coordinates, the result is

$$\frac{\partial g(\eta,z)}{\partial z}=-g(\eta,z)N\int_0^\infty \sigma(\chi)\chi d\chi+N\int_0^\infty dy\, g(y)\times\int_0^\infty \chi d\chi\, \sigma(\chi)J_0(\eta,\chi)\int_0^\infty y\theta J_0(y\theta)J_0(\theta\eta)d\theta. \quad \text{(A12)}$$

However, by Eq. (A7), the last integral in the second term is just $\delta(y-\eta)$; thus

$$\frac{\partial g(\eta,z)}{\partial z}=g(\eta,z)N\int \sigma(\chi)\chi d\chi\,[J_0(\eta\chi)-1], \quad \text{(A13)}$$

which is Bethe's Eq. (4).

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Theory of Longitudinal Plasma Instabilities

AKIRA HASEGAWA
Faculty of Engineering Science, Osaka University, Toyonaka, Osaka, Japan
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A stability theory of longitudinal disturbances in a plasma is presented. The theory states necessary conditions and a necessary and sufficient condition of the instability in terms of the mathematical nature of linear plasma conductivities on the real ω axis, by introducing an idea of passive and active conductivities combined with Nyquist diagram techniques. It is applied to obtain conditions of instability in both collision-dominated and collision-free plasmas in the presence of a uniform magnetic field. Effects of drift, density gradient, and temperature anisotropy appear as the causes of the active conductivities, whose presence is shown to be the basic necessary condition for an instability. Each of these effects contributes differently to the conductivity, giving a different necessary and sufficient condition for the associated instability.

I. INTRODUCTION

THE study of longitudinal plasma instabilities may be dated from Hahn's space-charge wave theory[1] in 1939. Since then, the development of traveling-wave-type amplifiers[2] and, more recently, research on nuclear fusion, have led to almost innumerable examples of such instabilities.

Although there are many instabilities, the basic causes are limited by the way in which a plasma can depart from thermal equilibrium; there may be drift, density, or temperature gradients, and temperature anisotropy. In an actual plasma, these not only may be coupled with each other, but may be coupled with many different kinds of stable plasma waves, and hence may lead to many different instabilities.

Ordinarily, an instability can be found from the zeros of the dispersion relation $D(\omega,\mathbf{k})=0$ in the complex ω plane. For a phasor of form $\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$, each zero in the $\mathrm{Im}(\omega)>0$ plane corresponds to an unstable wave. However, this method requires solution of the dispersion relation for each different case and sometimes obscures the physical origin of the instability. One alternative to this method is the use of energy principles. Since the causes of instabilities are limited, these methods may have more general meaning and, in addition, may be useful in understanding the physical mechanism. For example, the small signal-power theorem[3] and the principle of the negative-energy wave,[4] which is an alternative expression of the power theorem in $(\omega,\mathbf{k})$ space, are powerful methods for finding the necessary conditions for instabilities in a dissipation-free system. The variational principle,[5] which examines the change of potential energy, is also helpful in macroscopic instabilities, where conversion of the potential energy to kinetic energy is the essential cause. More recently, Hall and Heckrotte[6] have proposed to examine the sign of the imaginary part of the dielectric constant to obtain the necessary condition of instability in a plasma with anisotropic temperature.

These methods, although they have individual merits, tend to lack general applicability. In particular, the fact that the energy methods can present only the necessary condition for instability (or stability) is often unsatisfactory. In this paper, after noting that the dispersion relation of a longitudinal wave resembles the impedance function of an ordinary electrical circuit, we shall apply the well-established theorems in that field to our models. The reasoning for this possibility is based on the fact that even in a system with retardation or propagation, i.e., a system with a nonzero value of wave number $\mathbf{k}$, the existence of a zero of the dispersion relation in the $\mathrm{Im}(\omega)>0$ plane is necessary and sufficient for an in-

[1] W. C. Hahn, Gen. Elec. Rev. **48**, 258 (1939).
[2] J. R. Pierce, *Traveling Wave Tubes* (D. Van Nostrand, Inc., New York, 1950).
[3] L. J. Chu, in Proceedings of the IRE Conference on Electron Devices, Durham, New Hampshire, 1951 (unpublished).
[4] A. Bers and S. Gruber, Appl. Phys. Letters **6**, 27 (1965).
[5] I. Bernstein, E. Frieman, M. Kruskal, and R. Kulsrud, Proc. Roy. Soc. (London) **A244**, 17 (1958).
[6] L. S. Hall and W. Heckrotte, Phys. Rev. **134**, A1474 (1964).

stability.[7] We can treat $\mathbf{k}$ in the dispersion relation as merely a parameter.

We start by defining an active plasma conductivity on an energy basis in Sec. II, and show that its presence is the basic necessary condition of an instability by applying the classic theorems of electrical-circuit theory. The activeness of an equivalent conductivity can be stated purely in mathematical form, which can be reduced either to the form obtained by Hall and Heckrotte for a dissipative (nonconservative) system or, by the use of Foster's reactance theorem, to that of Bers[4] for a dissipation-free (conservative) system. The theorem is developed further to obtain a necessary and sufficient condition for instability by the use of a Nyquist-type criterion.[8] This condition can be stated in a form that requires only the solution for the real roots of an equation of much lower order than that of the original dispersion relation. These conditions give a fairly clear picture of the mathematical process of creation of unstable roots, which can be related to the physical causes of instability. For example, in a system of drifting electrons and stationary ions, we can show that the electrons contribute to the active conductivity, and we can obtain the frequency range over which the conductivity becomes active. Then the mathematical nature of the active conductivity automatically gives the mathematical condition on the conductivity of the ions that can lead to an instability. This condition can be related to physical behavior, such as ion-collision rate or temperature, etc.

In Sec. III, we give an application of the theory to both collision-dominated and collision-free plasmas in the presence of drift, density gradient, temperature anisotropy, and uniform magnetic field. We omit temperature gradients because they are not well defined physically. We show the cause of the active conductivities and their mathematical nature, and predict, or in some cases derive, the instability conditions. The purpose in Sec. III is not to derive all possible instability conditions, but rather to point out types of active sources. Once these types become clear, it is then sufficient to know which stable longitudinal waves are coupled in to predict instabilities.

Although we treat only instabilities of longitudinal disturbances, the method may be applied to those of more general disturbances, where the conductivity becomes a tensor.

II. STABILITY THEORY

We use the equivalent plasma conductivity given as a function of the angular frequency ω, the wave vector $\mathbf{k}$, and the various plasma parameters. We shall first define an active conductivity in terms of its behavior on the real ω axis. The presence of an active conductivity is essential for an instability; hence this definition is quite useful in finding the conditions on parameters in a plasma that lead to an instability. The second step is to obtain a necessary and sufficient condition by using the known nature of the active conductivity relative to a passive conductivity in the plasma. A Nyquist-type criterion is used to obtain this condition. Using the active and passive conductivities so found, the condition is stated in a unique form, which can also be written in several different forms if we use the arbitrariness of the choice of the active and passive conductivities in a given system.

We consider only longitudinal disturbances propagating in the x direction. The dispersion relation may be obtained in the following form:

$$-i\omega\epsilon_0+\sum_j \sigma_j(\omega,\mathbf{k})=0\,, \qquad (2.1)$$

where σ_j is the conductivity of the jth species and a phasor of form $\exp i(k\cdot x-\omega t)$ is used. Even for a group of electrons (or ions), we take the drifting part, for example, as different from the nondrifting part as long as it is possible to separate these. Thus the species type j is also based on its physical state. We treat $\mathbf{k}$ as a real constant or parameter, and therefore consider σ_j only a function of ω. We assume nonzero collision or recombination dissipation, which can sometimes be vanishingly small, in all the species, so that neither the poles nor the zeros of $\sigma_j(\omega)$ lie exactly on the real ω axis. Furthermore, we assume that none of the $\sigma_j(\omega)$ has poles in the $\mathrm{Im}(\omega)>0$ plane. Thus a system with any one of the σ_j is always stable when it is short-circuited, which is true in most cases in plasma conductivities.

