Gas Pumping in Discharge Tubes

ARTHUR N. CHESTER
Bell Telephone Laboratories, Murray Hill, New Jersey 07971
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Collisions between gas atoms and charged particles in the positive column of a dc-excited gas discharge tend to cause an axial transport of gas between the ends of a discharge tube. Since, because of approximate charge neutrality, essentially equal forces are exerted on the plasma ions and electrons by the axial field, any net force on the gas must be balanced by an axial force exerted on the wall by the plasma. Earlier workers have treated this pumping by assuming that ions near the wall give their cathode-directed momentum to the wall rather than to neutral atoms, thus producing a net anode-directed force on neutral atoms close to the tube wall, this being the predominant force. Such a treatment is shown here to be an incomplete discussion of the forces acting on the neutral gas; moreover, the "wall pumping" force itself is shown to be incorrectly considered in the earlier work. At higher pressures, the principal gas pumping force turns out to be one which acts throughout the discharge volume, caused by the unequal radial distances traveled by ions and electrons between collisions with gas atoms. The pressure and radius dependences of gas pumping predicted by the present treatment differ substantially from those of earlier work.

I. INTRODUCTION

It has been known for more than 40 years that a pressure difference is observed between the ends of a gas discharge tube due to net axial forces exerted on the gas in the positive column.[1] When the two ends of a discharge tube are maintained at nearly the same pressure (for example, by providing a path external to the discharge connecting the cathode and anode regions of the tube), the forces of the positive column instead cause an axial flow of gas. Early explanations[2,3] of this effect gave a poor account of the observed pressure differences, but little further theoretical work was done, possibly because the effect was small and difficult to measure and seemed to have no important bearing upon other measurements in gaseous discharge tubes.

Recent work on gaseous ion lasers has stimulated interest in high-current-density discharges in small-radius tubes. Under these conditions the flow of gas through the positive column is no longer a small effect[4] and often leads to visible axial nonuniformities in the discharge. By measurement of these gas pumping effects, one can gain insight into some of the fundamental atomic collision processes occurring in the plasma.

In Sec. II we summarize previous theoretical results by Langmuir[2] and Druyvesteyn,[3] with recent extensions by Halsted.[5,6] A correct expression for the volume forces in the plasma is derived from basic principles in Sec. III, Sec. IV considers forces near the wall, and Sec. V introduces corrections due to the positive ion sheath near the wall. In Sec. VI the results obtained in this paper are summarized.

[1] For a bibliography of experimental work on gas pumping, see C. C. Leiby, Jr., and H. J. Oskam, Phys. Fluids **10**, 1992 (1967).
[2] Irving Langmuir, J. Franklin Inst. **196**, 751 (1923)
[3] M. J. Druyvesteyn, Physica **2**, 255 (1935).
[4] For example, see E. I. Gordon and E. F. Labuda, Bell System Tech. J. **43**, 1827 (1964)
[5] W. B. Bridges and A. S. Halsted, Hughes Research Laboratories Technical Report No AFAL-TR-67-89, 1967, 171–209 (unpublished).
[6] W. B. Bridges, P. O. Clark, and A. S. Halsted, IEEE J. Quantum Electron. **QE-2**, xix No. 3B-1 (April 1966).

II. PREVIOUS THEORETICAL RESULTS

In this section we outline the theoretical results obtained by Langmuir,[2] Druyvesteyn,[3] and Halsted,[5,6] to which some minor corrections and caveats have been added.

Let λ_- and λ_+ denote the electron and ion momentum-transfer mean free paths, with $\lambda_+ \ll R$. Because of approximate charge neutrality, ions and electrons gain nearly equal momentum from the axial electric field. Langmuir and Druyvesteyn pointed out that the electrons deliver a negligible amount of momentum to the tube walls, so that essentially all their longitudinal momentum is transferred to the gas. However, ions within a mean free path of the walls give their momentum to the walls rather than to the gas, and so a net force is exerted on the gas within about λ_+ of the walls due to electron collisions. There is also an opposing force arising from the lack of strict charge neutrality. These forces set up an anode-directed motion of the neutral gas which is resisted by viscous damping at the tube walls. This gas flow is only slightly offset by the cathode-directed motion of atoms represented by the ion current, and causes the anode-cathode pressure difference to build up until the net gas flow vanishes.

The number density of electrons, ions, and neutral atoms is denoted $n_e(r,z)$, $n_i(r,z)$, and $n_a(z)$. The implicit assumption that the latter quantity is not a function of radial position is well satisfied for discharges such as that of the argon ion laser, as Webb's experimental measurements[7] have shown. The axial drift velocity of the neutral atoms is $v_a(r,z)$, and their kinetic temperature is $T_a(z)$. The axial electric field $E(z)$ is assumed to be independent of radial position. Laminar flow of neutral atoms with viscosity η is assumed, since the Reynolds number is typically only about unity. Considering the total force F_t per unit volume on the

[7] C. E. Webb, in Proceedings of the Nineteenth Annual Gaseous Electronics Conference, Atlanta, Georgia, 1966 (to be published).

gas gives the equation of motion

$$F_t = n_e(r,z)eE(z) - n_i(r,z)\{1-\exp[-(R-r)/\lambda_+(z)]\}eE(z) - \frac{1}{r}\frac{\partial}{\partial r}\left[r\eta\frac{\partial}{\partial r}v_a(r,z)\right] - \frac{\partial}{\partial z}[n_a(z)kT_a(z)] = 0, \quad (2.1)$$

in which $z=0$ is the cathode end.

Druyvesteyn performs three r integrations to obtain $V(z)$, the volume of gas pumped past position z per second toward the anode:

$$V(z) = \int_0^R v_a(r,z)2\pi r dr = V_1(z)+V_2(z) - (2\pi/16\eta)R^4(\partial/\partial z)[n_a(z)kT_a(z)], \quad (2.2)$$

in which

$$V_1(z) = (2\pi/\eta)\int_0^R r''dr''\int_{r''}^R (dr'/r') \times \int_0^{r'} rdr[n_e(r,z)-n_i(r,z)]eE(z) \quad (2.3)$$

and

$$V_2(z) = (2\pi/\eta)\int_0^R r''dr''\int_{r''}^R (dr'/r') \times \int_0^{r'} rdrn_i(r,z)\exp[-(R-r)/\lambda_+(z)]eE(z). \quad (2.4)$$

$V_1(z)$ is caused by a volume force arising from the lack of strict charge neutrality; $V_2(z)$ is due to forces exerted within a few ion mean free paths of the tube wall and is usually much larger[3,5] than $V_1(z)$. For subsequent manipulations note that

$$\int_0^R r''dr''\int_{r''}^R (dr'/r')\int_0^{r'} drf(r) = \frac{1}{4}\int_0^R (R^2-r^2)f(r)dr, \quad (2.5)$$

as may be shown by calculating a Green's function for the triple integral.

Gauss's law is applied to Eq. (2.3) to yield

$$V_1(z) = -\pi E(z)\epsilon_0\eta^{-1}\int_0^R r^2E_r(r,z)dr. \quad (2.6)$$

Druyvesteyn cut cut off this expression at radius $R-\lambda_+$ because his expression for E_r diverged at the wall, but such a treatment is not correct. The following approximation is more accurate.

The radial dependence of the electrostatic potential φ in the central part of the tube is

$$\varphi(r) = (kT_e/e)\ln J_0(2.4r/R) \quad (2.7)$$

since the electron density in this region, when the loss of ions and electrons is due to ambipolar diffusion,[8,9] is

$$n_e(r,z) = 2.3163n_e(z)J_0(2.4r/R), \quad (2.8)$$

in which $n_e(z)$ is the electron density at position z, averaged across the tube. T_e denotes the electron temperature.

Equation (2.7) is not valid close to the tube wall because of the presence of the ion sheath, which modifies the radial field. Calculations of the wall potential, both for the free-fall plasma (low pressures)[10–13] and for the ambipolar discharge (high pressures),[13] give $\varphi(R)\approx -6kT_e/e$.

The approximate evaluation of Eq. (2.6) is carried out by using Eq. (2.7) for the potential only out to the radius where a linear extrapolation in r, at the same slope, yields the correct wall potential. This linear extrapolation is used beyond that radius. One finds

$$V_1(z) = -4.07\pi\epsilon_0E(z)\eta^{-1}(kT_e/e)R^2. \quad (2.9)$$

This differs from Druyvesteyn's result by a factor of $\sim(\frac{1}{6})\ln(R/2\lambda_+)$.

To treat the term $V_2(z)$, Eq. (2.4), Dryuvesteyn expands for small ion mean free path:

$$V_2(z) = (2\pi/\eta)eE(z)(2.3163)n_e(z) \times\{1.2483R\lambda_+^3[1-3(\lambda_+/R)+O(\lambda_+^2/R^2)] + \lambda_+^2R^2\exp(-R/\lambda_+)[\tfrac{1}{4}+O(\lambda_+^2/R^2)]\}. \quad (2.10)$$

This differs slightly from Druyvesteyn's result because he failed to keep the second term in the Taylor series expansion of the Bessel function about the point $r=R$.

