We have also assumed that the $I=J=1$ channel is dominated by the ρ meson and the $I=0, J=2$ channel is dominated by the f^0 meson. It is also of interest to examine the pion-pion cross sections predicted by our model. In Fig. 6 we show the total cross sections in the $I=J=0$ and $I=J=1$ channels. It is of special interest to note that for such a broad s-wave resonance, although the phase shift passes through 90° at the ϵ mass, there is no "peak" in the cross section, but only a "shoulder." Thus it is not possible to observe the ϵ by merely looking at the pion-pion center-of-mass energy distribution. Hence we must rely on indirect, model-dependent determinations of the pion-pion phase shifts until direct pion-pion scattering "experiments" are possible.[22]

While the ϵ will not manifest itself as a typical resonance "peak," we have seen that it is very useful to introduce the ϵ parametrization for the low-energy pion-pion interaction to describe a variety of experimental data.

It would be extremely useful to have accurate measurements of quantities from which one could infer pion-pion phase shifts in order to better understand the pion-pion interaction.

[22] P. L. Csonka, CERN Report No. TH 836, 1967 (unpublished).

PHYSICAL REVIEW VOLUME 169, NUMBER 5 25 MAY 1968

Solutions of the Faddeev Equation for Short-Range Local Potentials*

James S. Ball†
University of California, Los Angeles, California

AND

David Y. Wong
University of California at San Diego, La Jolla, California
(Received 22 January 1968)

A systematic method for solving the Faddeev equation for three bodies interacting through two-body local potentials is presented. This method is then applied to the problem of three identical particles interacting through a Yukawa potential, and the convergence of the method is studied numerically. Solutions are obtained for one particle scattering off a bound state of the other two, as well as for the three-particle bound-state case.

The application of the Faddeev equation to nonrelativistic three-body problems has been of considerable interest, as seen in the literature.[1–8] In the present work we address ourselves to the question of how one would solve the equations systematically once the two-body potentials are given, local or otherwise.

The angular momentum decomposition of the Faddeev equations was first treated by Ahmadzadeh and Tjon,[6] resulting in a set of coupled integral equations in two variables. If the two-body potentials are taken to be separable (nonlocal) as was considered by a number of authors,[2–6] then the integral equations reduce to one variable and can be solved by ordinary numerical methods. It was suggested by Zambotti and one of us (DYW)[7] that even if the potentials were local, it would still be convenient to expand the two-body T matrix as a sum of separable terms. A generalized effective-range-type expansion was introduced and applied to the Yukawa-potential problem.[7] Although the method proved to be useful, the treatment was not entirely systematic. In the present paper, we suggest a systematic way of expanding the two-body T matrix for the solution of the Faddeev equation below the three-body threshold and show that the expansion converges fairly rapidly for potentials characteristic of strong interactions. This method is applicable to the calculations of bound-state energies and wave functions as well as the scattering of a particle by a bound state of two other particles. The problem of extending beyond the three-particle threshold is discussed at the end. The treatment of Coulomb potentials is reported in a separate paper.[9]

Although the method outlined below is applicable to any angular momentum state of the three-body system, we consider, for convenience, only states corresponding to zero total angular momentum and no spin. For this three-body state, the Faddeev equation can be written

* Work supported in part by the U. S. Atomic Energy Commission.
† Alfred P. Sloan Fellow.
[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].
[2] C. Lovelace, Phys. Rev. **135**, B1225 (1964).
[3] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **140**, B1291 (1965).
[4] M. Bander, Phys. Rev. **138**, B322 (1965).
[5] R. Omnes, Phys. Rev. **134**, B1358 (1964).
[6] A. Ahmadzadeh and J. A. Tjon, Phys. Rev. **139**, B1085 (1965).
[7] D. Y. Wong and G. Zambotti, Phys. Rev. **154**, 1540 (1967).
[8] Thomas A. Osborn (to be published).
[9] J. S. Ball, J. Chen, and D. Y. Wong (to be published).

