Low-Energy Pion-Pion Interaction

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Low-energy pion-pion scattering is assumed to be dominated by the ρ resonance in the $I=1, J=1$ channel, the f^0 resonance in the $I=0$, $J=2$ channel, and a broad resonance of mass about 700 MeV in the $I=0$, $J=0$ channel. This simple model is found to be in substantial agreement with: (a) Adler's sum rule for pion-pion scattering, (b) the $K_1^0-K_2^0$ mass difference, (c) t-channel contributions to backward pion-nucleon dispersion relation, (d) Malamud and Schlein's "experimental" determination of the pion-pion phase shifts, (e) constraints on the s-wave pion-pion phase shifts at the K mass deduced from the two pion decays of the neutral K mesons, (f) characteristics of the K_{e4} decay, and (g) odd-pion spectral shape for K decay into three pions.

I. INTRODUCTION

THE purpose of this paper is to explore the consequences of a simple model for the pion-pion interaction in the low-energy region. Low-energy pion-pion scattering is taken to be dominated in the $I=1$, $J=1$ and the $I=0, J=2$ channels by the ρ and f^0 resonances, respectively. We further assume that the $I=0, J=0$ channel is dominated by a single resonance, the ϵ resonance. We determine the parameters (mass and width) of this resonance by requiring that it provide the experimentally observed mass difference between the short- and long-lived neutral K mesons, and demanding that, together with the ρ and f^0 resonances, it should saturate the Adler sum rule for pion-pion scattering.[1] The s-wave pion-pion $I=0$ phase shifts obtained from this model agree favorably with the phase shifts obtained by Malamud and Schlein[2] by analyzing the data for pion production in pion-nucleon collisions. This model provides the required t-channel contributions to the backward pion-nucleon dispersion relations[3–5] and is also consistent with various data obtained from the K-meson decays.

No dynamical discussion of the above-mentioned dipion resonances is attempted in this paper and the pion-pion dispersion relations are not considered. Such considerations have led to many different and conflicting results in the past depending on the assumptions made.[6] We have therefore considered it more appropriate to apply a simple phenomenological model to various situations where pion-pion interactions could play an important role.

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[1] S. Adler, Phys. Rev. **140**, B736 (1965).

[2] E. Malamud and P. Schlein, Phys. Rev. Letters **19**, 1056 (1967).

[3] C. Lovelace, R. M. Heinz, and A. Donnachie, Phys. Letters **22**, 332 (1966).

[4] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962).

[5] D. Atkinson, Phys. Rev. **128**, 1908 (1962).

[6] For example, H. J. Rothe [Phys. Rev. **140**, 1421 (1965)] concludes on the basis of pion-pion forward-scattering dispersion relations that the $I=0$ scattering length is negative. On the other hand, F. T. Meiere and M. Sugawara, Phys. Rev. **153**, 1709 (1967) conclude that the $I=0$ scattering length is positive. See, also, J. R. Fulco and D. Y. Wong, Phys. Rev. Letters **19**, 1399 (1967).

II. K_1^0–K_2^0 MASS DIFFERENCE

We assume, as usual,[7] that the K_1^0–K_2^0 mass difference may be estimated by ascribing it to the weak coupling of K_1^0 to the two-pion system as shown in Fig. 1(a). The $\Delta I=\frac{1}{2}$ rule with CP conservation (for the present purposes the effects of CP violation are negligible) together with angular momentum conservation will restrict the two-pion state to the $I=0, J=0$ channel. If now we replace the two-pion intermediate state by the ϵ pole as shown in Fig. 1(b), the mass difference between the neutral K mesons is given by

$$\delta m = g_{K\epsilon}^2/2m_K(m_K^2-m_\epsilon^2)\,, \tag{1}$$

where $g_{K\epsilon}$ is the K-ϵ coupling constant, m_K is the K

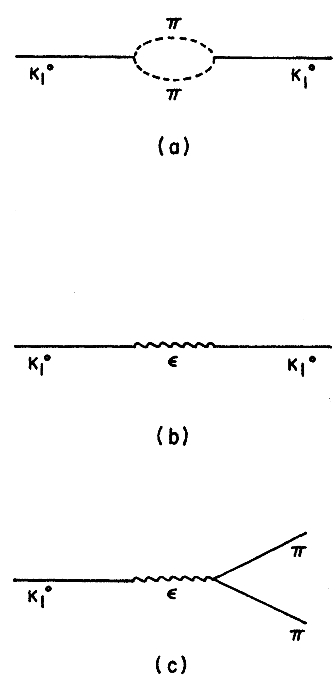

FIG. 1. (a) K_1^0 self-energy diagram with two-pion intermediate state. (b) K_1^0 self-energy diagram with ϵ intermediate state. (c) Diagram for $K_1^0\to 2\pi$ decay with ϵ intermediate state.