Halsted tried to extend Eq. (2.10) to lower pressures ($\lambda_+\sim R$) by performing the integral of Eq. (2.4) exactly, using Eq. (2.8). The $V_2(z)$ thus obtained is one dimensional and neglects the cylindrical geometry; however, it is obviously correct in the limit of very large ion mean free path since in that limit it says that

[8] Lewi Tonks and Irving Langmuir, Phys. Rev. 34, 876 (1929); see p. 889.

[9] Gordon Francis, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. XXII, pp. 53–208; see p. 123.

[10] Lewi Tonks and Irving Langmuir, Phys. Rev. 34, 876 (1929); see p. 898.

[11] Jerald V. Parker, Ph.D. thesis, California Institute of Technology, 1964 pp. 17, 113–118 (unpublished). This work also appears as report No. AD 603534, Office of Technical Services, Department of Commerce, Washington, D. C. (unpublished). See also Fig. 1 of Jerald V. Parker, Phys. Fluids 6, 1657 (1963). Parker's parameter β^2 lies in the range 10^5–10^6 in argon ionic discharges, according to electron-density and electron-temperature measurements made by C. E. Webb and reported by Labuda *et al.* (Ref. 12).

[12] E. F. Labuda, C. E. Webb, R. C. Miller, and E. I. Gordon, Bull. Am. Phys. Soc. 11, 497 (1966).

[13] S. A. Self and H. N. Ewald, Phys. Fluids 9, 2486 (1966). The algebraic sign of the normalized potential at the inner edge of the plasma sheath, as given in Sec. 4 of this reference, should be negative rather than positive.

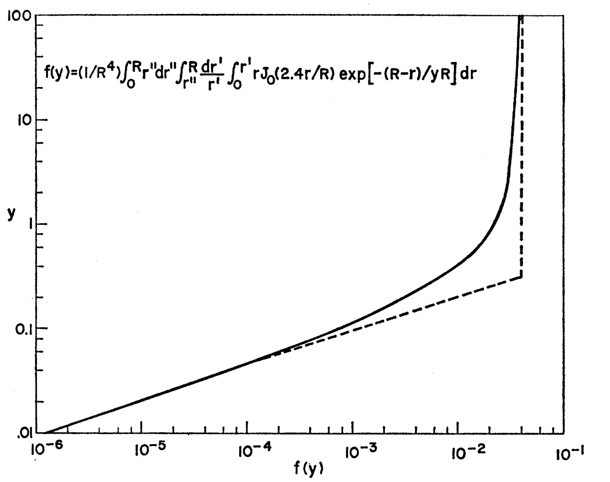

FIG. 1. Plot of $f(y)$. The dashed lines are extrapolated from the limits at small and large y.

the ions fall directly to the wall without making any collisions. Halsted's calculation gives

$$V_2(z) = (2\pi/\eta)eE(z)(2.3163)n_e(z)R^4 f(\lambda_+/R), \quad (2.11)$$

in which

$$f(y) = (1/R^4)\int_0^R r''dr''\int_{r''}^R (dr'/r') \times \int_0^{r'} rJ_0(2.4r/R)\exp[-(R-r)/yR]dr. \quad (2.12)$$

The asymptotic forms of $f(y)$ are

$$f(y) \to 1.2483y^3, \quad \text{when} \quad y \equiv \lambda_+/R \to 0+ \quad (2.13)$$

and

$$\lim_{y\to\infty} f(y) = 0.0373. \quad (2.14)$$

Figure 1 is a plot of the function $f(y)$. Because a radial averaging is performed in the integration, it is probably adequate to use Eq. (2.8) even at the lower pressures, as done here, rather than the more accurate radial profiles obtained by Tonks and Langmuir[14] or by Parker.[11] If only the asymptotic expressions (2.13) and (2.14) are used to approximate $f(y)$, the appropriate place to change from the high-pressure limit to the low-pressure limit is $y = \lambda_+/R = 0.31$; $f(y)$ will be overestimated by about a factor of 10 near this value.

A gas return path at temperature T_0, external to the discharge, is included as follows. Define Φ as the number of atoms per second passing any point in the gas return, multiplied by kT_0, and write the continuity equation

$$\Phi + \pi R^2 n_i(z) v_{di} kT_0 = n_a(z) kT_0 V(z), \quad (2.15)$$

in which $v_{di}(z)$ is the axial drift velocity of ions at position z (averaged over the cross section if the mobility varies with r). Dryuvesteyn then makes the approximation appropriate when the positive column pressure difference is small compared to its mean pressure p:

$$(\partial/\partial z)[n_a(z)kT_a(z)] \approx \Delta p/L, \quad (2.16)$$

in which L is the length of the positive column and Δp is the anode pressure minus the cathode pressure. In this approximation v_{di}, n_a, n_e, E, and λ_+ are no longer functions of z and, by combining Eqs. (2.2), (2.15), and (2.16), one obtains

$$\Delta p/L = (8\eta/\pi R^4)(V_1 + V_2) - (8\eta/n_a kT_0 \pi R^4)(\Phi + \pi R^2 n_i v_{di} kT_0). \quad (2.17)$$

By applying this equation to the case $n_i = n_e = 0$, one may also obtain an expression for the pressure difference appearing across the gas return tube:

$$\Delta p_{\text{ret}} = 8\eta_{\text{ret}} L_{\text{ret}} \Phi / \pi R_{\text{ret}}^4 n_{\text{ret}} kT_0. \quad (2.18)$$

In typical tube designs the end bottles offer negligible resistance to gas flow. This fact, plus the previous assumption of viscous flow, implies that

$$\Delta p_{\text{ret}} = \Delta p. \quad (2.19)$$

It should be noted that when the pressure is so low that molecular rather than viscous flow is a more appropriate description, Δp_{ret} will more nearly approach $(T_0/T_a)^{1/2}\Delta p$, but in a way that depends upon the detailed tube geometry and the temperatures of the end bottles.[15–18]

From Eqs. (2.17)–(2.19), one obtains

$$\Phi = (T_0/T_a)p[V_1 + \tfrac{1}{2}V_2 - (n_i/n_a)v_{di}\pi R^2](1+c)^{-1}, \quad (2.20)$$

with

$$c = (\eta_{\text{ret}}/\eta)(R/R_{\text{ret}})^4 (L_{\text{ret}}/L)(T_0/T_a). \quad (2.21)$$

When a gas return is used, usually $c \ll 1$, implying that the diffusive flow of neutral gas back through the discharge is negligible compared to the diffusive flow through the return tube, so that gas circulates through the system. With no gas return, Eq. (2.17) with $\Phi = 0$ yields Δp directly.

The theory outlined here underestimates the amount of gas flow and pressure difference, typically by a factor of 4 or more; more significantly, however, it predicts a dependence on pressure and tube radius that disagrees with experimental observations. This discrepancy was noted by Druyvesteyn, but he was not able to remove it. The only other theoretical treatment known to the author is very recent work by Leiby[1,19]; it uses a com-

[14] Lewi Tonks and Irving Langmuir, Phys. Rev. 34, 876 (1929); see pp. 883–886.

[15] J. P. Hobson, T. Edmonds, and R. Verreault, Can. J. Phys. 41, 983 (1963).

[16] T. Edmonds and J. P. Hobson, J. Vac. Sci. Technol. 2, 182 (1965).

[17] J. P. Hobson, Vacuum 15, 543 (1965).

[18] X-ray absorption measurements in argon by R. C. Miller (private communication).

pletely different approach leading to a special case of the results presented here, and will be discussed in a following section.

III. VOLUME FORCES

A. Kinetic Theory Derivation

The principal force obtained in previous theories, that of Eq. (2.4), acts only within a few ion mean free paths of the tube wall. The force treated in this section acts throughout the volume of the discharge, and usually causes much more gas flow than do the forces acting near the wall.

The principal physical process neglected in earlier treatments is the radial motion of ions and electrons. In the positive column ions and electrons both move radially outward with a common average velocity due to the combined effects of diffusion and the radial field. When a charged particle collides with a neutral gas atom, delivering to it axial momentum which it has gained from the axial electric field, it has traveled radially outward a distance of about $v_{dr}\tau$ since its last collision. Here, v_{dr} is the *radial* drift velocity of the charged particle, and τ is its mean free time (which may be controlled either by its radial drift velocity or by its thermal velocity). Consider here only the case of higher pressures: $v_{dr}\tau \ll R$. Then the axial momentum delivered to atoms at radius r is characteristic of the charged particle density at radius $r-v_{dr}\tau$. Since v_{dr} must be nearly the same for ions and electrons (because their wall currents are equal and their densities are nearly equal at any radius), and since usually $\tau_- \neq \tau_+$ (in fact, usually $\tau_- \ll \tau_+$), the effect on the transfer of axial momentum to the gas is the same as that of a radial charge separation, and so can result in a net force on the gas. Since this type of differential momentum transfer produces an axial force on the gas distributed throughout the discharge volume, it differs fundamentally from the major process described by Refs. 2, 3, 5, and 6.