in the form

$$\Psi_l^{(i)}(p,q;s)=\Phi_l^{(i)}(p,q;s)+\sum_{j\neq i}\sum_{l'=0}^{\infty}\int_0^\infty dq_j^2\int_{L_{ij}}^{U_{ij}}dp_j^2\frac{(-1)^{l+l'}[(2l+1)(2l'+1)]^{1/2}P_l(z_{ij})P_{l'}(z_{ij}')t_l^{(i)}(p,p_i;s-q^2)}{4\pi\alpha_{ij}\beta_{ij}q(p_j^2+q_j^2-s)}\times\Psi_{l'}^{(j)}(p_j,q_j;s)\quad\text{for } i=1,2,3,\quad(1)$$

where $\Psi_l^{(i)}(p,q;s)$ is the three-body T-matrix element with particles j and k ($j\neq k\neq i$) undergoing final-state interaction with relative angular momentum l. The quantity p is proportional to the magnitude of the relative momentum of j and k, and q is proportional to the magnitude of the momentum of particle i, in the three-body center-of-mass (c.m.) frame. The initial state is arbitrary. $\Phi_l^{(i)}(p,q;s)$ is the corresponding T-matrix element in the absence of interactions between particle i and particles j and k. $t_l^{(i)}(p,p';E)$ is the two-body T-matrix element for the interaction of particles j and k with angular momentum l, normalized to $t_l^{(i)}(p,p;p^2)=(e^{i\delta_l}\sin\delta_l)/p$; p^2= c.m. energy. s is the total energy of the three particles in the c.m. system.

$$\alpha_{ij}=(m_im_j)^{1/2}/[(m_i+m_k)(m_j+m_k)]^{1/2};\ i\neq j\neq k$$
$$\beta_{ij}=(1-\alpha_{ij}^2)^{1/2},$$
$$p_i^2=p_j^2+q_j^2-q^2,$$
$$z_{ij}=(-1)^P[\alpha_{ij}^2(q_j^2-q^2)+\beta_{ij}^2(q^2-p_j^2)]/(2\alpha_{ij}\beta_{ij}p_iq),$$
$$z_{ij}'=(-1)^P[-q^2+\beta_{ij}^2p_j^2+\alpha_{ij}^2q_i^2]/(2\alpha_{ij}\beta_{ij}p_jq_j),$$
P = cyclic permutation of i, j,
$$U_{ij}=(\alpha_{ij}q_j+q)^2/\beta_{ij}^2,$$
$$L_{ij}=(\alpha_{ij}q_j-q)^2/\beta_{ij}^2.$$

It is clear that if $t_l^{(i)}(p,p_i;s-q^2)$ is expanded in a sum of terms separable in p and p_i, then the p dependence of $\Psi_l^{(i)}(p,q;s)$ is explicit (p does not appear in the kinematic functions z_{ij}, z_{ij}', or the limits of integration) and the integral equations (1) can be reduced to equations involving q only.

In a problem where the two-body potentials are given, the two-body T-matrix can be obtained from the solution of the Lippmann-Schwinger equation:

$$t_l(p,p';E)=V_l(p,p')+\frac{1}{\pi}\int_0^\infty dp''^2\frac{p''V_l(p,p'')t_l(p'',p';E)}{p''^2-E}.\quad(2)$$

Since the argument E is replaced by $(s-q^2)$ in the Faddeev equation, it is negative definite provided the three-body total energy s is below the three-body threshold ($s=0$). For negative values of E, the $(p''^2-E)^{-1}$ factor in (2) is nonsingular, and it is well known that the solution for t_l can be expressed in terms of eigenfunctions of the homogeneous equation as follows. The solutions ϕ_n of the homogeneous equation and the corresponding eigenvalues λ_n are defined by

$$\lambda_n(E)\phi_n(p,E)=\frac{1}{\pi}\int_0^\infty dp''^2\frac{p''V_l(p,p'')}{p''^2-E}\phi_n(p'';E),\quad(3)$$

with the orthonormality property

$$\frac{1}{\pi}\int_0^\infty dp''^2\frac{p''\phi_n(p'';E)\phi_m(p'',E)}{p''^2-E}=\delta_{nm}.\quad(4)$$

Since ϕ_n constitutes a complete set, the two-body T matrix can be expanded in the form

$$t_l(p,p';E)=\sum_n C_n(p';E)\phi_n(p;E).\quad(5)$$

Substituting (5) into (2) and making use of (3) and (4), one finds

$$t_l(p,p';E)=\sum_n\frac{\lambda_n(E)\phi}{1-\lambda_n(E)}\phi_n(p;E)\phi_n(p';E).\quad(6)$$