The conductivity shown in Eq. (2.1) may be obtained by transforming the ordinary conductivity tensor $\boldsymbol{\sigma}^0$, obtained in coordinates with one of its axes directed along the magnetic field, to coordinates with one of its axes directed along the propagation vector $\mathbf{k}$, by

$$\sigma=[(\mathbf{T}\boldsymbol{\sigma}^0\mathbf{k})\cdot(\mathbf{T}\mathbf{k})]/k^2\,, \qquad (2.2)$$

where $\mathbf{T}$ is the transforming matrix. Alternatively, the conductivity σ may be obtained from the ratio of perturbed density n_1 and potential ϕ_1 as

$$\sigma=i\omega e n_1/k^2\phi_1\,. \qquad (2.3)$$

We now introduce the idea of passive and active conductivities as used in electrical-circuit theory.[9] These are convenient to use because they lead immediately to a necessary condition for instability. *A passive conductivity* $\sigma_p(\omega)$ has

$$\mathrm{Re}[\sigma_p(\omega)]\geq 0\,,\quad \mathrm{Im}(\omega)\geq 0\,. \qquad (2.4)$$

It can be shown that if a conductivity satisfies the above condition, it can be constructed from a combination of passive elements, i.e., resistors, capacitors, and induc-

[7] P. A. Sturrock, Phys. Rev. **112**, 1488 (1958).

[8] See, for example, J. D. Jackson, J. Nucl. Energy **1**, 171 (1960).

[9] Concerning theorems used in relation to circuit theory, see, for example, W. Cauer, *Synthesis of Linear Communication Networks* (McGraw-Hill Book Co., New York, 1958).

tances, which cannot provide any energy to an external element. Furthermore, it can be shown that such a conductivity possesses neither a zero nor a pole in the $\mathrm{Im}(\omega)>0$ plane. Thus a system with only passive conductivities is always stable.[9] *An active conductivity* $\sigma_a(\omega)$ has

$$\mathrm{Re}[\sigma_a(\omega)]<0 \text{ for at least some region of } \mathrm{Im}(\omega)\geq 0. \tag{2.5}$$

If $\sigma_a(\omega)$ is regular in the $\mathrm{Im}(\omega)>0$ plane as assumed, and also because $\sigma(\omega)$ will become the conductivity in free space as $\omega\rightarrow\infty$, the definitions in Eqs. (2.4) and (2.5) may be written as follows: A conductivity is *passive* if and only if for all real ω,

$$\mathrm{Re}[\sigma(\omega)]\geq 0. \tag{2.4$'$}$$

A conductivity is *active* if and only if for at least some real ω,

$$\mathrm{Re}[\sigma(\omega)]<0. \tag{2.5$'$}$$

If a conductivity is active, it can be shown that energy can be delivered to an external element; or, an active conductivity implies a source of free energy. The presence of an active conductivity presents the possibility of instability. Thus we have the following theorem:

Theorem 1. A plasma is always stable when all of its conductivities σ_j as shown in Eq. (2.1) are passive.

A plasma can be unstable only when at least one of its σ_j is active, which is the basic necessary condition for an instability. When $\mathrm{Re}(\sigma_a)$ is negligibly small, that is, when the system is *conservative*, this condition can be shown to be the same as that of the *negative energy* derived by Bers[10] by the use of Foster's reactance theorem.[9] Namely, for a *conservative* system, an active conductivity has

$$\partial\, \mathrm{Im}[\sigma(\omega)]/\partial\omega>0 \text{ for at least some real } \omega. \tag{2.5$''$}$$

The active conductivity in a conservative system as defined by Eq. (2.5″) is *reactively active.* On the other hand, a conductivity that is active in a nonconservative system is *dissipatively active.* The physical meaning of these two active conductivities will become clear when we consider the equation of energy conservation for linearized quantities in a system with only one of these active conductivities. Such an equation, after integration over a unit volume taking into account our assumption of real wave number k, can be written in the following form:

$$\frac{\partial}{\partial t}\int \mathcal{W} dV=\int \mathcal{S} dV. \tag{2.6}$$

For a conservative system, the source term $\mathcal{S}$ in Eq. (2.6) is zero, and hence the conductivity in such a system can become active only when the energy $\mathcal{W}$ itself is negative. An example of such a case was derived by Chu[3] for an electron beam. On the other hand, the conductivity of a system can become active even when the energy $\mathcal{W}$ is positive, if (and only if) there exists a source term $\mathcal{S}$ which is positive. Such a case we call "dissipatively active." An example is a collision-dominated drifting beam with drift velocity smaller than the thermal velocity. The reactively active conductivity corresponds to a system with a negative-energy wave. For such a system dissipation will contribute to a positive-source term, but, because the wave energy itself is negative, the positive-source term contributes damping to the wave, while the same positive source would cause a positive-energy wave to grow.

The first step to be taken in studying the stability of a plasma is to find an active conductivity and the frequency at which it becomes active. The existence of an active conductivity can be found physically by considering the way in which a plasma is away from its thermal equilibrium, such as those associated with a drifting part, a nonuniform part, etc. One of the merits of doing this is that we can find a parameter (or parameters) in a plasma that causes the instability. Once this is done, we can write the dispersion relation (2.1) as

$$W_0\equiv\sigma_a(\omega)+\sigma_p(\omega)=0. \tag{2.7}$$

Note that the space conductivity $-i\omega\epsilon_0$ is included in $\sigma_p(\omega)$ in Eq. (2.7), following the definition of Eq. (2.5″). Mathematically, to escape from unnecessary complications, ϵ_0 should be assumed to possess a small positive imaginary part.

Now, it is well known that the simple presence of an active element does not necessarily give rise to an instability when the whole system is considered. Instability occurs only when a suitable feedback is provided by the passive part of the conductivity, σ_p. Depending on the type of $\sigma_a(\omega)$, the $\sigma_p(\omega)$ that leads to an instability may either be inductive, resistive, or capacitive, or a combination of these. In many cases, σ_p should either be inductive or resistive for an instability, but there are cases where the system is unstable when σ_p is capacitive. Instability occurs when and only when at least one zero of W_0 in Eq. (2.7) exists in the $\mathrm{Im}(\omega)>0$ plane. The number of zeros in the $\mathrm{Im}(\omega)>0$ plane may be found by the Nyquist theorem. The theorem states that when ω moves on the boundary of the $\mathrm{Im}(\omega)>0$ plane as shown by c in Fig. 1, the number of times that

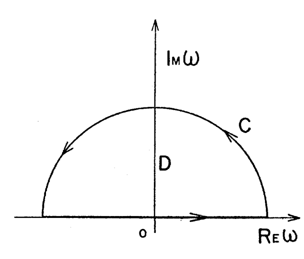

FIG. 1. Contour c, showing the boundary of the $\mathrm{Im}(\omega)>0$ plane.

[10] Actually, Eq. (2.5″) is not exactly the form derived by Bers, who included the space conductivity. If it is included, Eq. (2.5″) gives $\partial\, \mathrm{Im}\sigma/\partial\omega-\epsilon_0>0$, reducing to his result.

the locus in the W_0 plane encircles the origin gives the number of zeros in the $\mathrm{Im}(\omega)>0$ plane, provided that W_0 does not possess any poles in the $\mathrm{Im}(\omega)>0$ plane. The proof can be easily made by considering the Cauchy integral of $1/W_0$ along the contour in the W_0 plane corresponding to the locus c on the ω plane. The Nyquist method has the merit that the stability of a system can be found simply by an explicit process, i.e., there is no need for solving for roots of a complex function, which is an implicit process. However, it fails to give the relation of the parameters to the stability of the system.