This qualitative discussion indicates that the term representing pumping throughout the discharge volume (that is, several ion mean free paths distant from the tube wall), Eq. (2.3), should be replaced by

$$V_1'(z) = (2\pi/\eta)\int_0^R r''dr''\int_{r''}^R (dr'/r') \times\int_0^{r'} dr\{(r-s_-)n_e(r-s_-,\, z)eE(z) - (r-s_+)n_i(r-s_+,\, z)eE(z)\}\,, \quad (3.1)$$

with

$$s_\pm \approx v_{dr}\tau_\pm\,. \quad (3.2)$$

This in fact turns out to be approximately correct, as will be shown below. Note that it is not suggested that the charged-particle densities at position r are simply equal to those at radius $r-s_\pm$, spread out to fill the differential volume at r. The number of charged particles in a radial volume element at r is indeed just $rn(r)dr$, and the difference between this number and the quantity $(r-s_\pm)n(r-s_\pm)dr$ is made up for by charged particles produced in the vicinity of that point (this serves to give the charged particle distributions their characteristic radial profiles). However, these particles just produced by ionization have not yet had time to acquire the radial and axial drift velocity characteristic of "older" particles; thus an equation dealing with directed momentum transferred to the atoms, such as Eq. (3.1), is correct in omitting these particles. Usually v_{dr} is much smaller than the mean electron velocity, so that the *electron term* inside the braces may be approximated simply by $rn_e(r,z)eE(z)$, as in Eq. (2.3). Therefore, it will not be necessary to consider electron radial motion.

A rigorous derivation of Eq. (3.1) and the correct version of Eq. (3.2) will now be given. Assume that the mean free path of a charged particle is small compared with the distance over which its radial drift velocity changes:

$$\lambda[dv_{dr}(r)/dr] \ll v_{dr}(r)\,. \quad (3.3)$$

The derivation will first be carried through assuming that no ion-atom charge-exchange collisions occur. The effects of charge exchange will be included later.

Consider the charged particles colliding with a particular gas atom at time $t=0$, giving to that atom, on the average, all[20] of the axial momentum which they have gained since their formation or since their last collision. In reality, of course, the strong radial field causes the charged particles to move along curved paths between collisions and strongly alters their velocity distribution between collisions; however, the effects of the radial field will be approximated by superimposing an average radial drift velocity v_{dr} on the Maxwellian velocity distribution of the charged particles, neglecting the more detailed effects of the radial fields. First, obtain the force contributed by the ions to atoms at some position **r** in the tube.[21] Let ions of velocity **v** reach position **r** at time $t=0$. Assume that if an ion collides with a neutral atom at **r**, then it gives up all of the axial momentum that it has gained since its formation or since its last elastic collision with a neutral atom, whichever occurred later. Let **dJ** denote the flux of ions of velocity **v** (tolerance d^3v) originating at

[19] C. C. Leiby, Jr., and H. J. Oskam, in Proceedings of the Eighth International Ionization Conference, Vienna, 1967 (unpublished).

[20] It is only true "on the average" that a charged particle gives up *all* the axial momentum it has gained since its last collision. Actually, an ion still has some axial momentum following an elastic collision, gains more during its motion, and loses about half its *total* axial momentum upon colliding with a gas atom; but, in steady state, this must on the average leave it with the same axial momentum it had following its previous collision.

[21] This derivation is also applicable to the electron force. However, ordinarily the electrons move a much smaller distance radially during one mean free time than the ions do, and the effect considered here is negligible for electrons. This is also discussed following Eq. (3.2).

time $t<0$ (tolerance dt) which reaches position $\mathbf{r}$ at time $t=0$. This flux will be equal to the number of such ions per unit volume formed from times t to $t+dt$ at the originating position $\mathbf{r}+\mathbf{v}t$, multiplied by the probability $\exp(vt/\lambda_+)$ that they arrive at $\mathbf{r}$ without suffering a collision, multiplied by their velocity $\mathbf{v}$:

$$\mathbf{dJ}=R(\mathbf{r}+\mathbf{v}t,\ \mathbf{v})d^3v dt\ e^{vt/\lambda_+}\mathbf{v}. \quad (3.4)$$

In this expression, $R(\mathbf{r},\mathbf{v})d^3v$ denotes the rate of origination of ions (due to ionization or to collisions with neutral atoms) of velocity $\mathbf{v}$ (tolerance d^3v) per unit volume at position $\mathbf{r}$. Since the collision rate per neutral gas atom at $\mathbf{r}$ is equal to the magnitude of the ion flux $\mathbf{dJ}$ multiplied by the ion-atom momentum-transfer cross section, the total axial force F per unit volume acting on the gas at $\mathbf{r}$ due to these ions is

$$Fd^3vdt=(dJ/\lambda_+)(-eEt), \quad (3.5)$$

in which $(-eEt)$ is the axial momentum gained by an ion in time $|t|$.

One now obtains an expression for the total rate of origination of ions by collisions per unit volume, $R(\mathbf{r},\mathbf{v})$. This rate consists of the rate of formation of ions of velocity $\mathbf{v}$ by ionization of neutral atoms by electron collisions, $R_i(\mathbf{r},\mathbf{v})$, plus the rate of formation of ions of velocity $\mathbf{v}$ from ion-atom elastic collisions, $R_c(\mathbf{r},\mathbf{v})$:

$$R(\mathbf{r},\mathbf{v})=R_i(\mathbf{r},\mathbf{v})+R_c(\mathbf{r},\mathbf{v}). \quad (3.6)$$

R_i is simply the product of the ionization rate per unit volume $K(\mathbf{r})$ and a normalized ion velocity distribution:

$$R_i(\mathbf{r},\mathbf{v})=K(\mathbf{r})f_1(\mathbf{v}). \quad (3.7)$$

In Eq. (3.7) the distribution $f_1(\mathbf{v})$ of newly formed ions ("new" ions) will be very nearly the same as the velocity distribution of the neutral atoms themselves.

To determine the form of the collision term R_c, let ρ be the number density of ions in a particular volume of phase space, d^3vd^3r. In a steady-state plasma, $\partial\rho/\partial t=0$, and the various terms contributing to $\partial\rho/\partial t$ will be considered.

First of all, ions enter and leave a velocity element d^3v becauase of the action of the axial and radial electric fields in accelerating ions. However, by taking ion velocity distributions which contain a superimposed average drift velocity in addition to the usual thermal velocities the average effects of the fields are accounted for. Then $\partial\rho/\partial t$ may be treated as if there were no accelerating fields present.

Next, there are terms contributing to $\partial\rho/\partial t$ from newly formed ions [accounted for by the term R_i of Eq. (3.7)]. However, since the spatial ion density in d^3r does not change in time, ions must be leaving d^3r at a rate equal to the volume ionization rate, $K(\mathbf{r})$. This occurs by an outward radial drift and diffusion of ions which causes a net ion loss rate from the volume d^3r. However, the ions newly formed by ionization have very accurately a thermal velocity distribution at the neutral gas temperature T_a (plus the small neutral gas drift velocity); the ions leaving the volume d^3r have a thermal velocity distribution at the ion temperature T_i (plus the over-all drift velocity common to all the ions in steady state). Because of the difference of drift velocities, these two rates do not exactly cancel and lead to some rearrangement of ions in velocity space.

Finally, there are two terms contributing to $\partial\rho/\partial t$ which are caused by ion-atom collisions. The term giving the rate of production of ions into $d^3v\ d^3r$ is merely the desired term R_c; however, it would be difficult to write down its form directly without a detailed consideration of the elastic collision processes. The other term is the rate of loss of ions in phase space due to collisions with neutral atoms, which is the product of the density of ions in $d^3v\ d^3r$, $n_i(\mathbf{r})f_2(\mathbf{r},\mathbf{v})$, and the collision frequency for an ion of velocity $\mathbf{v}$, v/λ_+. The ion velocity distribution is denoted $f_2(\mathbf{r},\mathbf{v})$, and it will be taken as Maxwellian at the ion temperature T_i, but displaced in velocity space by the mean ion drift velocity. Since the radial component of the ion drift velocity changes with the radial coordinate r the velocity distribution is therefore dependent upon position $\mathbf{r}$, as indicated in its argument. The mean free path λ_+ will be assumed independent of ion velocity for the present purposes.