Having obtained $\phi_n(p;E)$ by solving (3), we can use the expression (5) for t_l in the Faddeev equation (1). Since the p dependence is now explicit, $\Psi_l^{(i)}(p,q;s)$ takes the form

$$\Psi_l^{(i)}(p,q;s)=\Phi_l^{(i)}(p,q;s)+\sum_n\frac{\lambda_{nl}^{(i)}(s-q^2)}{1-\lambda_{nl}^{(i)}(s-q^2)}\times\phi_{nl}^{(i)}(p;s-q^2)\chi_{nl}^{(i)}(q;s).\quad(7)$$

Note that the two-body eigenvalues λ_n and eigenfunctions ϕ_n must also carry the indices i and l.

Substituting (7) into (1), one obtains a set of coupled single-variable integral equations for $\chi_{nl}^{(i)}(q;s)$:

$$\chi_{nl}^{(i)}(q;s)=\eta_{nl}^{(i)}(q;s)+\sum_{\substack{n',l'\\ j\neq i}}\int_0^\infty dq_j^2\,K_{nl,n'l'}^{(i,j)}(q,q_j;s)\chi_{n'l'}^{(j)}(q_j;s),\quad(8)$$

$$\eta_{nl}^{(i)}(q;s)=\sum_{l',j\neq i}\int_0^\infty dq_j^2\int_{L_{ij}}^{U_{ij}}dp_j^2\times\frac{(-1)^{l+l'}[(2l+1)(2l'+1)]^{1/2}P_l(z_{ij})P_{l'}(z_{ij}')\phi_{nl}^{(i)}(p_i,s-q^2)}{4\pi\alpha_{ij}\beta_{ij}q(p_j^2+q_j^2-s)}\Phi_{l'}^{(j)}(p_j,q_j;s),\quad(9)$$

$$K_{nl,n'l'}{}^{(i,j)}(q,q_j;s)=\int_{L_{ij}}^{U_{ij}} dp_j{}^2 \frac{(-1)^{l+l'}[(2l+1)(2l'+1)]^{1/2}P_l(z_{ij})P_{l'}(z_{ij}')}{4\pi\alpha_{ij}\beta_{ij}q(p_j{}^2+q_j{}^2-s)[1-\lambda_{n'l'}{}^{(j)}(s-q_j{}^2)]} \times\phi_{nl}{}^{(i)}(p_i,s-q^2)\lambda_{n'l}{}^{(j)}(s-q_j{}^2)\phi_{n'l'}{}^{(j)}(p_j,s-q_j{}^2). \quad (10)$$

Let us first examine the singularities of the kernel K given by (10). If the total energy s is positive (above three-particle threshold), then there is a region $0<q^2<s$ where the two-body energy $(s-q^2)$ is positive and the expansion (6) in general fails to converge. We shall return to this problem later. For negative values of s, there are two possibilities: (a) There exist two-body bound states. For each bound state, say of energy $-E_B$, there is a corresponding eigenvalue λ which is equal to unity at $-E_B$. The denominator $[1-\lambda_{n'l'}{}^{(j)}(s-q_j{}^2)]$ in (10) then vanishes at $q_j{}^2=s+E_B$ for $s>-E_B$, therefore creating a branch point for $\chi_{nl}{}^{(i)}(q;s)$ at $s=-E_B$. Three-body bound states can only occur below the branch points. The region between the lowest and the next branch point is the energy range for purely elastic scattering of a particle by the ground state of a two-body system. A single inelastic process occurs above the second threshold, and so forth. (b) There is no two-body bound state. In this case, the kernel is purely real below $s=0$ and possible three-body bound-state energies and wave functions are determined by solving Eq. (8).

So far the initial states of the three-body system are left unspecified. This is possible because the kernel of the integral equation is independent of the initial states. For a physical scattering process, one would have an initial state consisting of two particles; in the present case, a particle plus a bound state. One finds that an initial state of particle 1 plus a bound state of (2,3) with energy s_0 and angular momentum l_0 corresponds to an inhomogeneous term

$$\Phi_{l_0}{}^{(1)}(p,q) \underset{q_0{}^2\to s-s_0}{=} -(4/\pi q)t_{l_0}{}^{(1)}(p,p_0,\, s-q_0{}^2)\delta(q^2-q_0{}^2), \quad (11)$$