If we combine an implicit process, we can use the Nyquist theorem in a somewhat more useful way. For example, for the locus to encircle the origin in the W_0 plane, it should cross a radial axis in that plane at least once; by solving for such a condition, we may be able to obtain the stability condition without an actual plot of the locus. Now, what radial axis will be most suitable for this reference axis? In view of the fact that the locus can enter into the $\mathrm{Re}(W_0)<0$ area only in the presence of an active conductivity, some axis existing in that plane may be suitable. If we choose negative $\mathrm{Re}(W_0)$ axis as the reference, the condition that the locus crosses the axis may be stated as at ω on c, such that $\mathrm{Im}(W_0)=0$, $\mathrm{Re}(W_0)<0$. But, why not choose some other axis? We shall show that a suitable axis may be chosen if we take advantage of the presence of the passive conductivity σ_p.

Since we know that the passive conductivity σ_p does not possess any zero in the $\mathrm{Im}(\omega)>0$ plane, we can apply the Nyquist criterion to a new function W defined as

$$W(\omega)\equiv\frac{W_0(\omega)}{\sigma_p(\omega)}=1+\frac{\sigma_a(\omega)}{\sigma_p(\omega)}\equiv U+iV, \qquad (2.8)$$

and the number of zeros in the $\mathrm{Im}(\omega)>0$ plane is given by

$$I=\frac{1}{2\pi i}\int_c \frac{d}{d\omega}\left(\frac{\sigma_a}{\sigma_p}\right)\left(1+\frac{\sigma_a}{\sigma_p}\right)^{-1}d\omega$$

$$=\frac{1}{2\pi i}\int_{c_W}\frac{dW}{W}, \qquad (2.9)$$

where c_W is the locus of c on the W plane. Thus the same criterion used in W_0 applies to the new function W, and the number of times that the locus c_W encircles the origin gives the number of zeros in the $\mathrm{Im}(\omega)>0$ plane.

As can be seen from Eq. (2.8), the arcs of W and W_0 are related through arc (σ_p) as

$$\mathrm{arc}\ (W)=\mathrm{arc}\ (W_0)-\mathrm{arc}\ (\sigma_p), \qquad (2.10)$$

where

$$|\mathrm{arc}\ (\sigma_p)|<\tfrac{1}{2}\pi;$$

therefore, when arc (σ_p) varies in its full scale as ω moves along contour c, there is only a single radial axis in the W plane which can stay in the $\mathrm{Re}(W_0)<0$ plane,

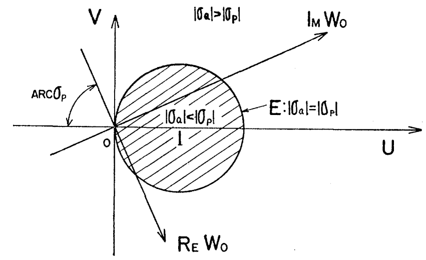

FIG. 2. W and W_0 planes and the circle E, given by $|W-1|=1$. If the locus remains inside the circle, the system is always stable.

that is, the negative U axis. Hence the negative U axis may be most suitable for the reference axis in the W plane.

Now we study the relation between the encircling of the origin and the crossing of the locus with the negative U axis. One of the great benefits that the transformation (2.8) has is that we can apply Rouche's theorem to the function W. As can be immediately seen in Eq. (2.8), if $|\sigma_a|/|\sigma_p|<1$, the locus stays always inside the circle $|W-1|=1$, shown by E in Fig. 2; hence the locus can never encircle the origin. This applies even for a range of ω such that $\mathrm{Re}(\sigma_a)<0$; thus the theorem may be used as a higher necessary condition for stability of a system. An interesting point here is that if σ_a itself does not possess a zero in $\mathrm{Im}(\omega)>0$, which can be made so in most cases, we can apply the same argument to a function W', which is obtained by

$$W'(\omega)\equiv\frac{W_0(\omega)}{\sigma_a(\omega)}=1+\frac{\sigma_p(\omega)}{\sigma_a(\omega)}; \qquad (2.11)$$

thus, if $|\sigma_p|/|\sigma_a|<1$, the locus can never encircle the origin. Therefore, we obtain the following theorem:

Theorem 2. A plasma is always stable when, for all ω on c, $|\sigma_p|\geq|\sigma_a|$. In particular, if σ_a does not itself possess any zero in the $\mathrm{Im}(\omega)>0$ plane, the plasma is stable when, for all ω on c, $|\sigma_p|\neq|\sigma_a|$.

For example, we can see that if σ_a does not possess any zero in the $\mathrm{Im}(\omega)>0$ plane, and if $\mathrm{Re}(\sigma_a)<0$ occurs for only one continuous range of ω on c, the plasma can be unstable only when $|\sigma_a|=|\sigma_p|$ for such a range of ω.

Next we study the way in which the locus encircles the origin. Suppose we follow the locus starting from $\omega=\omega_s$ on c, such that the corresponding point on the W plane, as shown by S in Fig. 3, does not lie exactly on the negative U axis. As is obvious from a topological consideration, for the locus to encircle the origin, it should cross the negative U axis at least once. When the locus starts from S, and crosses the negative U axis only once, one can see that the locus encircles the origin once. When it crosses twice, there are two possibilities: Either it encircles the origin twice, or it does not encircle the origin as shown by A and B in Fig. 3. To dis-

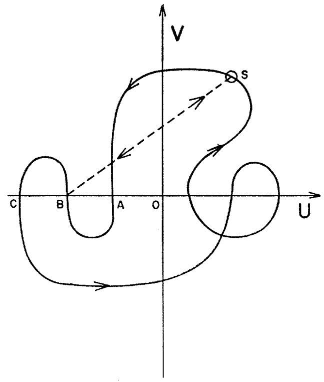

FIG. 3. Possible locus in W plane. When the locus encircles the origin, the system is unstable, which may be found by the way it crosses the negative U axis.

tinguish these two possibilities, we return the locus to S artificially, when the locus meets the negative U axis at the second time without letting the return path cross the negative U axis and create a closed loop, as shown by the dashed line BS in the figure. We can see that if the locus crosses the axis twice in opposite directions, the closed loop can never encircle the origin; if it crosses the axis twice in the same direction, it encircles the origin once. If we define a number λ such that when the locus crosses the negative U axis once in the positive direction (counter clockwise), $\lambda=1$, and when it crosses the same axis once in the negative direction, $\lambda=-1$, the total number of times N that the locus encircles the origin is consequently given by

$$N=\sum \lambda. \tag{2.12}$$

Thus we have the following theorem:

Theorem 3. A plasma is unstable when and only when $\sum \lambda$ is not zero.

The above theorem gives a form of the necessary and sufficient condition for an instability. The condition that the locus crosses the negative U axis is given by the following: For $\omega=\omega_0$ on c such that

$$\text{arc}\,[\sigma_a(\omega_0)]-\text{arc}\,[\sigma_p(\omega_0)] = (2n+1)\pi, \quad n \text{ is an integer}, \tag{2.13}$$

and

$$|\sigma_a(\omega_0)| > |\sigma_p(\omega_0)|. \tag{2.14}$$

Or, as an alternative form: For $\text{Re}[\sigma_a(\omega_0)]<0$ and

$$\text{Re}[\sigma_a(\omega_0)]\,\text{Im}[\sigma_p(\omega_0)] - \text{Re}[\sigma_p(\omega_0)]\,\text{Im}[\sigma_a(\omega_0)]=0, \tag{2.13'}$$

$$-\text{Re}[\sigma_a(\omega_0)]>\text{Re}[\sigma_p(\omega_0)]$$

or

$$|\text{Im}[\sigma_a(\omega_0)]| > |\text{Im}[\sigma_p(\omega_0)]|. \tag{2.14'}$$

In many cases, $\text{Re}[\sigma_a(\omega)]$ becomes negative only for one range of frequency ω, say, $\omega_1<\omega<\omega_2$. For an instability in such a case, $|\sigma_a(\omega)|$ should be equal to $|\sigma_p(\omega)|$ at least once, for ω in this range. We shall see, in Sec. III, that there are many problems in which this is true. Let us classify the types of σ_a into four groups, depending on the behavior of the locus on the $\sigma_a(\omega)$ plane when ω moves along on c in the ω plane. We assume that $\sigma_a(\omega)$ does not itself have a zero in the $\text{Im}(\omega)>0$ plane; thus the locus does not encircle the origin in σ_a plane.