Thus one obtains

$$\begin{aligned}\partial\rho/\partial t&=0\\ &=K(\mathbf{r})f_1(\mathbf{v})-K(\mathbf{r})f_2(\mathbf{r},\mathbf{v})+R_c(\mathbf{r},\mathbf{v})\\ &\quad-(v/\lambda_+)n_i(\mathbf{r})f_2(\mathbf{r},\mathbf{v}). \quad (3.8)\end{aligned}$$

To compare terms in this equation, note that the ionization rate $K(\mathbf{r})$ is proportional to the local electron density, and that its integral over the discharge volume cross section must be equal to the ion loss rate at the walls. With Francis's equation for the wall current[22] one therefore obtains

$$K(\mathbf{r})=5.79(\mu_+kT_e/eR^2)n_e(\mathbf{r}),$$

valid in the case of ambipolar diffusion. To estimate the size of this term one uses the Langevin expression[23] for the ion mobility μ_+. One finds for the relative size of the terms in Eq. (3.8) the expression

$$K(r)/[vn_i(r)/\lambda_+]\approx 2.9(T_e/T_i)(\lambda_+/R)^2. \quad (3.9)$$

Since $\lambda_+\ll R$ was assumed for this derivation, one finally obtains from Eq. (3.8) the desired expression $R_c(\mathbf{r},\mathbf{v})$ which was required in Eq. (3.6):

$$R_c(\mathbf{r},\mathbf{v})\approx n_i(\mathbf{r})(v/\lambda_+)f_2(\mathbf{r},\mathbf{v}). \quad (3.10)$$

The ions produced by collisions described by Eq. (3.10) will be referred to as "old" ions (in contrast to the "new" ions just formed by ionization).

[22] Gordon Francis, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. XXII, p. 124.

[23] Earl W. McDaniel, *Collision Phenomena in Ionized Gases* (John Wiley & Sons, Inc., New York, 1964), p. 429.

In calculating $\partial\rho/\partial t$ above the effect of ion-electron collisions was neglected. These do not affect the ion velocity distribution significantly in most gas discharges, including those of concern here, as evidenced by the fact that the ion mobility in such discharges does not depend upon the electron density.

The neglect of ion-ion collisions deserves some comment. The most important point to notice about these collisions is that they do not change the mean axial momentum of a given ion, and therefore cannot greatly influence the transfer of axial momentum from ions to neutral atoms. However, one should also note that the ion-ion momentum-transfer cross section is inversely proportional to the fourth power of the relative velocity.[24] Therefore the cross section will be greatly reduced for a high ion kinetic temperature compared with its value for a plasma near room temperature.

It will be shown later that the force arising from the ionization term R_i in Eq. (3.6) is smaller than the force caused by the collision term R_c, and an estimate will be obtained for its magnitude. For the present, neglect the ionization term entirely in Eq. (3.6) and use Eqs. (3.5) and (3.10) to write

$$Fd^3vdt = v^2 n_i(\mathbf{r}+\mathbf{v}t) f_2(\mathbf{r}+\mathbf{v}t,\,\mathbf{v})(-eEt/\lambda_+^2) \times \exp(vt/\lambda_+)d^3vdt. \quad (3.11)$$

Then the total force per unit volume F_t acting on the gas at position r due to ions is

$$F_t = \int d^3v \int_{-\infty}^{0} dtF = \int d^3v \int_{-\infty}^{0} dtv^2 n_i(\mathbf{r}+\mathbf{v}t) f_2(\mathbf{r}+\mathbf{v}t,\,\mathbf{v}) \times(-eEt/\lambda_+^2)\exp(vt/\lambda_+). \quad (3.12)$$

A useful physical check is provided by the fact that

$$F_t = eEn_i, \quad (3.13)$$

when n_i is independent of r; that is, when the ion density is uniform, all the force exerted on the ions by the longitudinal field is eventually passed on to the neutral gas atoms. This is of course consistent with the previous work, such as Eq. (2.1).

The natural approximation one would make for $f_2(\mathbf{r},\mathbf{v})$ would be a Maxwellian distribution at the ion temperature, displaced by the mean ion drift velocity. Because the axial drift velocity of the ions is independent of radius at small radii, and though variable is much less than the radial drift velcoity at large radii, its effect on $f_2(\mathbf{r},\mathbf{v})$ may be neglected. Thus, for ions with radial drift velocity $\mathbf{v}_{dr}(r)$, assume for $f_2(\mathbf{r},v)$ the displaced Maxwellian

$$f_2(\mathbf{r},\mathbf{v}) = (m_i/2kT_i\pi)^{3/2} \times\exp(-m_i|\mathbf{v}-\mathbf{v}_{dr}(r)|^2/2kT_i). \quad (3.14)$$

This velocity distribution is the simplest plausible form which, when substituted into Eq. (3.12) yields the correct force [given by Eq. (3.13)] in the limit of uniform density. That it is a reasonable approximation may be seen by using it to calculate the total ion production rate:

$$R_{\text{total}} = \int d^3vR(\mathbf{r},\mathbf{v}) = \lambda_+^{-1}n_i v_{dr}[(1/x\sqrt{\pi})\exp(-x^2) + \tfrac{1}{2}(2+x^{-2})\,\text{erf}x], \quad (3.15)$$

where $x = v_{dr}/v_{\text{th}i}$ and $v_{\text{th}i} = (2kT_i/m_i)^{1/2}$. This has exactly the correct limiting forms for the effective ion-atom collision rate for momentum transfer:

$$R_{\text{total}} \to n_i v_{dr}/\lambda_+ \quad \text{as} \quad v_{dr} \to \infty \quad (3.16)$$

and

$$R_{\text{total}} \to (n_i/\lambda_+)(2v_{\text{th}i}/\sqrt{\pi}) = n_i\bar{v}_{ai}/\lambda_+ \quad \text{as} \quad v_{dr} \to 0, \quad (3.17)$$

since

$$\bar{v}_{ai} = (2/\sqrt{\pi})v_{\text{th}i} \quad (3.18)$$

when neutral-atom motion is neglected.

Now one must put in the dependence of f_2 on $\mathbf{r}$ arising in Eq. (3.14) through the radial drift velocity. Not only must the variation in the magnitude of $\mathbf{v}_{dr}$ with $\mathbf{r}$ be considered, but also the change in direction of $\mathbf{v}_{dr}$ between the position $\mathbf{r}$ where the force is calculated and the position $\mathbf{r}+\mathbf{v}t$ from which the ion comes. Choose polar coordinates v, θ, and φ for the v integration, with the polar axis directed radially outward at the point $\mathbf{r}$, and with the discharge tube axis located at $\varphi=0$. Then

$$|\mathbf{v}-\mathbf{v}_{dr}(r)|^2 = v^2 + v_{dr}^2(r) - 2vv_{dr}(r)(vt\sin^2\theta\sin^2\varphi + vt\cos^2\theta + r\cos\theta) \times[(vt\sin\theta\sin\varphi)^2 + (vt\cos\theta+r)^2]^{-1/2}. \quad (3.19)$$

When $\lambda_+|\nabla n_i(\mathbf{r})| \ll n_i(\mathbf{r})$, as will be assumed here, one may expand $n_i f_2$ in a Taylor series about the point $\mathbf{r}$, taking $n_i(\mathbf{r})$ to be only a function of r:

$$n_i(\mathbf{r}+\mathbf{v}t)f_2(\mathbf{r}+\mathbf{v}t,\,\mathbf{v}) = (m_i/2kT_i\pi)^{3/2} \times\exp[-m_i(v^2 - 2vv_{dr}(r)\cos\theta + v_{dr}^2(r))/2kT_i] \times\{n_i(r) + vt\cos\theta[dn_i(r)/dr] + 2v_{dr}(r)t(m_iv^2/2kT_i)\sin^2\theta\sin^2\varphi[n_i(r)/r] + 2n_i(r)[dv_{dr}(r)/dr](m_i/2kT_i)vt \times\cos\theta[v\cos\theta - v_{dr}(r)] + \cdots\}. \quad (3.20)$$

Substitution into Eq. (3.12) then gives

$$F_t(\text{old}) = eE\{n_i(r) - s_+(r)[dn_i(r)/dr] - s_+(r)[n_i(r)/r] - [ds_+(r)/dr]n_i(r) + \cdots\}, \quad (3.21)$$

[24] David J. Rose and Melville Clark, Jr., *Plasmas and Controlled Fusion* (The MIT Press, Cambridge, Mass., 1961), p. 163.

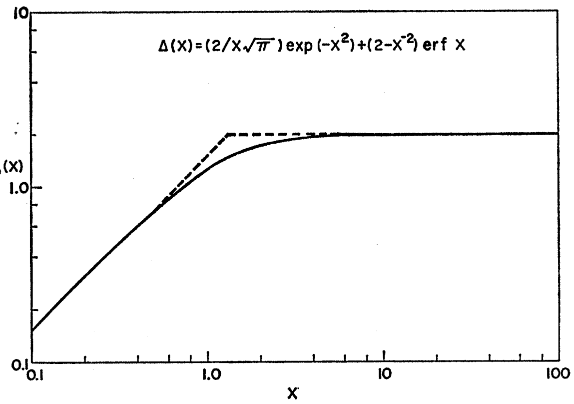

FIG. 2. Plot of $\Delta(x)$. The dashed lines are extrapolated from the limits at small and large x.

in which

$$s_+=\lambda_+\Delta(x)\,, \tag{3.22}$$

$$\Delta(x)=(2/x\sqrt{\pi})\,\exp(-x^2)+(2-x^{-2})\,\mathrm{erf}x\,, \tag{3.23}$$

and

$$x(r)\equiv v_{\mathrm{d}r}(r)/v_{\mathrm{th}}=(mv_{\mathrm{d}r}^2(r)/2kT)^{1/2}. \tag{3.24}$$

The meaning of the designation "old" will be explained shortly.