where p_0 and q_0 are the p and q of the initial state. Since $t_{l_0}{}^{(1)}$ has a pole at $s-q_0{}^2=s_0$, $\Phi_{l_0}{}^{(1)}(p,q)$ can be written as

$$\Phi_{l_0}{}^{(1)}(p,q) \underset{q_0{}^2\to s-s_0}{=} -\frac{4}{\pi q} \times\left[\frac{\phi_{n_0l_0}{}^{(1)}(p;\,s_0)\phi_{n_0l_0}{}^{(1)}(p_0;\,s_0)}{-\lambda_{n_0l_0}{}^{(1)}(s_0)(s-q_0{}^2-s_0)}\right]\delta(q^2-s+s_0). \quad (12)$$

Now multiply both sides of Eq. (1) by $[(s-q_0{}^2-s_0)/\phi_{n_0}{}^{(1,l_0)}(p_0;s_0)]$ and take the limit $q_0{}^2\to s-s_0$. It is easily seen that all the inhomogeneous terms vanish except $\Phi_{l_0}{}^{(1)}$, and that the wave function of the initial (2,3) bound state $\phi_{n_0l_0}{}^{(1)}(p_0;s_0)$ is factored out of the equation. The substitution of this Φ function into (9) gives an explicit inhomogeneous term $\eta_{nl}{}^{(i)}$ for the scattering problem, and Eq. (8) can be solved by standard numerical methods. For s above the lowest branch point, the kernel must be taken as the limit of s approaching the real axis from above. One can either use numerical methods for complex arithmetic or the Fredholm reduction method given by Noyes[10] and Kowalski.[11]

The example of a Yukawa interaction for three identical spinless particles is investigated in detail to test the rate of convergence with respect to n (like a principal quantum number) as well as to l. It is found that the solutions converge rapidly with respect to l but less rapidly with respect to n for potentials characteristic of strong interactions. Specifically, we examine a unit-range attractive Yukawa potential $V=-g^2e^{-r}/r$ with strengths g^2 varying from 1 to 2.4. The masses of the particles are taken to be unity. The first two-body bound state appears at $g^2=1.67$ while a three-body system has a bound state at $g^2=1.4$. The curve for binding energy versus g^2 is plotted on Fig. 1 for various values of n_0 with $l=0$. The quantity $q\cot\delta_0$ for the scattering of a particle by the bound state of the other two is shown in Fig. 2 for $g^2=2.373$, and also with various values of n_0. It is found that the $l=2$ contribution is negligible for both of these plots. The bound-state curve is in good agreement with corresponding results of Osborn.[8]

From the results of the preceding analyses one can draw the following conclusions. Since the three-body problem at a given energy s requires the knowledge of the two-body T matrix at energies $(s-q^2)$, where

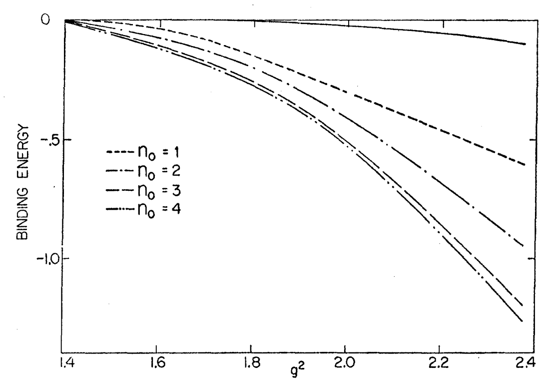

FIG. 1. Binding energy as a function of g^2 and n_0, the number of terms kept in the expansion of the two-body amplitude. The solid curve is the binding energy of the second three-body bound state.

[10] H. P. Noyes, Phys. Rev. Letters 15, 538 (1965).
[11] K. L. Kowalski, Phys. Rev. Letters 15, 798 (1965).

$\infty > q^2 \geq 0$, it is not certain, *a priori*, how well the eigenfunction expansion converges. The examples we have studied show that the convergence is sufficiently rapid and therefore the mathematical complexity of solving this kind of three-body problem is comparable to that of solving a problem for a few coupled two-body channels.

For energies lying in the continuum of the three-body spectrum, the method discussed above must be modified. Here one can first obtain a separable expansion of the two-body T matrix using the Fredholm reduction method suggested by Kowalski.[11] The substitution of the two-body T matrix into the Faddeev equation now yields a complex kernel for all values of $(s-q^2)>0$. In addition, there is a singularity due to the three-body Green's function $(p_j^2+q_j^2-s)^{-1}$. Numerically, the equations are more difficult to solve but the mathematical structure is basically the same as the problem below the three-body threshold. As for the rate of convergence with respect to the number of terms in the two-body T matrix, one must again test it against some examples.

