Omega+2iZ_2\omega),$$
$$\alpha=2\Delta^2(1+Z_2^2)/(\Omega+2iZ_2\omega),$$
$$\gamma=-i\Omega A_0f_0'[\Omega+2iZ_2\omega(1+\Delta^2/\omega^2)]/(\Omega+2iZ_2\omega),$$
$$E^2=\epsilon^2+\Delta^2. \quad (5)$$

In the above, primes denote derivatives with respect to $\omega$, and $\delta$ is the deviation of the gap from equilibrium given by

$$\delta=-i\lambda\Sigma_k\int\frac{d\omega}{2\pi}[f^{<(1)}+f^{(2)}]. \quad (6)$$

We remark that while $\Delta$ and $\delta$ are real, $f^{(2)}\neq f^{\dagger(2)}$ but

$$\Sigma_k\int d\omega\,[f^{(2)}-f^{\dagger(2)}]=0.$$

We shall proceed by discussing Eqs. (4)–(6) in the no-scattering BCS limit $Z_2\to0$ and in the scattering limit $|\Omega|\ll Z_2\Delta$.

### A. BCS Limit $Z_2\to0$

We let $Z_2\to0$ everywhere in Eqs. (4) and (5). We insert the result for Eq. (4) into Eq. (6) and use the BCS expressions for $A_0$ and $G_3^<$:

$$A_0=\pi[\delta(\omega+E_k)+\delta(\omega-E_k)],$$
$$G_3^<=iA_0f_0\epsilon/\omega. \quad (7)$$

After some algebra we find, near $T_c$, an equation of relaxation form

$$M\delta=-iN\Omega\delta+\delta+O(\Omega/kT_c)^2\delta,$$
$$M=-(\lambda N_0\Delta^2/2)\int E^{-3}\tanh(E/2kT_c)\,d\epsilon,$$
$$N=\pi\lambda N_0/8kT_c, \qquad \Omega>\Delta$$
$$=0, \qquad \Omega<\Delta \quad (8)$$

where $N_0$ is the density of states of one spin at the Fermi surface in the normal metal. The relaxation rate of the gap is then given by $M/N$ which is approximately $4\Delta$ near $T_c$. This result will be compared with that of AT [23] in the next section.

[23] Reference 4, Sec. IV A.

### B. Scattering Limit $\Omega \ll Z_2\Delta$

Here we consider the limit of small $\Omega$. When the inequality $\Omega \ll Z_2\Delta$ is satisfied we may expand the right-hand side of Eqs. (4) and (5) to the first power of $\Omega$. We insert the result into Eq. (6) and use Eq. (7). The result has the same relaxation form as Eq. (4). In the present case, however, the second-order term is $O(\Omega/Z_2\Delta)^2$ and $M$, $N$ are given by

$$M=-(\lambda N_0\Delta^2/2)\int E^{-3}[\tanh(E/2kT) - (E/2kT)\,\mathrm{sech}^2(E/2kT)]d\epsilon,$$

$$N=-N_0\int d\epsilon\int\frac{d\omega}{2\pi}f_0' A_0(1+\Delta^2/\omega^2) \times\frac{Z_2\omega+\Delta^2/Z_2\omega}{\epsilon^2+(\omega^2-\Delta^2)\Delta^2/\omega^2+\Gamma^2\omega^2/4(\omega^2-\Delta^2)},$$

where, from Eq. (1), $\Gamma$, the quasiparticle decay rate, is given by

$$\Gamma=2Z_2(\omega^2-\Delta^2)/\omega. \tag{9}$$

The relaxation rate is again given by $M/N$.

We now point out some features of our result. If we increase the amount of scattering, $Z_2$ increases and $N$ decreases so that the relaxation time shortens. When $Z_2\to 0$, $N$ goes to infinity and there is no relaxation. At zero temperature $N$ is zero but as the temperature is lowered we must go to higher order in our expansion in $\Omega$. Near $T_c$, however, the second-order contribution is smaller than the first by $\Omega/Z_2\Delta \ll 1$. The dominant temperature dependence of the relaxation rate is contained in $M$ through the factor $\Delta^2$. It is not hard to show that near $T_c$, $M\simeq -\lambda N_0 7\zeta(3)\Delta^2/(2\pi kT_c)^2$ where $\zeta$ is the Riemann zeta function, while $N$ depends only weakly on temperature. Therefore the relaxation time is proportional to $(T_c-T)^{-1}$ just below the transition point. This result differs from that of Lucas and Stephen[8] who found the behavior $[\ln(1-T/T_c)]^2$. We shall discuss their work in the next section.