The function $\Delta(x)$ is plotted in Fig. 2. Its asymptotic limits are

$$\Delta(x)\rightarrow 8x/3\sqrt{\pi}=1.51x\,,\quad \text{when}\quad x\rightarrow 0 \tag{3.25}$$

and

$$\Delta(x)\rightarrow 2\,,\quad \text{when}\quad x\rightarrow \infty\,. \tag{3.26}$$

A similar result is valid for electrons, but, as previously stated, electron radial motion may usually be neglected because $\tau_-\ll\tau_+$.

The changes which must be made in Eq. (2.3) to include this result are simply to replace the ion density $n_i(r,z)$ by the term in braces in Eq. (3.21), and similarly for the electron term. This yields an equation that is just the lowest-order expansion of Eq. (3.1) in powers of s/r, with $s_\pm$ being given by Eqs. (3.22)–(3.26). The third term of Eq. (3.21), which leads to the cylindrical volume element in Eq. (3.1), is of course absent in the analogous problem using plane-parallel geometry.

B. Corrections Due to Newly Ionized Atoms

Now determine the correction which is introduced by including the term involving ionization in Eq. (3.6). Refer to ions arriving at **r** without suffering any collisions since their formation by ionization as "new" ions; they produce the first term in Eq. (3.6). Ions arriving at **r** which have just experienced an elastic collision with a neutral atom will be called "old" ions; they produce the second term in Eq. (3.6).

Now note this fact: If "new" ions and "old" ions each gave the same average amount of axial momentum to neutral atoms at **r**, the total force per unit volume F_t would have to depend on the ion density *only* at position **r**. This should be clear, since F_t would just be given by the average momentum transferred times the collision rate of ions and neutral atoms at position **r**, and this depends only upon their densities at **r**. Thus, since Eq. (3.21) gives an expression like

$$F_t(\mathrm{old})=eE[n_i(r)-\Delta n_i] \tag{3.27}$$

for the force due to "old" ions alone, the "new" ions in that case would yield just

$$F_t(\mathrm{new})=eR\Delta n_i. \tag{3.28}$$

Physically, the volume production term consisting of the "new" ions is responsible for the ion spatial distribution, and therefore must be just sufficient to maintain it against the effects of drift and diffusion. If the effects of the ionization term in Eq. (3.6) were included accurately, but the momentum transferred in Eq. (3.5), $(-eEt)$, were replaced by some average value, the ion production rate could be expressed in terms of the ambipolar diffusion coefficient and hence in terms of the gradient of the ion density, and the total force F_t obtained would be simply proportional to $n_i(\mathbf{r})$.

The key point here, which has been overlooked in previous theories, is that the "new" ions give much less momentum to the gas on the average than the "old" ions do. The new ions have not yet acquired their full radial and axial drift velocities by the time they have their first collision; thus in reality the effects of the ionization term in Eq. (3.6) only partially offset the collision term, and quite a large residual force still remains. Now estimate the magnitude of the correction in the limits of small x and large x. The discussion will be guided by the heuristic derivation of Eq. (3.1).

When x is small (high-pressure limit), the ion radial drift velocity is much smaller than the average ion thermal velocity, so that the ion thermal velocity controls the ion mean free time. Since new ions and old ions have approximately the same thermal velocity (because $T_i\approx T_a$, and the new ions have roughly the neutral atom temperature), the average time each has traveled when it reaches **r** since its origination is the same interval τ. The average old ion loses one-half its radial drift velocity at each collision; thus it starts with velocity $\frac{2}{3}v_{\mathrm{d}r}$ after its last collision with a neutral atom and arrives at **r** with velocity $\frac{4}{3}v_{\mathrm{d}r}$. A new ion, on the other hand, has no radial drift velocity when first formed and thus arrives at **r** with radial velocity $\frac{2}{3}v_{\mathrm{d}r}$. It is easy to show from these considerations that if the average old ion arrives at **r** having traveled radially outward a distance s since its last collision, the average new ion arriving has only traveled radially a distance of $\frac{1}{3}s$ since its formation. Thus Eq. (3.28) should be changed by replacing s_+ by $\frac{1}{3}s_+$ [using Eqs. (3.21) and (3.27) to define Δn_i]. This would then yield $F_t(\mathrm{new})=\frac{1}{3}eE\Delta n_i$. However, another correction must still be made. It was assumed in Eq. (3.5) that all the axial momentum gained by the old ion since its last collision was transferred to the neutral atom upon collision at **r**. As previously pointed out in Ref. 20, this statement is really only true on the average, since an old ion originates with some axial momentum, gains more on the trip to **r**, and loses (on the average) half of its total axial momentum upon making an elastic collision, which must leave it on the average with the same axial momentum it had following its previous collision. However, a new ion is formed with *no* axial momentum (the small average axial momentum carried by new ions is, in fact, just the axial momentum of the neutral gas atoms as modified by the electron collision producing the ionization). Therefore, instead yielding *all* the axial momentum it has gained upon collision at **r**, it can give up on the average only half that amount. This reduces the contribution to the force due to new ions by this additional factor, and one finally obtains

$$F_t(\text{new}) \approx (\tfrac{1}{6})eE\Delta n_i, \qquad (3.29)$$

when x is small. Thus, for small x, the right-hand side of Eq. (3.22) should be multiplied by $\frac{5}{6}$ to allow for new ions.

When x is large, which is important experimentally at lower values of pR, the ion thermal velocities may be neglected compared to the ion radial drift velocity, and the correction required is larger. Now both new ions and old ions move outward radially an effective distance of $2\lambda_+$ between their origination and their arrival at **r**, as shown by Eqs. (3.1), (3.22), and (3.26). The old ions fall through a radial potential that increases their velocity from $\frac{2}{3}v_{dr}$ to $\frac{4}{3}v_{dr}$. The new ions are formed with no radial drift velocity, and, falling through the same potential, acquire a radial velocity of $(2/\sqrt{3})v_{dr}$. The average time spent by the old ions in gaining axial momentum between their last previous collision and their arrival at **r** is thus $t=2\lambda_+/v_{dr}$, but the new ions spend a longer time, by a factor of $\sqrt{3}$. However, as pointed out in the previous paragraph, the old ions on the average give up all the axial momentum gained during this time, but the new ions give up only one half. Thus one obtains for the new ion contribution, at large values of x,

$$F_t(\text{new}) \approx (\tfrac{1}{2}\sqrt{3})eE\Delta n_i. \qquad (3.30)$$

Thus for large x, the right-hand side of Eq. (3.22) should be mulitplied by $(1-\frac{1}{2}\sqrt{3})\approx 0.134$.

Thus the effects of "new" ions can be allowed for by replacing Eq. (3.22) by

$$s_+=\lambda_+\Delta'(x), \qquad (3.31)$$

in which $\Delta'(x)$ is a function with asymptotic limits

$$\Delta'(x)\rightarrow \tfrac{5}{6}\Delta(x), \quad \text{when} \quad x\rightarrow 0, \qquad (3\ 32)$$

and

$$\Delta'(x)\rightarrow (1-\tfrac{1}{2}\sqrt{3})\Delta(x), \quad \text{when} \quad x\rightarrow \infty. \qquad (3.33)$$

It will be desirable to interpolate between these forms when predictions are made for comparison with experiment. For this purpose the following analytic approximation is useful:

$$\Delta'(x)\approx \Delta(x)-(\tfrac{1}{2}\sqrt{3})\Delta(x/3\sqrt{3}). \qquad (3.34)$$

This approximation not only has the correct limiting forms, but also another intuitively correct property: Even at the largest values of x (the largest values of v_{dr}), the new ions have no radial drift velocity when first formed, and thus at least for a short time their differential collision rate is determined by their thermal velocities rather than their radial drift velocity. Thus their collision rate is slightly increased even at large values of x over the collision rate appropriate to the old ions, and they will not travel quite as far radially as the old ions in a mean free time. In other words, as the pressure is decreased (that is, as v_{dr} is increased), the average radial distance traveled by the old ions between collisions should approach the limiting value $2\lambda_+$ more quickly than the corresponding distance for the new ions (which is the mean distance they travel radially between their formation and their first collision). Since Eq. (3.34) states that old ions travel a distance $\lambda_+\Delta(x)$ but that new ions travel a distance of only $\lambda_+\Delta(x/3\sqrt{3})$, this feature of the approximation (3.34) is also qualitatively correct.