Type 1. The locus crosses both the negative $\text{Im}(\sigma_a)$ axis and the negative $\text{Re}(\sigma_a)$ axis [but does not cross the positive $\text{Im}(\sigma_a)$ axis because of the assumed nature of σ_a] [Fig. 4(1)].

Type 2. The locus crosses the negative $\text{Im}(\sigma_a)$ axis but does not cross the negative $\text{Re}(\sigma_a)$ axis [Fig. 4(2)].

Type 3. The locus crosses the positive $\text{Im}(\sigma_a)$ axis but does not cross the negative $\text{Re}(\sigma_a)$ axis [Fig. 4(3)].

Type 4. The locus crosses both the positive $\text{Im}(\sigma_a)$ axis and the negative $\text{Re}(\sigma_a)$ axis [Fig. 4(4)].

We know that for an instability to occur, arc (σ_a) $-$arc $(\sigma_p)=\pi$; hence, depending on the types of σ_a, we can find the type of σ_p that leads to an instability. If we define σ_p as inductive, resistive, and capacitive if $\text{Im}(\sigma_p)>0$, $\text{Im}(\sigma_p)=0$, and $\text{Im}(\sigma_p)<0$, respectively [note that we use a phasor of $\exp(-i\omega t)$], the pair of σ_p to each type of σ_a that leads to an instability is given by Table I.

Here the following remark may be worth making. As has been stated previously, the grouping of σ_j in Eq. (2.1) into σ_a and σ_p can be done in an arbitrary fashion, as long as σ_a and σ_p satisfy the conditions of the active and passive conductivities, respectively. Depending on the choice of σ_a and σ_p, the function W changes; thus the form of the condition of crossing of the locus with the negative U axis changes. For example, if we include the $\text{Im}(\sigma_a)$ into σ_p, $\text{Im}(\sigma_a)$ becomes zero in the new σ_a; thus the crossing condition becomes $\text{Im}(\sigma_a)+\text{Im}(\sigma_p)=0$ and $-\text{Re}(\sigma_a)>\text{Re}(\sigma_p)$, which is the same condition obtainable from the condition of crossing of the locus with the negative $\text{Re}(W_0)$ axis in the W_0 plane as shown previously.

III. LONGITUDINAL PLASMA INSTABILITIES

In this section, we give applications of the stability theory. The objectives are to show examples of the way

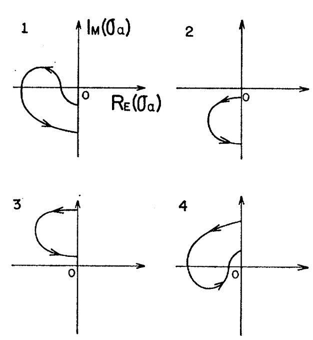

FIG. 4. Possible behavior of the locus of active conductivities.

in which the theory is applied to obtain instability conditions, to study the nature of the instabilities based on our definition, and to obtain hints for their stabilization. This section consists of two subsections. In subsection A, we treat collision-dominated plasmas, and we discuss the effects of drift and density gradient of electrons, using the diffusion equation. In subsection B, we treat collision-free plasmas, and we consider the effects of drift, stream, density gradient, and anisotropy of temperature, using the Vlasov equation.

We shall derive only the necessary and sufficient conditions of these instabilities in terms of the related plasma parameters. The instabilities treated cover many of the longitudinal instabilities so far found by many authors. Some results are previously found conditions, but reconfirmed with our method, and some, to the author's knowledge, are new.

A. Instabilities in Collision-Dominated Plasmas

Here we consider instabilities in collision-dominated plasmas, where $\omega\tau_e$ (τ_e is electron-collision time) is much smaller than unity. We assume that the plasma is imbedded in uniform magnetic and electric fields $\mathbf{B}_0$ and $\mathbf{E}_0$. The plasma has its density gradient in the direction transverse to the magnetic field. We take $\mathbf{B}_0$ in the z direction, as shown in Fig. 5. The electric field E_{0z} is applied to maintain the discharge. The transverse electric field E_{0x}, which exists in the direction parallel to that of the density gradient, is either an ambipolar field or that plus a field applied from an external source. We consider a longitudinal wave propagating in an arbitrary direction in such a plasma.

First, let us study the behavior of electrons. The density gradient and the drift may contribute to create an active conductivity. The equations are the equation of motion

$$n\mathbf{v}=-\mu_e n(\mathbf{E}+\mathbf{v}\times\mathbf{B})-D_e\nabla n \tag{3.1}$$

and the equation of continuity

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{v})=-\delta n, \tag{3.2}$$

where μ_e is the electron mobility $(=e/\nu_e m)$, D_e is the electron-diffusion constant $(=v_T^2/\nu_e)$, ν_e is the electron-neutral collision rate, v_T is the electron thermal velocity, and δ is the recombination frequency, which we assume to be negligibly small but not zero.

TABLE I. Kinds of passive conductivities that lead to instability in the presence of different types of active conductivities (see Fig. 4).

Type of σ_a	1	2	3	4
Can become unstable only when σ_p is	Inductive Resistive	Inductive	Capacitive	Capacitive Resistive

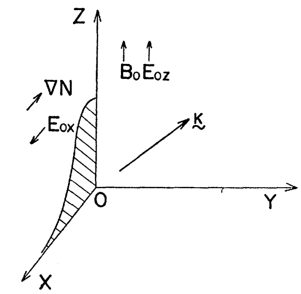

FIG. 5. Coordinate system used to study instabilities.

The unperturbed quantities, for which we use suffix 0, are obtained from Eqs. (3.1) and (3.2) as

$$v_{0z}=-\mu_e E_{0z}, \tag{3.3}$$

$$v_{0x}=\frac{\kappa D_e-\mu_e E_{0x}}{1+\mu_e^2B_0^2}, \tag{3.4}$$

$$v_{0y}=\mu_e B_0 v_{0x}, \tag{3.5}$$

where κ shows the magnitude of the density gradient:

$$\kappa=-\frac{d(\ln n_0)}{dx}>0. \tag{3.6}$$

For the perturbed quantities, we assume a phasor of $\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$, and use subscript 1. The assumption of a longitudinal disturbance allows us to use a scalar potential ϕ_1 for the electric field:

$$E_1=-ik\phi_1. \tag{3.7}$$

Then we can solve Eqs. (3.1) and (3.2) for the perturbed density n_1 in terms of the perturbed potential ϕ_1. Substituting the result into the definition of the longitudinal conductivity shown in Eq. (2.3), we can obtain the conductivity of the electrons as

$$\sigma_e=\frac{\omega\epsilon_0\omega_{pe}^2(ik_0^2+\kappa k_y/\mu_e B_0)}{k^2[i\nu_e(\omega-\mathbf{k}\cdot\mathbf{v}_E)-v_T^2k_0^2]}, \tag{3.8}$$

where ω_{pe} is the electron plasma frequency $[=(e^2n_0/\epsilon_0 m)^{1/2}]$, $\mathbf{v}_E$ is the unperturbed velocity due to the electric field, namely,