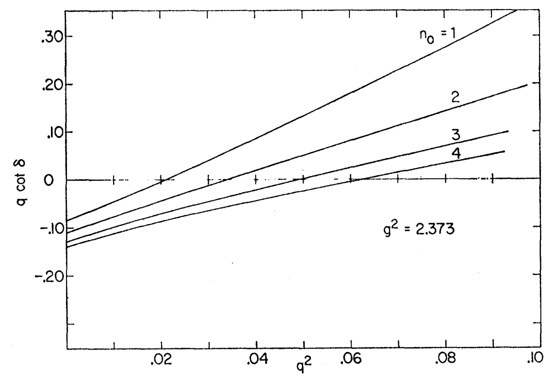

FIG. 2. Plot of $q\cot\delta$ for $g^2=2.373$ as a function of energy for various values of n_0.

PHYSICAL REVIEW VOLUME 169, NUMBER 5 25 MAY 1968

Dispersive Approach to the Study of Infinite-Momentum Limits and Equal-Time Commutators*

J.-L. GERVAIS†

Department of Physics, New York University, New York, New York

(Received 17 October 1967; revised manuscript received 19 December 1967)

Recently, a difficulty was pointed out by the author in the proof of the so-called Fubini–Dashen–Gell-Mann sum rule given by the last two authors using an infinite-momentum limit. This difficulty, which is connected with the locality of the current, is solved here assuming certain regularity properties for the weight functions of the Jost-Lehmann-Dyson representation which one can write for the one-particle matrix elements of the commutators involving the current and/or its divergence. This makes it possible to express the equal-time commutator of the time components as a sum of ordinary *covariant* dispersion integrals where the dependence in the momentum is explicit, thus leading to a straightforward proof of the sum rule from the $p\to\infty$ limit. As is required by Lorentz invariance, this limit in fact turns out to be unnecessary, since the result is also obtained for finite p. Also, the same technique, applied to the equal-time commutator of the time and the space components of the current, shows that they must be of the form usually assumed in current-algebra calculations.

IN a recent paper by the present author,[1] the proof of the Fubini[2] and Dashen and Gell-Mann[3] (FDG) sum rule given in Ref. 3 was criticized by means of the Jost-Lehmann-Dyson representation.

If one introduces, as in Ref. 1,

$$t_{\mu\nu}{}^{\alpha\beta}=(1/i)\int dx\, e^{iq\cdot x}\langle p|[A_\mu{}^\alpha(x),A_\nu{}^\beta(0)]|p\rangle, \quad (1)$$

which can be expanded in terms of invariants as

$$t_{\mu\nu}=ap_\mu p_\nu+b(p_\mu q_\nu+p_\nu q_\mu)+cq_\mu q_\nu+dg_{\mu\nu}, \quad (2)$$

the proof of the FDG sum rule given in Ref. 3 starts from the relation

$$2\pi p_0 f^{\alpha\beta\gamma}G^\gamma=\int_{-\infty}^{+\infty}dq_0 \times[a^{\alpha\beta}p_0^2+2b^{\alpha\beta}p_0q_0+c^{\alpha\beta}q_0^2+d^{\alpha\beta}]. \quad (3)$$

In (3) one integrates for fixed $\mathbf{q}$. It is easily obtained from (1) and (2) by assuming the usual commutation relations between $A_0{}^\alpha$ and $A_0{}^\beta$ at equal time.

In the method of Dashen and Gell-Mann,[3] one assumes that, if one lets $p\to\infty$, with $\mathbf{p}\cdot\mathbf{q}=0$, (3) can

* Research supported by the National Science Foundation.

† On leave from Laboratoire de Physique Théorique et Hautes Energies, Orsay S et O, France.

[1] J.-L. Gervais, Phys. Rev. Letters 19, 50 (1967); unpublished report.

[2] S. Fubini, Nuovo Cimento 43A, 475 (1966).

[3] R. F. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy*, edited by B. Kürsunoglu, A. Perlmutter, and I. Sakman (W. H. Freeman and Co., San Francisco, 1966).