Since the correction introduced by the term involving new ions is appreciable in one of the asymptotic limits (low pR), it would be desirable to give a more exact treatment of this term in Eq. (3.6). However, it is clear that this term depends crucially upon the changes in the ion velocity distribution between collisions. A treatment taking account of the ion acceleration terms in detail and the resulting curvature of ion trajectories would be rather difficult and has not been attempted. It is felt that this discussion, although heuristic, provides a good estimate for the correction by relating the new term directly to the carefully treated contribution from the old ions, and that therefore these limiting forms are probably fairly accurate. Moreover, in argon at least, there is less than a 25% difference in using $\Delta'(x)$ and $\Delta(x)$ in the region where the viscous flow criterion is satisfied (above $pR\sim 1$ Torr mm).

C. Corrections Due to Charge-Exchange Collisions

When charge-exchange collisions between atoms and ions occur, some further changes must be made in this analysis. For this discussion, a charge-exchange collision will be considered to change the charge state but not the velocities of the colliding particles.

It is not a convenient approach to redefine $R_e(\mathbf{r},\mathbf{v})$ [in Eq. (3.6)] to be the rate of formation of ions in both charge-exchange and elastic collisions. Not only do charge-exchange collisions produce ions with no radial drift velocity, which must then be accelerated; such collisions also produce neutral atoms with radial drift velocities characteristic of ions, and these drift velocities must be dissipated through collisions. Treating the problem by this approach would require putting in the acceleration of the ions produced by charge-exchange collisions in the radial electric field, and the resulting curvature of ion path. It would also require treating the loss of radial drift momentum by neutrals.

Instead, continue to define R_c as the rate of formation of ions through elastic collisions, so that the ions this term represents have their full share of radial drift velocity. In Eq. (3.10), this requires using, instead of λ_+, the mean free path against elastic collisions only, λ_{el}. Since the electric fields are included by taking an average drift velocity for the ions, ion acceleration need not be considered; thus those ions which are formed without drift velocity do not acquire drift velocity and lead to no axial force on the atoms.

Now consider what happens if an ion originating in an elastic collision at $\mathbf{r}+\mathbf{v}t$ suffers one or more charge-exchange collisions on its way to $\mathbf{r}$. Since the effects of the accelerating fields have been taken out by giving the ions an average drift velocity, charge-exchange collisions do not affect the velocity and path of an ion, but only its momentary charge. Thus, if the ion becomes a neutral atom on its way to $\mathbf{r}$, it leads to the same arrival flux at $\mathbf{r}$ as if no charge-exchange occurred—except that part of the arriving flux now has zero charge instead. Thus the probability of arrival at $\mathbf{r}$ without loss of momentum, which was given by the factor $\exp(vt/\lambda_+)$ in Eq. (3.4), should instead be $\exp(vt/\lambda_{el})$—thus charge-exchange collisions are not considered an interruption of the flight of the ion.

Finally, one must consider possible changes in Eq. (3.5). Now the arriving flux $\mathbf{dJ}$ consists of two parts; a fraction a_+ consists of ions, and a fraction a_0 consists of neutrals:

$$dJ=a_+dJ+a_0dJ. \qquad (3.35)$$

It may be shown from elementary considerations that because of charge-exchange collisions an atom that starts from $\mathbf{r}+\mathbf{v}t$ as an ion has a probability

$$a_+=\tfrac{1}{2}[1+\exp(-2vt/\lambda_{ce})] \qquad (3.36)$$

of being an ion upon arrival at $\mathbf{r}$, with λ_{ce} defined as the mean free path against charge-exchange collisions. Since ions and neutrals may have different elastic momentum-transfer cross sections for collisions with neutral atoms, the collision rate per neutral gas atom at $\mathbf{r}$ is given now by a_+dJ/λ_{el} for collisions with arriving ions, and by $a_0dJ/\lambda_a=(1-a_+)dJ/\lambda_a$ for collisions with arriving neutrals produced by charge exchange. As usual, λ_{el} is the momentum-transfer mean free path for ions against elastic collisions, and λ_a is the momentum-transfer mean free path for neutrals in collisions with neutrals.

In two useful cases one can obtain a simpler expression for the collision rate per neutral gas atom. One is, of course, when one may take $2vt\ll\lambda_{ce}$. In terms of macroscopic discharge parameters, this requires

$$v_{thi},\ v_{dr}\ll\lambda_{ce}/2\tau_+, \qquad (3.37)$$

with τ_+ the ion mean free time, as before. In this case $a_0\approx 0$, $a_+\approx 1$, and the neutral-atom collision rate at $\mathbf{r}$ is just dJ/λ_{el}. The other special case (and one which applies to argon, for example) is when

$$\lambda_{el}\approx\lambda_a. \qquad (3.38)$$

In this case, since $a_0+a_+=1$, the neutral-atom collision rate is again dJ/λ_{el}.

The other term required to include charge exchange in Eq. (3.5) is the average amount of axial momentum given up per collision. This momentum is again $(-eEt)$, as in Eq. (3.5), since the total rate at which axial momentum is acquired from the field is not changed, even though part of this momentum is vested in neutral atoms rather than in ions because of charge exchange. Thus, when either Eq. (3.37) or Eq. (3.38) is satisfied, Eq. (3.5) may be used even when charge exchange collisions occur, merely by replacing λ_+ by λ_{el}.

The result of including ion-atom charge-exchange collisions is therefore to replace λ_+ by the momentum-transfer mean free path against elastic collisions λ_{el} in Eqs. (3.4), (3.5), and (3.10), whenever either Eq. (3.37) or Eq. (3.38) holds. Thus all the results previously derived may be used provided that Eq. (3.31) is replaced by

$$s_+=\lambda_{el}\Delta'(x). \qquad (3.39)$$

D. Summary of Volume Forces

In summary, we have shown in this section that a more correct treatment of the forces operating within the volume of the plasma is to replace Eq. (2.3) by Eq. (3.1), using Eqs. (3.31), (3.34), (3.23), and (3.24) to evaluate the quantity s_+. When charge-exchange collisions are important, and either Eq. (3.37) or Eq. (3.38) holds, Eq. (3.31) is replaced by Eq. (3.39).

To do the integrals in Eq. (3.1), an expression is required for the radial drift velocity $v_{dr}(r)$ appearing in Eq. (3.24). At any radial position this velocity has two components, arising from diffusion and from the radial electric field[25,26]:

$$v_{dr}(r)=-[D_+/n_i(r)][dn_i(r)/dr]+\mu_+E_r(r), \qquad (3.40)$$

where the ion-diffusion constant D_+ and mobility μ_+ are evaluated at radius r. The relation[26] $D_+/\mu_+=kT_i/e$ (T_i is the ion kinetic temperature) and Eqs. (2.7) and (2.8) allow this to be written

$$\begin{aligned}v_{dr}&=-(\mu_+/e)k(T_i+T_e)[1/n_i(r)][dn_i(r)/dr]\\&=+\mu_+E_r(r)[1+(T_i/T_e)]\\&=(2.4048/R)(\mu_+/e)k(T_i+T_e)\\&\qquad\times[J_1(2.4r/R)/J_0(2.4r/R)]. \qquad (3.41)\end{aligned}$$

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[25] Earl W. McDaniel, *Collision Phenomenon in Ionized Gases* (John Wiley & Sons, Inc., New York, 1964), p. 513.

[26] Gordon Francis, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. XXII, pp. 123–124.

Close to the positive-ion sheath, v_{dr} becomes very large, but not infinite, of course, as discussed when treating E_r in Sec. II.

The integrals of Eq. (3.1) were performed by analytic approximation for the two cases in which x is either much greater or much less then unity over most of the discharge volume, assuming $s_- \ll s_+ \ll R$. The result, which is known to vary smoothly as a function of the relevant parameters, was connected between these limits after the manner of Eq. (3.34). This leads to the following expressions for the gas pumping due to volume forces in the plasma:

$$V_1'(z) = V_1(z) + V_3(z), \tag{3.42}$$

in which $V_1(z)$ is given by Eq. (2.9), and

$$V_3(z) = (2\pi eE/\eta)(2.3163)n_e(0.054R^3\lambda_+)\Delta'(\bar{x}). \tag{3.43}$$

$V_1'(z)$ should be used in Eq. (2.2) instead of $V_1(z)$. The mean value of x for the discharge $\bar{x}$ is defined by

$$\bar{x} \equiv v_{dr}(0.740R)/v_{thi} = x(0.740R). \tag{3.44}$$

When evaluating $\bar{x}$ using Eq. (3.41) note that

$$J_1(0.74)/J_0(0.74) = 1.66. \tag{3.45}$$

The new term $V_3(z)$ is always non-negative, i.e., always leads to a flow of neutral gas towards the anode.

A result which can be shown to be essentially equivalent to the high-pressure ($\bar{x} \to 0$) limit of Eq. (3.43) was independently obtained by Leiby[1,19] by a completely different method, starting with the Boltzmann transport equation. His derivation in effect treats ion radial motion to lowest order. However, to correctly treat the case in which the ion radial drift velocity can be an appreciable fraction of the ion thermal velocity it is necessary to examine the collisions in detail, as was done in this section.