$$\mathbf{v}_E=-\mu_e E_{0z}\mathbf{e}_z-\frac{\mu_e E_{0x}}{1+\mu_e^2B_0^2}\mathbf{e}_x-\frac{\mu_e B_0\mu_e E_{0x}}{1+\mu_e^2B_0^2}\mathbf{e}_y,$$

and

$$k_0^2=k_z^2+\frac{k_x^2+k_y^2}{1+\mu_e^2B_0^2}.$$

The active range of σ_e can be immediately obtained from the condition that $\mathrm{Re}(\sigma_e)<0$, and is given by

$$0<\omega<\mathbf{k}\cdot\mathbf{v}_E+\omega_e^*, \tag{3.9}$$

where

$$\omega_e^*=\kappa v_T^2k_y/\omega_{ce}, \quad \omega_{ce}=eB_0/m.$$

Thus, from Eq. (3.9) we can see that both the electric-field drift and the density gradient contribute to make σ_e active. To find the type of σ_e, we now look at $\mathrm{Im}(\sigma_e)$. From Eq. (3.9),

$$\mathrm{Im}(\sigma_e) \propto \omega[\omega_e^*(\mathbf{k}\cdot\mathbf{v}_E-\omega)-k_0^4D_e^2]. \qquad (3.10)$$

Noticing that $\sigma_e \to 0$ at $\omega \to 0$, we can see that σ_e belongs either to type 1 or to type 2, depending on the sign of $\mathrm{Im}(\sigma_e)$ at $\omega \to 0$; namely, if

$$\omega_e^*\mathbf{k}\cdot\mathbf{v}_E-k_0^4D_e^2>0, \qquad (3.11)$$

σ_e belongs to type 1, so that an instability occurs even with pure resistive σ_p. On the other hand, if

$$\omega_e^*\mathbf{k}\cdot\mathbf{v}_E-k_0^4D_e^2<0, \qquad (3.12)$$

σ_e belongs to type 2; thus the instability occurs only when σ_p is inductive. Note that σ_e becomes active either owing to an electric-field drift or to a density gradient, but it belongs to type 1 only when both the drift *and* the density gradient are present.

Let us now see if σ_e can become a type-1 conductivity even in the absence of a drift parallel to the magnetic field. Assuming $\mu_eB_0\gg1$ and $k_z=k_y=0$, Eq. (3.9) becomes

$$0<\omega<\omega_e^*-k_y(E_{0x}/B_0), \qquad (3.13)$$

while Eq. (3.11) becomes

$$-\omega_e^*k_y\frac{E_{0x}}{B_0}-\frac{k_y^4D_e^2}{1+\mu_e^2B_0^2}>0. \qquad (3.14)$$

Thus, if E_{0x} is the ambipolar field, which may be given, say, for $T_i=0$, by

$$E_{0x}=\frac{\omega_e^*B_0}{k_y(1+\mu_i\mu_eB_0^2)}>0, \qquad (3.15)$$

then σ_e, though it can be active, can never become type-1 conductivity. However, if we apply an electric field externally in the negative x direction, then σ_e can be made to belong to type 1.

Now, what kinds of instability will be expected to result from the active conductivity σ_e? We presume that cold ions in the plasma will constitute the passive conductivity; by assuming a quasineutral condition, we can ignore the conductivity of the space. First, we consider a low-frequency instability, where $\omega\ll\omega_{ci}\ll\nu_i$ (ω_{ci} is the ion cyclotron frequency and ν_i is the ion-neutral collision frequency). For such a case, ions constitute a simple resistive medium whose conductivity is given by

$$\sigma_i=\epsilon_0\omega_{pi}^2/\nu_i, \qquad (3.16)$$

where ω_{pi} is the ion plasma frequency $[=(e^2n_0/\epsilon_0M)^{1/2}]$. Instability is only possible when σ_e is a type-1 conductivity, requiring the presence of both the density gradient and the drift. The condition for the instability is given from Eq. (2.12) as

$$-\mathrm{Re}(\sigma_e)>\epsilon_0\omega_{pi}^2/\nu_i, \quad \text{at } \mathrm{Im}(\sigma_e)=0, \quad \text{for } \omega\neq0;$$

or, explicitly, as

$$\frac{\omega_{pe}^2k_0^2}{\nu_ek^2}\left(\frac{\omega_e^*\mathbf{k}\cdot\mathbf{v}_E}{k_0^4D_e^2}-1\right)-\frac{\omega_{pi}^2}{\nu_i}>0. \qquad (3.17)$$

When E_{0x} is the ambipolar field, Eq. (3.17) reduces to the condition of the helical instability as derived by Kadomtsev.[11] Note that an instability is possible even in the absence of E_{0z}, if E_{0x} is applied externally. The related instabilities have been discussed by Buneman[12] and by Sato and Hatta.[13]

Next, we consider a relatively higher frequency region: $\omega\gg\nu_i\gg\omega_{ci}$, but $\omega<\omega_{pi}$, where the ions constitute an inductive medium whose conductivity is given by

$$\sigma_i=\frac{i\epsilon_0\omega_{pi}^2}{\omega+i\nu_i}. \qquad (3.18)$$

If $\omega<\omega_{pi}$, we can still use the quasineutral assumption, and then σ_i contributes to the passive conductivity σ_p. Because σ_p is now inductive, instability occurs even with type-2 conductivity, which means that conductivity with "drift" or "density gradient" is sufficient to cause an instability. Let us first consider the effect of the density gradient, and assume that there exists no electron drift parallel to $\mathbf{B}_0$. Because $\mathrm{Re}(\sigma_p)$ is very small under our assumption, the condition of the crossing of the locus with the negative U axis is given by

$$|\mathrm{Im}(\sigma_e)|>|\sigma_i|, \quad \text{at } \mathrm{Re}(\sigma_e)\to0. \qquad (3.19)$$

From Eq. (3.9) we can see that there are two ω's that satisfy $\mathrm{Re}(\sigma_e)=0$, i.e.,

$$\omega=0 \qquad (3.20)$$

and

$$\omega=\mathbf{k}\cdot\mathbf{v}_E+\omega_e^*. \qquad (3.21)$$

At $\omega\to0$, as can be seen in Eq. (3.8), $\sigma_e\to0$, while $\sigma_i\to\infty$; thus the above condition, Eq. (3.19), can never be satisfied. This shows that at $\omega\to0$ the locus on the W plane can never cross the negative U axis. If Eq. (3.19) is satisfied at the frequency given by Eq. (3.21), the locus crosses the negative U axis only once, and hence it gives the necessary and sufficient condition. In the absence of the parallel drift and with only the ambipolar electric field perpendicular to $\mathbf{B}_0$, Eq. (3.21) becomes

$$\omega=\frac{\mu_i\mu_eB_0^2}{1+\mu_i\mu_eB_0^2}\omega_e^*. \qquad (3.21')$$

Consequently, the instability condition is given by

[11] B. B. Kadomtsev, *Plasma Turbulence* (Academic Press Inc., New York, 1965), p. 13.
[12] O. Buneman, Phys. Rev. Letters **10**, 285 (1963).
[13] N. Sato and Y. Hatta, J. Phys. Soc. Japan **21**, 1801 (1966).