We close this section with a numerical estimate of the relaxation time. We have evaluated $M/N$ for tin for the two temperatures $T/T_c=0.99$ and 0.90. It may be shown that $M$ is related to the temperature-dependent penetration depth by $M=-\lambda N_0\Lambda(0)/\Lambda(T)$.[24] Values of $\Lambda(0)/\Lambda(T)$ are available from the review of Bardeen and Schrieffer.[25] We have evaluated $N$ numerically using the jellium model of Tewordt[20] to obtain the quasiparticle damping $\Gamma(\omega)$. Equation (9) relates $\Gamma(\omega)$ to $Z_2(\omega)$.

In order of magnitude, we find for the relaxation rate $\Omega=M/N$:

$$T/T_c=0.99, \quad \Omega\approx 10^8\ \mathrm{sec}^{-1},$$
$$T/T_c=0.90, \quad \Omega\approx 10^9\ \mathrm{sec}^{-1}.$$

The ratio $\Omega/\Gamma(\Delta)$ is of some interest. We find

$$T/T_c=0.99, \quad \Omega/\Gamma\approx 0.08,$$
$$T/T_c=0.90, \quad \Omega/\Gamma\approx 0.6.$$

We discuss the results further in the following section.

## IV. DISCUSSION

### A. Scattering-In Terms

We will argue that the scattering-in terms are unimportant near $T_c$ for phonon scattering. In general, scattering-out terms tell us how to calculate the lifetime of a particle with wave vector $\mathbf{k}$ when all the other particles are in equilibrium. Scattering-in terms tell us how to change the lifetime to take into account the deviations from equilibrium of all the other particles. Thus, if we write the collision term of Eq. (2) in the notation of Sec. II (upper case quantities are equilibrium values and lower case ones are departures therefrom),

$$\Sigma^>G^<-\Sigma^<G^>\to g^{(2)}(\Sigma^>-\Sigma^<)+\sigma^>G^<-\sigma^<G^>, \tag{10}$$

then the first term is scattering-out and the rest is scattering-in. The scattering-in terms may be taken into account by altering the density of final states in the calculation of the lifetime. If the change in distribution function is such that the density of final states is appreciably altered then scattering-in terms will be important. Thus for electrical conductivity, the displacement of the distribution in the field direction appreciably alters the density of states, the small-angle scattering is inhibited while the large-angle scattering is enhanced. In this case the scattering-in terms must be included in order to obtain the correct transport lifetime. For thermal conductivity, on the other hand, neither small- nor large-scale scattering is significantly changed so that the scattering-in terms are unimportant.

Let us consider the present problem near $T_c$. We pointed out in Sec. II that only $g^{(2)}$ enters into the scattering terms. It is therefore unnecessary to consider $g^{(1)}$ in this discussion. We need to examine the scattering-in terms of Eq. (10), $\sigma^>G^<-\sigma^<G^>$. The nonequilibrium part of the self-energy matrix $\sigma^{\gtrless}$ is given by integrals over the kinetic energy and frequency of $g^{(2)}$ (the change in distribution function) multiplied by an equilibrium phonon propagator.[20] Since the transport equations (3) decoupled to order $\Delta^2$ near $T_c$, we need only examine the off-diagonal part of the matrix $\sigma^{\gtrless}$ which depends on $f^{(2)}$ if we want to investigate the scattering-in contribution to the motion of the energy-gap function. We shall use the

[24] N. R. Werthamer, Phys. Rev. **132**, 663 (1963).

[25] J. Bardeen and J. R. Schrieffer, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Co., Amsterdam, 1961), Vol. III.