IV. FORCES NEAR THE WALL

This section will be primarily concerned with the forces acting on the neutral gas within a few ion mean free paths of the tube wall.

It was stated at the beginning of Sec. III that the principal defect of earlier theories was the neglect of ion radial drift motion. However, another shortcoming may be seen by examining the form of Druyvesteyn's force, Eq. (2.1), near the tube wall.

It is easy to see that Eq. (2.1) must be incorrect, even when radial drift motions are neglected compared with thermal velocities.[27] That equation says that the force per unit volume on the neutral gas at position r, caused by collisions with a *single ion* at that radial position, is the quantity

$$eE(z)(1-\exp[-(R-r)/\lambda_+(z)]).$$

[27] This fact was first noticed by E. I. Grodon (private communication).

However, this force *per ion* vanishes as r approaches R because of the exponential factor. But this force cannot vanish, since even ions arriving at the wall must be continually receiving momentum from the axial field and transferring it to neutral atoms through collisions.

The correct form for the force on the neutral gas due to collisions with ions may be obtained by a modification of the derivation of Sec. III. Consider Eq. (3.12), letting the point r of that equation be a distance d from the tube wall, with $d \ll R$. Ions arriving at $\mathbf{r}$ with the radial component of their velocities directed outward have traveled a distance $|vt|$ since their origination. Since the tube is large compared to the ion mean free path, the limits on the t integral are essentially $-\infty$ to 0, as in Eq. (3.12).

However, ions arriving at r with the radial component of their velocities directed inward cannot have originated further away than the tube wall, since ions lose their axial momentum at the tube wall. Since the ion has traveled a radial distance $|v_r t| = |vt| \cos\theta$, the time elapsed since its origination must be restricted to keep this distance less than or equal to the distance d from position $\mathbf{r}$ to the tube wall. Thus for ions with these velocities the limits on the t integral must be $-d/v\cos\theta$ to 0.

Thus, near the tube wall, Eq. (3.12) is replaced by

$$F_t(\text{near wall}) = \int_0^\infty v^2 dv \int_0^{2\pi} d\varphi \bigg[\int_0^{\pi/2} \sin\theta d\theta \int_{-\infty}^0 dt + \int_{\pi/2}^{\pi} \sin\theta d\theta \int_{-d/v\cos\theta}^0 dt\bigg] v^2 n_i(\mathbf{r}+\mathbf{v}t) \times f_2(\mathbf{r}+\mathbf{v}t, \mathbf{v})(-eEt/\lambda_+^2)\exp(vt/\lambda_+). \tag{4.1}$$

The calculation is simplified by the fact that near the wall only the first term in the Taylor expansion of $n_i f_2$ need be kept. Equation (4.1) then yields for the ion force

$$F_t(\text{near wall}) = n_i(r)eE(1-\alpha), \tag{4.2}$$

in which α is the dimensionless quantity

$$\alpha = (4/\pi)^{1/2} \int_0^\infty y^2 dy \int_0^1 du(1+a/u) \times \exp(-x^2-2uxy-y^2-a/u). \tag{4.3}$$

In this equation, x is the qunatity $x(r)$ of Eq. (3.24), applied in this case to the ions, and a is defined as

$$a \equiv d/\lambda_+ = (R-r)/\lambda_+. \tag{4.4}$$

At higher pressures one may take $x \to 0$ and Eq. (4.2) for the ion force becomes

$$F_t(\text{near wall}) = n_i(r)\{1-\tfrac{1}{2}\exp[-(R-r)/\lambda_+]\}eE, \tag{4.5}$$

which differs from Druyvesteyn's theory [Eq. (2.1)] by the presence of the factor $\frac{1}{2}$. This not only overcomes the objection to the use of Eq. (2.1) expressed at the beginning of this section, but is physically reasonable, since a neutral gas atom located near the wall only sees ions arriving from one-half the solid angle because of the presence of the wall.

The result expressed in Eq. (4.2) should be physically clear to the reader in the limit that the ions have no thermal velocities at all, but only a uniform radial drift motion. The following observation will be found helpful: It is true that within a mean free path of the wall, the momentum given to the ions by the axial field is transmitted by them to the walls rather than to the neutral atoms. However, the axial momentum received by ions at smaller radii will be given to neutral atoms in the region close to the wall on their way to the wall.

When the ion radial drift velocity becomes large compared with the ion thermal velocity, as occurs at lower pressures, $x\rightarrow\infty$. In this limit Eq. (4.2) becomes

$$F_t(\text{near wall})=n_i(r)eE. \tag{4.6}$$

If this limit were used for the ion force in Eq. (2.1), the principal pumping term treated by earlier theories, Eq. (2.4), vanishes entirely. Thus the effect of ion radial motion can be to eliminate the wall forces previously thought to dominate the gas pumping.

By combining the treatment of Sec. III with that presented here one obtains a more accurate expression for the gas pumping near the tube wall. The result is that $V_2(z)$ in Eq. (2.2) should be replaced by $V_2'(z)$, with

$$V_2'(z)=(2\pi/\eta)\int_0^R r''dr''\int_{r''}^R (dr'/r') \times\int_0^{r'}(r-s_+)dr\,n_i(r-s_+,z)\alpha eE(z). \tag{4.7}$$

At lower pressures, where x may be taken large in Eq. (4.3), $V_2'(z)$ vanishes entirely because of the factor α. At higher pressures, where x is small, a simple change of variables in Eq. (4.7) yields, to lowest order in λ_+/R,

$$V_2'(z)=V_2(z)\quad\text{at high pressure}. \tag{4.8}$$

[The factor of $\frac{1}{2}$ in α turns out to be effectively canceled by the factor of 2 in s_+ which arises from Eq. (3.26).] It can also be shown that, because $V_2'(z)$ increases with increasing s_+, Eq. (4.8) constitutes an upper bound for $V_2'(z)$.

Because $V_1'(z)$ is usually much larger than $V_2'(z)$ in practical cases, no attempt has been made to evaluate Eq. (4.7) numerically for intermediate or low pressures. Doing so accurately would be difficult because the main contributions to the integral come from regions close to the wall ($r\sim R-\lambda_+$), and in just these regions Eq. (3.41) for $v_{dr}(r)$ loses validity. Not only does the ion mobility used in that expression become dependent upon radial position because of the strong radial fields, but it may be questioned whether in this region of strong radial ion acceleration it is valid to replace the acceleration by a simple average drift velocity as was done in Eq. (3.14).

However, these objections do not apply to the case of higher pressure, where x is small and $\lambda_+\ll R$; at such pressures it appears to be valid to use Eq. (4.8) to evaluate $V_2'(z)$, with $V_2(z)$ being given by Eq. (2.11). Thus, as far as gas pumping near the wall is concerned, Druyvesteyn's omitted factor of $\frac{1}{2}$ in the exponential term in Eq. (2.1) is compensated by the effects of ion radial motion, and at higher pressures Eq. (2.4) may be retained as it stands.

V. EFFECTS OF THE ION SHEATH

An effect which may modify the results of the previous sections arises from the presence of the ion sheath at the wall of the discharge tube. The manner in which this sheath forms will be summarized briefly, assuming that electron and ion mean free paths are both much smaller than the tube radius, and assuming that the ion mean thermal velocity is much less than the electron mean thermal velocity. Within an ion mean free path of the wall one may think of ions as falling freely to the wall instead of slowly diffusing outward as they do near the center of the tube. The ion concentration near the center of the tube is depleted by the flow of ions to the wall, which is retarded only by the ion space charge near the wall. This charge builds up until it is sufficient to limit the ion wall current to the ion production rate in the tube. At the same time, within an electron mean free path of the wall electrons may fall freely to the wall, and in fact they do so until retarded by the electronic charge which builds up on the wall. The potential difference across the sheath is about $5kT_e/e$.[10,11,13]

Two conditions must be satisfied for equilibrium in the tube. First, the wall charge must build up until it is so high as to allow only the fastest electrons to reach the wall, since the ion and electron wall currents must be equal, and otherwise the much greater thermal velocities of the electrons would make the electron current much greater. Secondly, since there is approximate charge neutrality in the central regions of the tube ($r<R-\lambda_+$), the excess of ion charge near the wall must be approximately balanced by the excess electronic charge on the wall.

Solving Poisson's equation inside the sheath, assuming the ion current to be limited only by the space charge, yields Child's equation[28] for the ion wall current per unit length of the discharge:

$$i_W^+=2\pi R(4\epsilon_0/9)(2e/m_i)^{1/2}V_{\text{Wall}}^{3/2}\kappa^{-2} \times[1+2.66(eV_{\text{Wall}}/kT_i)^{-1/2}]. \tag{5.1}$$

[28] James Dillon Cobine, *Gaseous Conductors* (Dover Publications, Inc., New York, 1958), pp. 125, 126, and 136.