Eqs. (3.19) and (3.21′) as

$$\frac{\mu_i\mu_e B_0^2}{1+\mu_i\mu_e B_0^2}\omega_e^* > k_y c_s\,, \tag{3.22}$$

where c_s is the ion sound velocity $[=(mv_T^2/M)^{1/2}]$. The related instabilities have been discussed by Timofeev[14] and by Moiseev and Sagdeev.[15]

We next consider the effect of drift alone. We neglect the density gradient, and instead introduce a uniform electric field $\mathbf{E}_0$ in the x, z plane. The same argument as used in the derivation of Eq. (3.22) gives the condition of instability as

$$-\mu_e(E_{0z}k_z+E_{0x}k_y/\mu_e B_0) > (k_z^2+k_y^2)^{1/2}c_s\,. \tag{3.23}$$

Ordinarily, the excitation of the ion sound wave due to an electron drift is considered to be caused by resonant interaction[16] (contribution of the Landau pole); however, our analysis shows that excitation is also possible with a collision-dominated electron drift. Under the fluid approximation, a collision-free electron stream cannot become active unless the drift velocity exceeds the thermal velocity; hence it cannot cause the present instability. In this sense, the collisions in the electron stream are helping the instability. We shall see in the next subsection, however, that in a collision-free stream the Landau damping creates an effect similar to that of collision damping and can cause this instability. Excitation of the sound wave due to the collision-dominated stream has been discussed by Kuckes[17] for gaseous plasmas and by White[18] and many others for solid-state plasmas.

B. Instabilities in Collision-Free Plasmas

We take a coordinate system similar to that shown in Fig. 5. We have a density gradient in the x direction and a magnetic field in the z direction. We assume no unperturbed electric field in this case. We use the Vlasov equation to describe such a plasma. For a density function $f(\mathbf{v},\mathbf{r},t)$ of a species with charge e and mass m, and in the presence of a longitudinal disturbance, the equation may be written as

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{e}{m}\left(-\frac{\partial\phi}{\partial\mathbf{r}}+\mathbf{v}\times\mathbf{B}_0\right)\cdot\frac{\partial f}{\partial\mathbf{v}}=0. \tag{3.24}$$

For the unperturbed density $f_0(x,\mathbf{v})$, we take, as a solution of Eq. (3.24),

$$f_0(x,\mathbf{v})=n(x)f_{0v}[(v_z-v_0)^2,\, v_x^2+v_y^2](1-\kappa v_y/\omega_c)\,, \tag{3.25}$$

which is valid only if $\kappa v_y/\omega_c<1$ and if the "temperature" is constant in space. The equation for the perturbed density f_1 is obtained by linearizing Eq. (3.24):

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\frac{\partial f_1}{\partial\mathbf{r}}+\frac{e}{m}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial f_1}{\partial\mathbf{v}} =\frac{e}{m}\left(\frac{\partial f_{0v}}{\partial\mathbf{v}}\cdot\frac{\partial\phi_1}{\partial\mathbf{r}}-\frac{\kappa f_{0v}}{\omega_c}\frac{\partial\phi_1}{\partial y}\right)n(x). \tag{3.26}$$

Using a phasor of $\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$, and considering a disturbance propagating in the y, z plane, we can integrate Eq. (3.26) easily and obtain the perturbed charge density en_1 as

$$en_1=e\int f_1 d\mathbf{v}. \tag{3.27}$$

In particular, when f_{0v} has a Maxwellian distribution along the direction of the applied magnetic field,

$$f_{0v}(\mathbf{v})=\frac{f_0^{\perp}(v_x^2+v_y^2)}{(2\pi)^{1/2}v_{\parallel}}\exp\left[-\frac{(v_z-v_0)^2}{2v_{\parallel}^2}\right], \tag{3.28}$$

the result can be expressed using the plasma dispersion function Z.[19] Using the perturbed density, we can write the conductivity as follows. For electrons:

$$\sigma_e=\frac{-i\omega\epsilon_0\omega_{pe}^2}{k^2v_{\parallel}^2}\times\sum_{n=-\infty}^{\infty}F_n\left[1+\frac{\omega-\omega_e^*-k_zv_0-n\omega_{ce}(1-v_{\parallel}^2R_n/v_{\perp}^2)}{\sqrt{2}k_zv_{\parallel}}\times Z\left(\frac{\omega-n\omega_{ce}-k_zv_0}{\sqrt{2}k_zv_{\parallel}}\right)\right]. \tag{3.29}$$

For ions, by assuming $v_0=0$ and using capital letters for the thermal velocities:

$$\sigma_i=\frac{-i\omega\epsilon_0\omega_{pi}^2}{k^2V_{\parallel}^2}\times\sum_{n=-\infty}^{\infty}F_n\left[1+\frac{\omega+\omega_i^*-n\omega_{ci}(1-V_{\parallel}^2R_n/V_{\perp}^2)}{\sqrt{2}k_zV_{\parallel}}\times Z\left(\frac{\omega-n\omega_{ci}}{\sqrt{2}k_zV_{\parallel}}\right)\right], \tag{3.30}$$

where

$$F_n=\int_0^\infty J_n^2\left(\frac{k_yu}{\omega_c}\right)f_0^{\perp}(u)2\pi u du>0\,, \tag{3.31}$$

$$R_n=-\frac{v_{\perp}^2}{F_n}\int_0^\infty J_n^2\left(\frac{k_yu}{\omega_c}\right)\frac{1}{u}\frac{\partial f_0^{\perp}}{\partial u}2\pi u du>,<0\,, \tag{3.32}$$

$$u^2=v_x^2+v_y^2\,, \tag{3.33}$$

[14] A. V. Timofeev, Zh. Tech. Fiz. **33**, 909 (1963) [English transl.: Soviet Phys.—Tech. Phys. **8**, 682 (1964)].
[15] S. S. Moiseev and R. Z. Sagdeev, Zh. Eksperim. i Teor. Fiz. **44**, 763 (1963) [English transl.: Soviet Phys.—JETP **17**, 515 (1963)].
[16] See, for example, T. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Co., New York, 1962), p. 213.
[17] A. Kuckes, Phys. Fluids **7**, 511 (1964).
[18] D. L. White, J. Appl. Phys. **33**, 2547 (1962).

[19] B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

ω^* is the drift wave frequency defined in Eq. (3.9), with v_T replaced by $v_{\parallel}$, and $v_{\perp}^2$ is the variance of $f_0^1(u)$. In Eqs. (3.29) and (3.30), it is understood that ω includes some small dissipation.

Note here that, when $f_0^1(u)$ is also Maxwellian, F_n and R_n reduce to

$$F_n=\exp\left(-\frac{k_y^2v_{\perp}^2}{\omega_c^2}\right)I_n\left(\frac{k_y^2v_{\perp}^2}{\omega_c^2}\right), \qquad (3.31')$$

$$R_n=1. \qquad (3.32')$$

Note, also, that R_n can become negative when $\partial f_0^1/\partial u$ is locally positive, that is, when $f_0^1(u)$ has a hump at u not equal to zero.

It is known that $\text{Im}(Z)$ is always positive for real argument and becomes maximum when the argument is equal to zero; thus, to make σ_e active, i.e., to make $\text{Re}(\sigma_e)$ negative, near $\omega\sim n\omega_{ce}+k_zv_0$,

$$\omega<\omega_e^*+n\omega_{ce}(1-v_{\parallel}^2R_n/v_{\perp}^2)+k_zv_0. \qquad (3.34)$$

Equation (3.34) shows that the effect of a density gradient, the first term, the effect of anisotropy of temperature, $v_{\parallel}^2R_n^2/v_{\perp}^2<1$, the second term, and the effect of drift parallel to $\mathbf{B}_0$, the third term, are all contributing similarly to make σ_e active. We treat those effects separately.