result for $f^{(2)}$ in the absence of scattering-in terms to give an iterative discussion. From Eqs. (4) and (5), we find that near $T_c$, $f^{(2)}$ is given by

$$f^{(2)}=-(\pi/kT)\delta(\omega-E)\left\{\operatorname{sech}^2(\omega/2kT)\times\left[\frac{1}{\Gamma}\left(\frac{\Gamma^2/4+\Delta^2\xi^2}{\xi^2(\omega^2+\Delta^2)+\Gamma^2/4}\right)\right](1+\Delta^2/\omega^2)+Q(\omega,\epsilon)\right\}\Omega\delta, \tag{11}$$

where $\xi=(1-\Delta^2/\omega^2)$. The term $Q(\omega, \epsilon)$ is odd in $\epsilon$ and even in $\omega$ so it makes no contribution to $\sigma^{\gtrless}$ after the integral over $\epsilon$. The remaining part of $f^{(2)}$ contains the product of two peaked functions of $\omega$, the first is of width $\approx kT_c$, while the other in square brackets is narrower and becomes more so as $T_c$ is approached. For example, at $T/T_c=0.99$ we find for tin that $f^{(2)}(kT_c)/f^{(2)}(\Delta)\sim 1/40$. Now, the important frequencies which appear in the integrand for $\sigma^{\gtrless}$ are of order $kT_c$ since this is the typical phonon frequency. Thus phonon emission or absorption moves a quasiparticle away from the Fermi surface by about $kT_c$. We see that this state lies at a frequency for which the change in distribution function $f^{(2)}$ is very small. Then $\sigma^{\gtrless}$ is small and we may safely neglect the scattering-in terms. That is, the final states are not appreciably altered by the departure from equilibrium represented by $f^{(2)}$. [The argument is not so good for $T/T_c=0.9$, where $\Delta/kT_c$ is of order 1.]

For impurity scattering, the situation is quite different. Since the scattering is elastic, all scatterings are inhibited and scattering-in terms become quite important. In fact, as was shown in AT,[26] the scattering-in terms just cancel the scattering-out terms for the impurity scattering case and there is no relaxation in the absence of spatial inhomogeneity.

Below $T_c$, the situation is much more complicated since all the equations are coupled. If we look at the change in distribution function $g^{(2)}(\omega)$ we can see that besides the peaked function at $\omega=0$, there is a part coming from the $\tau_0$ component of $g^{(2)}$ which is odd in $\omega$ and of width $kT$. This is also a description of the distribution function in the problem of thermal conductivity where scattering-in terms may be neglected. We therefore suggest that they are negligible for the present problem as well, although we have not investigated this point in detail for general temperatures.

### B. Comparison with BRT

In 1959, BRT[9] gave a transport equation for a weak-coupling superconductor. It has been used recently by Lucas and Stephen[8] to discuss the relaxation of the order parameter. We shall comment on its applicability to this case.

[26] Reference 4, Sec. VI. In this formulation, the scattering-in terms appear in the vertex corrections.

The BRT equation is a generalized Boltzmann equation for the quasiparticle distribution function. The approximations concerning the driving terms are the same as those made in the present paper. There is an important difference, however, in that the BRT equation concerns only the quasiparticles and the superfluid is assumed to be in local equilibrium with the lattice vibrations so that its transport equation is not needed. That is to say, the energy gap is taken to be constant at its equilibrium value for the local temperature. To obtain the scattering term of BRT it is necessary to assume that the spectral function is the local equilibrium one and furthermore that the change in distribution function is the same for all components of the Green's function.[13]

From the solution we have derived in Sec. III, it is clear that even the assumption of local equilibrium of the condensate is invalid for the present problem. The different matrix components of the spectral function are not related as they would be if local equilibrium obtained. Furthermore, the distribution function is different for each component of the matrix Green's function. We conclude that the BRT equation cannot be applied to this problem. Its validity will be doubtful whenever the characteristic time in a problem becomes short enough to be comparable to the relaxation time we have calculated in Sec. III.

### C. Comparison with AT

In the work of AT[4] it was assumed that the quasiparticles maintained a state of local equilibrium with respect to the instantaneous energy gap due to the fast quasiparticle-phonon relaxation rate $\Gamma$. It is of interest to discuss the extent to which this is true. The results presented at the end of Sec. III indicate that the gap relaxes more slowly than the quasiparticle lifetime which would make it appear that the quasiparticles can achieve local equilibrium at a rate faster than that of the relaxation of the gap. However, the discussion of the scattering-in terms in Sec. IV A shows that in a narrow region near the gap edge, the quasiparticle distribution function is appreciably different from equilibrium. An examination of the terms entering the quasiparticle decay rate[20] shows that the contribution of terms involving another quasiparticle and the condensate (recombination and excitation of pairs) give contributions just as large as single quasiparticle scattering from phonons. Therefore if none of the components of the matrix distribution function $g^{(2)}$ have the equilibrium form near the gap edge, we do not expect that the fast recombination-excitation processes will drive the quasiparticles to local equilibrium but rather to a distribution determined by the details of $g^{(2)}$ itself. We conclude that near $T_c$, the assumption of AT concerning local equilibrium of the quasiparticles is not very good near the gap edge and probably is not good anywhere at lower temperatures. We may remark, however, that if one is satisfied with

small deviations from thermodynamic equilibrium then the local equilibrium assumption of AT is unnecessary.