Here, κ is the distance from the wall at which the ion density would have to become infinite to sustain the wall current (at which distance, of course, ion collisions must be taken into account) and is to be identified as the approximate sheath thickness. This equation is strictly valid only when $\kappa \ll R$ because a factor correcting for the cylindrical geometry has not been included. The quantity V_{Wall} is the potential of the wall relative to the point $r=R-\kappa$. The sheath thickness may be either larger or smaller than the ion mean free path. It may be determined numerically by equating the wall current given by Eq. (5.1) to the appropriate expression obtained from theories of the positive column, at low,[11] intermediate,[13] or high[22] pressures.

In the sheath the Bessel function distribution of Eq. (2.8) for ions and electrons at higher pressures is no longer valid, since the radial diffusion equation is not valid within a mean free path of the wall. When the sheath thickness becomes an appreciable fraction of either the tube radius or the ion mean free path an accurate calculation of the gas pumping force requires a knowledge of the charged-particle densities and the radial electric field in the sheath. The idea is that the radial field in the sheath is very high and thus ions move through this region very rapidly on their way to the wall. Since they have not had as long an exposure to the axial electric field, they therefore arrive at the wall with less axial momentum than one would have expected in the absence of an ion sheath (process A). Moreover, charge neutrality does not hold in the sheath region, and because of the excess of ions there is a net force toward the cathode on the neutral gas in the sheath region (process B).

In the free-fall limit (low pressure), recent numerical computations by Parker[11] have yielded accurate curves for the potential and the charged-particle densities everywhere in the discharge, following the transition through the sheath in detail.

For the high-pressure case, as considered here, take for the electrostatic potential

$$\varphi(r)=(kT_e/e)\ln J_0(2.4r/R), \quad r<R-\epsilon \tag{5.2}$$

as in Eq. (2.7), and for the sheath solution[29]

$$\varphi(r)=V_{\mathrm{Wall}}\kappa^{-4/3}(r-R+\kappa)^{4/3}+V_\kappa, \quad R-\epsilon<r<R. \tag{5.3}$$

In the latter expression, V_κ is the potential at $r=R-\kappa$ relative to $r=0$, and V_{Wall} is the potential of the tube wall relative to the potential at $r=R-\kappa$. The value of ϵ is chosen to make the radial electric field continuous at $r=R-\kappa$. If the typical value[10,11,13] $V_{\mathrm{Wall}}\approx 5(kT_e/e)$ is chosen, this yields

$$\epsilon/\kappa\approx 0.16. \tag{5.4}$$

The ion and electron density are assumed to agree with the ambipolar case, Eq. (2.8), for $r<R-\epsilon$. In the sheath region the charged-particle densities are instead assumed to be

$$n_i(r)=\gamma(r+\kappa-R)^{-2/3}, \quad R-\epsilon<r<R \tag{5.5}$$

and

$$n_e(r)=0, \quad R-\epsilon<r<R \tag{5.6}$$

with

$$\gamma=(4\epsilon_0/9)(V_{\mathrm{Wall}}/e)^{-4/3}. \tag{5.7}$$

This is the simplest form for the charged-particle densities which appears to retain the essential physics of the sheath.

The effects of the ion sheath on gas pumping may now be determined by calculating V_1, V_2, and V_3 from Eq. (2.3), (2.4), (3.1), and (3.42), but using Eqs. (5.5) and (5.6) rather than the ambipolar solutions whenever $r>R-\epsilon$. The quantities so obtained will be denoted V_{1s}, V_{2s}, and V_{3s} to indicate that the sheath correction has been included. This paper will treat only the higher-pressure case, where $\lambda_+\ll R$ and $\kappa\ll R$, but will not restrict the ratio κ/λ_+. At lower pressures usually $\kappa\ll\lambda_+$, and the ion-sheath corrections may be neglected entirely.

As in Sec. II, $V_{1s}(z)$ is calculated from Eq. (2.6) rather than directly from Eq. (2.3). The result is that

$$V_{1s}(z)=[\ln(R/\epsilon)-1.02]V_1(z)/4.07, \tag{5.8}$$

yielding a major change in the small term $V_1(z)$. However, even with this change V_{1s} may usually be neglected.

For V_{2s} one obtains the somewhat more complicated form

$$V_{2s}(z)=-0.0025V_1(z) + V_2(z)(1+\epsilon\lambda_+^{-1}+0.5\epsilon^2\lambda_+^{-2})\exp(-\epsilon/\lambda_+). \tag{5.9}$$

The term involving V_1 in this equation represents process B [described preceding Eq. (5.2)] and the term involving V_2 and the exponential factor arises from process A.

Finally, as a consequence of the fact that $\Delta(x)$ can never exceed 2, one may show that

$$|V_{3s}(z)-V_3(z)|<|0.22(\lambda_+/\kappa)V_1(z)|. \tag{5.10}$$

In the frequently encountered case that $V_1(z)$ is much less than $V_2(z)+V_3(z)$, Eqs. (5.8)—(5.10) show that the ion sheath essentially affects only the gas pumping term $V_2(z)$, and that the effect of the ion sheath upon that term is a function only of the quantity $\epsilon/\lambda_+\approx 0.16\kappa/\lambda_+$. In the case of charge-exchange collisions λ_+ should be replaced by λ_{el}, as before.

VI. SUMMARY

The principal result of the work reported here is the discovery of a new physical effect, caused by the radial motion of ions, leading to gas pumping in gas discharges. This new effect, as well as those previously recognized, were treated in some detail in Secs. III and IV, and corrections due to the ion sheath were

[29] James Dillon Cobine, *Gaseous Conductors* (Dover Publications, Inc., New York, 1958), p. 125.

introduced in Sec. V. The result of this work is to replace the Druyvesteyn[3] expression, Eq. (2.2), by

$$V(z)=V_{1s}(z)+V_{2s}(z)+V_{3s}(z)-(2\pi/16\eta)R^4(\partial/\partial z)[n_a(z)kT_a(z)]. \quad (6.1)$$

Evaluating the quantities appearing in this expression requires Eqs. (5.8)–(5.10), (2.9), (2.11), (2.12), (3.43), (3.34), (3.23), (3.24), and (3.41). The gas flow and pressure difference between the ends of the discharge tube may then be obtained from Eqs. (2.15), (2.18), and (2.19). The following paper[30] compares the theory developed here with experimental results in gas discharges.

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[30] Arthur N. Chester, following paper, Phys. Rev. **169**, 184 (1968).

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Experimental Measurements of Gas Pumping in an Argon Discharge

ARTHUR N. CHESTER
Bell Telephone Laboratories, Murray Hill, New Jersey 07971

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Measurements have been made of the pressure difference between the cathode and anode bottles of a standard cw argon ion laser tube, with a positive column of 30-cm length and 1.25-mm radius. At several amperes current the pressure difference (anode minus cathode) is approximately constant at 0.25 Torr below 1.2 Torr, and is inversely proportional to pressure above that pressure. The results are compared with the predictions of various theories. The theory presented in the preceding paper accurately predicts both the magnitude of the pumping effect and its dependence upon pressure and current.

I. INTRODUCTION

It is well known that a pressure difference is developed between the anode and cathode in a dc gaseous plasma, but at the current densities usually used in past work this has been a small effect and difficult to measure. However, in high current density discharges ($1\ \mathrm{A/mm^2}$) in small bore tubes (a few millimeters radius) gas pumping by the plasma is so strong that an external gas return[1] connected between cathode and anode must be used to keep the discharge from extinguishing itself. Even with such a return path for the gas, the pressure at the anode often builds up to several times the cathode pressure.

Recent discharge diagnostic work on the argon ion laser provides a unique opportunity to study the gas pumping effect in detail at high current densities in small-radius tubes. Various groups have recently measured electron, ion, and neutral-atom kinetic temperatures, number density and radial distribution of ions and neutral atoms, and axial electric field under a great variety of discharge conditions. Such data make it possible to compare experimental measurements in argon with the predictions of various theories in some detail.

In Sec. II we discuss the range of discharge parameters for which theoretical treatments of the gas pumping effect should apply; these considerations show that the theory presented in the preceding paper should be applicable to the argon discharge. In Sec. III we gather together certain experimental measurements of discharge parameters which are seen to be necessary in calculations using the theory, and in Sec. IV we give specific numerical predictions for argon discharges in various pressure regimes. In Sec. V we describe the experimental tube used in the present investigation and compare measured gas pumping with the predictions of various theoretical models. A long-standing discrepancy between theory and early experimental measurements in argon is examined in the Appendix.

II. VALIDITY OF THEORETICAL MODELS IN ARGON DISCHARGES

It is desirable to determine the ranges of pressure, current, and tube radius for which various theoretical models for the gas pumping effect may apply, since direct comparison of theory and experiment is to be undertaken. Relevant theoretical treatments of the problem have been given by Langmuir,[2] Druyvesteyn,[3]

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[2] Irving Langmuir, J. Franklin Inst. **196**, 751 (1923).

[3] M. J. Druyvesteyn, Physica **2**, 255 (1935).