Effect of Drift Parallel to B_0

Let us first study the effect of the drift parallel to the magnetic field. It is well known that such a drift gives rise to an instability, but we wish to show that there are two different kinds of instabilities. Assuming propagation parallel to the magnetic field, Eq. (3.29) reduces to

$$\sigma_e=-\frac{i\omega\epsilon_0\omega_{pe}^2}{k^2v_{\parallel}^2}\left[1+\frac{\omega-k_zv_0}{\sqrt{2}k_zv_{\parallel}}Z\left(\frac{\omega-k_zv_0}{\sqrt{2}k_zv_{\parallel}}\right)\right]. \qquad (3.35)$$

First, we consider the low-frequency region where $|\omega-k_zv_0|\ll|\sqrt{2}k_zv_{\parallel}|$ can be assumed. For such a region, we can use the power-series expansion of Z function, and σ_e becomes

$$\sigma_e=\frac{\omega\epsilon_0\omega_{pe}^2}{v_{\parallel}^2k^2}\left[(\tfrac{1}{2}\pi)^{1/2}\frac{\omega-k_zv_0}{k_zv_{\parallel}}-i\right]. \qquad (3.36)$$

Therefore, σ_e becomes active for ω such that

$$0<\omega<k_zv_0,$$

and it belongs to type 2 and is *dissipatively active;* we can show that the wave energy associated with this condition is positive, but σ_e becomes active because of the nonconservative nature of the Landau damping, producing a positive-source term in Eq. (2.6). Hence instability can occur only with inductive σ_p. If, by assuming quasineutrality, σ_p is composed of cold ions, the conductivity is given by Eq. (3.18), and the condition of instability can easily be obtained as

$$-\text{Im}(\sigma_e)>\text{Im}(\sigma_i)\,,\quad \text{at}\quad \omega=k_zv_0\,,$$

or, explicitly, as

$$v_0>c_s\,, \qquad (3.37)$$

which is the well-known result.[16] Here it is worth noting that the condition is exactly the same as the case for collision-dominated drift. In fact, σ_e obtained in Eq. (3.36) can be reduced exactly to the same form as that obtained for the collision-dominated example Eq. (3.8) by expanding in powers of $(\omega-k_zv_0)/kv_T$, and by simply equating the collision term ν_e/kv_T to $(\frac{1}{2}\pi)^{1/2}$ in Eq. (3.36).

Next, we consider a higher-frequency region where $|\omega-k_zv_0|/|\sqrt{2}k_zv_{\parallel}|\gg1$. We use the asymptotic expansion of the Z function, finding that σ_e may be written as

$$\sigma_e=\frac{i\omega\epsilon_0\omega_{pe}^2}{(\omega-k_zv_0)^2-3k_z^2v_{\parallel}^2}. \qquad (3.38)$$

Because σ_e in Eq. (3.38) is pure imaginary, we use the criterion of Eq. (2.5″); the σ_e is shown to become active when

$$|\omega|<[k_z^2(v_0^2-3v_{\parallel}^2)]^{1/2}.$$

Thus, unlike σ_e in Eq. (3.36), σ_e here can become active only when the drift velocity is larger than the thermal velocity. Another difference between σ_e in the high-frequency region and σ_e in the low-frequency region is that σ_e here is *reactively active*, and the wave energy can be shown to be *negative*; thus instability occurs even with a pure resistive medium,[20,21] as can be expected from the argument in Sec. II. The instability caused by σ_e in Eq. (3.38) is known as the two-stream instability, and its nature has been well studied by many authors.

Effect of Density Gradient

Now we consider the effect of a density gradient, which is responsible for the drift-wave instabilities. We treat the interaction between the drift wave of electrons and ions, and assume the frequency range of $\omega\ll\omega_{ce}$. For such a frequency range, the conductivity of electrons may be obtained by taking only the first term in the summation in Eq. (3.29) (an isotropic temperature is assumed here), i.e.,

$$\sigma_e=\frac{\omega\epsilon_0\omega_{pe}^2}{v_T^2k^2}\left[(\tfrac{1}{2}\pi)^{1/2}\frac{\omega-\omega_e^*}{k_zv_T}-i\right]. \qquad (3.39)$$

Thus σ_e has exactly the same form as that in Eq. (3.36), if we equate ω_e^* to k_zv_0. σ_e belongs to type 2, and instability occurs only when σ_p is inductive. However,the passive conductivity differs significantly, because here

[20] C. K. Birdsall, G. R. Brewer, and A. V. Haeff, Proc. IRE **41**, 865 (1953).

[21] C. K. Birdsall and J. R. Whinnery, J. Appl. Phys. **24**, 314 (1953).

we should consider the conductivity of the ions with a component of propagation perpendicular to the magnetic field. If we consider the frequency range of $\omega<\omega_{ci}$, we can again assume quasineutrality. The conductivity of the ions may be obtained by taking only the $n=0$ term in Eq. (3.30) as

$$\sigma_i=\frac{-i\omega\epsilon_0\omega_{pi}^2}{k^2V_T}\left[1+\frac{\omega+\omega_i^*}{\sqrt{2}k_zV_T}F_0\left(\frac{k_yV_T}{\omega_{ci}}\right)Z\left(\frac{\omega}{\sqrt{2}k_zV_T}\right)\right]. \tag{3.40}$$

If the ion temperature is small and the ion Landau damping is negligible, the condition of instability becomes

$$-\operatorname{Im}(\sigma_e)>\operatorname{Im}(\sigma_i), \quad \text{at } \omega=\omega_e^*,$$

or, explicitly,

$$\frac{T_i}{T_e}+1+\frac{\omega_i^*+\omega_e^*}{\sqrt{2}k_zV_T}F_0\left(\frac{k_yV_T}{\omega_{ci}}\right)\operatorname{Re}\left[Z\left(\frac{\omega_e^*}{\sqrt{2}k_zV_T}\right)\right]>0. \tag{3.41}$$

The above condition is satisfied only for large values of k_y/k_z. It may be worth noting that the condition is satisfied even for cold ions. The related instability has been discussed by Kadmtsev and Timofeev,[22] but Eq. (3.41) gives a more general form.

An interesting feature of the drift wave in a collisionless plasma is that it presents a reactively active (negative-energy-wave) conductivity for the perturbation propagating exactly perpendicular to the magnetic field. Assuming an isotropic temperature ($v_{\parallel}=v_{\perp}\equiv v_T$) and no drift parallel to the magnetic field, the electron conductivity given in Eq. (3.29) becomes, for the perpendicular propagation,

$$\sigma_e=\frac{-i\omega\epsilon_0\omega_{pe}^2}{k^2v_T^2}\sum_{n=-\infty}^{\infty}\frac{\omega_e^*-n\omega_{ce}}{\omega-n\omega_{ce}}F_n. \tag{3.42}$$

Using the criterion of Eq. (2.5′), we can obtain the condition that σ_e becomes active at $\omega\sim n\omega_{ce}$, as[23]

$$\omega_e^*>n\omega_{ce}, \quad n=1,2,\cdots. \tag{3.43}$$

Careful study of the locus of σ_e in Eq. (3.42) shows that it is stable against a pure reactive loading, but is unstable against a resistive loading.[24]

Effect of Anisotropic Temperature

In this subsection, we consider the effect of anisotropic temperature. There are two features in this effect. One is of the case when the transverse distribution $f_0^{1}(u)$ has a hump and R_n can become negative, while the other is when $\partial f_0^{1}/\partial u<0$ for all u and thus R_n is always positive. In both cases, the conductivity of ions, as well as that of electrons, becomes active.

We first treat the latter case. As a typical example, we assume a bi-Maxwellian distribution for $f_{0v}(\mathbf{v})$. We consider only the case where the instability condition is satisfied most easily, which is the case where the anisotropy of the ion distribution contributes to create an active conductivity, while relatively cold electrons contribute to the passive part.