We now compare our result for the BCS limit Eqs. (7) and (8) with the result of AT.[27] It is stated there that AT Eq. (4.6) is valid when $\Omega > 2\Delta$ but this is incorrect. An additional requirement is $v_F q > \Omega$, where $v_F$ is the Fermi velocity and $1/q$ is the characteristic length of the spatial variation of the order parameter. Our result for the BCS limit is for the case $\Omega > 2\Delta$, $q=0$. The expression of AT must be modified in this case to include a term of order $\Delta^2$ which they have dropped in their kernel $L(0, 0)$. If this correction is made, the results agree. The value for the relaxation rate of AT as quoted by Lucas and Stephen[28] is therefore incorrect and should be replaced by the expression $1/\tau_a = 4\Delta$.

We conclude by commenting on the result of AT in the diffusion regime $v_F q > \Delta > \Omega$. If we linearize their Eq. (4.6) about equilibrium near $T_c$ we find

$$-\lambda \delta\Delta = (\partial/\partial t - D\nabla^2)\Delta, \qquad (12)$$

where

$$\lambda = 14\zeta(3)\Delta^2/\pi^3 k T_c, \qquad D = 7\zeta(3) v_F^2/6\pi^3 k T_c.$$

Since $\lambda/Dq^2 = 12(\Delta/v_F q)^2$, we may neglect the left-hand side of Eq. (12). We then have a simple diffusion equation with a constant diffusion coefficient and a diffusion rate $\Omega = Dq^2$. The range of validity is then $kT_c > v_F q > \Delta > Dq^2$. There is no anomalous behavior of $D$ as $T \to T_c$ but it should be pointed out that the maximum value of $q$ for which the diffusion equation is valid decreases to zero as the critical point is approached.

[27] Reference 4, Eq. (4.6).
[28] Reference 8, Eq. (38).

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# Cylindrically Symmetric Solutions of the Ginzburg-Landau Equations*

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The Ginzburg-Landau questions have been solved for an isolated magnetic flux line enclosing a single flux quantum. The radial dependence of the magnetic field $H$, the order parameter $n_s$, the current density $J$, and the resulting free energy per unit length $F/L$, are obtained for values of the Ginzberg-Landau parameter $\kappa = 20$, 5, 1.0, 0.5, and 0.2. For $\kappa \lesssim 5$, the axial magnetic field $H(0)$ is approximately given by $H(0) = 0.62\kappa^{-0.43}\sqrt{2}H_{cb}$. The maximum value of the current density is approximately $J = 0.33\sqrt{2}H_{cb}/\lambda$, where $H_{cb}$ is the bulk critical field and $\lambda$ is the superconducting penetration depth.

## I. INTRODUCTION

THE quantized flux tube is central to the present understanding of the behavior of type-II superconductors.[1–5] The structure of the flux tube was first treated by Abrikosov.[6] He set up the Ginsburg-Landau[7] (GL) equations for the cylindrically symmetric case, which corresponds to an isolated flux tube. Analytic solutions were then obtained for values of the GL parameter $\kappa \gg 1$. If this condition is not met, numerical analysis is required. Computer solutions have been obtained by Fink and Presson[8] and by Doll and Graf[9] for a fluxoid in a material for which $\kappa \lesssim 1$. However, these solutions were for a fluxoid inside a wire of finite diameter and therefore represent a different set of boundary conditions than that for an isolated flux tube.

We are presently investigating the behavior of thin, type-I, superconducting films in the presence of a perpendicular magnetic field. In this situation a flux tube structure is also exhibited.[10,11] We have investigated the behavior of the solutions of Abrikosov's equations for the isolated flux tube for several values of $\kappa$ in the range from 0.2 to 20. We are primarily interested in the radial behavior of the order parameter and of the magnetic field for values of $\kappa < 1/\sqrt{2}$. However, we have also determined the current density distribution and the free energy per unit length of the isolated flux tube. Harden

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