We assume a frequency range of $\omega>\omega_{ci}$, and that $V_{\parallel}$ is relatively small, such that $\omega_{ci}/k_zV_{\parallel}\gg1$. Then we can use the asymptotic expansion of Z function in Eq. (3.30) for all but the $n=l$ term, and the corresponding conductivity can be written as

$$\sigma_l=-\frac{i\omega\epsilon_0\omega_{pi}^2}{k^2V_{\parallel}^2}\left[1+\left(\frac{V_{\parallel}}{V_{\perp}}\right)^2\sum_{n\neq l}\frac{n}{n-l}\left(\frac{F_n}{F_l}\right)+\left\{\frac{\omega-l\omega_{ci}}{\sqrt{2}k_zV_{\parallel}}+\frac{l\omega_{ci}}{\sqrt{2}k_zV_{\parallel}}\left(\frac{V_{\parallel}}{V_{\perp}}\right)^2\right\}Z\left(\frac{\omega-l\omega_{ci}}{\sqrt{2}k_zV}\right)\right]F_l. \tag{3.44}$$

The upper limit of ω, such that σ_l becomes active, may be obtained immediately, by using the fact that for real argument $\operatorname{Im}(Z)$ is always positive, as

$$\omega<l\omega_{ci}\left[1-\left(\frac{V_{\parallel}}{V_{\perp}}\right)^2\right].$$

The lower limit may be obtained from the condition that the real part of σ_l is cancelled by that of σ_{l-1}. Combining the result with the upper limit, we can obtain the active range of σ_l as

$$(l-\tfrac{1}{2})\omega_{ci}<\omega<l\omega_{ci}[1-(V_{\parallel}/V_{\perp})^2], \quad l=1,2,\cdots. \tag{3.45}$$

This is the same relation as that of the necessary condition of instability derived by Soper and Harris.[25]

Now, to obtain the condition of instability, let us study the nature of σ_l. For ω at the lower limit in Eq. (3.45), the conductivity σ_l takes the form

$$\sigma_l^1=-\frac{i\omega\epsilon_0\omega_{pi}^2}{k^2V_{\parallel}^2}\left(\frac{V_{\parallel}}{V_{\perp}}\right)^2\left[2l+\sum_{n\neq l}\frac{n}{n-l}\left(\frac{F_n}{F_l}\right)\right]F_l, \tag{3.46}$$

while for ω at the upper limit,

$$\sigma_l^2=-\frac{i\omega\epsilon_0\omega_{pi}^2}{k^2V_{\parallel}^2}\left[1+\left(\frac{V_{\parallel}}{V_{\perp}}\right)^2\sum_{n\neq l}\frac{n}{n-l}\left(\frac{F_n}{F_l}\right)\right]F_l. \tag{3.47}$$

Hence, if $\operatorname{Im}(\sigma_l^1)\operatorname{Im}(\sigma_l^2)<0$, σ_l itself possesses zero in the $\operatorname{Im}(\omega)>0$ plane, unlike all the previous examples.

As for σ_p, we consider the contributions of electrons and the space. The effect of the temperature of electrons

[22] B. B. Kadomtsev and A. V. Timofeev, Dokl. Akad. Nauk SSSR **146**, 581 (1962) [English transl.: Soviet Phys.—Doklady **7**, 826 (1963)].

[23] Exactly the same condition applies to the ion conductivity with the propagation direction in the negative y direction.

[24] It is interesting to compare Eq. (3.42) with the conductivity of the flute disturbance, i.e., $\sigma\sim-i/(\omega+\omega_0)$, which is also reactively active, but stable against a pure inductive loading.

[25] G. K. Soper and E. G. Harris, Phys. Fluids **8**, 984 (1965).

causes the Landau damping which contributes to a positive real part in σ_p. If the damping is small, and if $\mathrm{Im}(\sigma_l^1)\,\mathrm{Im}(\sigma_l^2)<0$, we may still expect an instability. If $\mathrm{Im}(\sigma_l^1)\,\mathrm{Im}(\sigma_l^2)>0$, because $\mathrm{Im}(\sigma_l^2)<\mathrm{Im}(\sigma_l^1)$ and $\mathrm{Im}(\sigma_l^2)$ is negative for large value of $(V_\perp/V_{\parallel})^2$, the instability will occur most easily with an inductive σ_p, requiring relatively cold electrons for which $l\omega_{ci}/k_z v_{\parallel}\gg 1$ can be assumed. Then σ_p may be written as[26]

$$\sigma_p=\sigma_e-i\omega\epsilon_0$$

$$=\frac{i\omega\epsilon_0\omega_{pe}^2}{k^2v_{\parallel}^2}\left[\frac{k_z^2v_{\parallel}^2}{\omega^2}-\frac{k_yv_\perp^2}{\omega_{ce}^2}-\frac{k^2v_{\parallel}^2}{\omega_{pe}^2}\right]. \quad (3.48)$$

If we write σ_p at $\omega=(l-\frac{1}{2})\omega_{ci}$ and $\omega=l\omega_{ci}(1-V_{\parallel}^2/V_\perp^2)$ as σ_p^1 and σ_p^2, respectively, the instability condition can be written as

$$-\mathrm{Im}(\sigma_l^2)>\mathrm{Im}(\sigma_p^2)>0\,,$$
$$-\mathrm{Im}(\sigma_l^1)<\mathrm{Im}(\sigma_p^1)\,. \quad (3.49)$$

Equation (3.49) may be satisfied if σ_p is almost zero. If $v_\perp=0$, this condition may be satisfied by $l\omega_{ci}k\sim\omega_{pe}k_z$. If $v_\perp\neq 0$, a large k_y/k_z ratio will also satisfy the condition. For example, if $\omega_{pe}\gtrsim\omega_{ce}$, the latter condition gives

$$\frac{k_y}{k_z}\sim\left(\frac{v_{\parallel}}{v_\perp}\right)\left(\frac{M}{m}\right)\frac{1}{l(1-V_{\parallel}^2/V_\perp^2)}. \quad (3.50)$$

Instabilities associated with anisotropy of temperature have been discussed by many authors,[27] who have derived the former condition, i.e., $l\omega_{ci}k\sim\omega_{pe}k_z$. The condition shown in Eq. (3.50) is not generally known.

We now study the case where $f_0^1(u)$ has a hump at u not equal to zero. The instabilities associated with such a distribution have been studied by Gruber, Klein, and Auer,[28] by Hall and Heckrotte,[29] and by several others.[30] The active conductivity with such a distribution has several different features from that with the bi-Maxwellian distribution. One interesting feature is that it can become reactively active when the direction of propagation is exactly perpendicular to the magnetic field. In the same way as is used in the derivation of Eq. (3.42), assuming $k_z\to 0$, we can show that at $\omega\sim n\omega_c$, if

$$R_n<0\,,\quad n=1,2,\cdots, \quad (3.51)$$

the conductivity becomes active.

Another interesting feature of this conductivity with negative R_n is that, when the direction of propagation has a component parallel to the magnetic field, it becomes active even right at $\omega=n\omega_c$, as can be seen by replacing $(v_{\parallel}/v_\perp)^2$ by $R_n(v_{\parallel}/v_\perp)^2$ in Eq. (3.45). Moreover, since the $Z(x)$ function takes its maximum value (~ 1) at $x\sim -1$, if R_n is negative, the imaginary part of σ_l can more easily be made positive at $\omega\lesssim l\omega_c$; thus σ_l may belong to type 1 easily, even for $k_z\neq 0$. This fact shows that an instability should be expected even with hot electrons whose Landau damping contributes to a large positive real part of the passive conductivity, but helps to change it from a pure capacitive conductivity to a capacitive-resistive conductivity. Such an instability was shown to exist by Hall and Heckrotte.[29] Here we only state the nature of the active conductivities and do not discuss the conditions of instabilities.

IV. CONCLUSION

We have presented a new stability theory which may be useful in finding causes of instability and obtaining conditions of stabilization of a plasma in the presence of longitudinal disturbances. By applying the theory to collision-dominated and collision-free plasmas imbedded in a uniform magnetic field, we have shown that drift, density gradient, and temperature anisotropy can all become the causes of instabilities, but under considerably different conditions. These conditions, which are stated in terms of the direction of propagation relative to the applied magnetic field, may be useful in considering methods of stabilization which use magnetic shear.

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[26] Previous authors neglect the second term in Eq. (3.48).
[27] Major works are listed in Ref. 25.
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