[7] For example, K. Kang and D. J. Land, Phys. Rev. Letters **18**, 503 (1967).

mass, and m_ϵ is the ϵ mass. We note immediately that the mass of the ϵ must be greater than the K mass to give the observed sign of the mass difference. Furthermore, in this pole model the rate for K_1^0 decay is given by the diagram shown in Fig. 1(c) as

$$\Gamma_K=\frac{g_{K\epsilon}^2 g_{\epsilon\pi\pi}^2}{16\pi}\frac{(m_K^2-4m_\pi^2)^{1/2}}{m_K^2(m_K^2-m_\epsilon^2)^2}, \tag{2}$$

where the coupling constant $g_{\epsilon\pi\pi}$ is related to the width of the ϵ by

$$\Gamma_\epsilon=\frac{g_{\epsilon\pi\pi}^2}{4\pi}\frac{(m_\epsilon^2-4m_\pi^2)^{1/2}}{4m_\epsilon^2}, \tag{3}$$

and m_π is the π mass.

The experimental mass difference is given by[8]

$$2\delta m/\Gamma_K=-0.96\pm0.04\approx-1. \tag{4}$$

Combining Eqs. (1)–(4), we obtain a relationship between the mass and width of the ϵ, namely,

$$\Gamma_\epsilon=m_K\left(\frac{m_\epsilon^2-m_K^2}{m_\epsilon^2}\right)\left(\frac{m_\epsilon^2-4m_\pi^2}{m_K^2-4m_\pi^2}\right)^{1/2}. \tag{5}$$

Equation (5) is shown by curve (a) in Fig. 2. It should be noted that in the region of interest the curve is almost a straight line.

We may obtain essentially the same results through a dispersion relation approach. We assume, as usual,[7] that the self-energy operator $\Sigma(s)$ for the K_1^0 meson obeys an unsubtracted dispersion relation

$$\Sigma(s)=\frac{1}{\pi}\int_{4m_\pi^2}^\infty ds'\frac{\mathrm{Im}\Sigma(s')}{s'-s-i\epsilon}, \tag{6}$$

where $\mathrm{Im}\Sigma(s)$ is related through unitarity to the denominator function $D(s)$ in the N/D decomposition

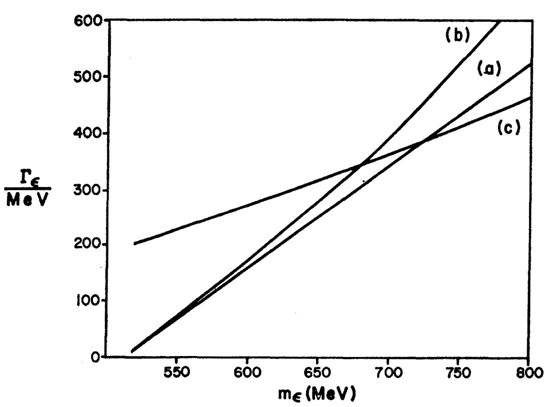

FIG. 2. Relationship between ϵ mass (m_ϵ) and width (Γ_ϵ). (a) Plot of Eq. (5). (b) As obtained from Eq. (8). (c) Saturating Adler sum rule.

[8] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).

of the s-wave pion-pion amplitude by the equation

$$\mathrm{Im}\Sigma(s)=\zeta\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}\frac{1}{|D(s)|^2}, \tag{7}$$

where ζ is a constant. On the other hand, $\mathrm{Im}\Sigma(s=m_K^2)$ is related to the decay rate for $K_1^0\to 2\pi$. Thus one obtains

$$2\delta m/\Gamma_K=-P\int_{4m_\pi^2}^\infty ds'\left[\frac{(s'-4m_\pi^2)m_K^2}{(m_K^2-4m_\pi^2)s'}\right]^{1/2}\times\left|\frac{D(m_K^2)}{D(s')}\right|\frac{1}{(s'-m_K^2)}. \tag{8}$$

The D function that appears in the above equations is for pion-pion scattering in the $I=0, J=0$ channel. Since we assume a resonance to dominate this channel, we may parametrize the denominator function as

$$D(s)=(s-m_\epsilon^2)+i\gamma_\epsilon\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}, \tag{9}$$

where

$$\gamma_\epsilon=m_\epsilon^2\Gamma_\epsilon(m_\epsilon^2-4m_\pi^2)^{-1/2} \tag{10}$$

and Γ_ϵ is the width of the ϵ resonance. Using the experimental value for the mass difference given in Eq. (4), we obtain a relation between the mass and the width of the ϵ. This is shown by curve (b) in Fig. 2. We again note that the mass of the ϵ resonance must be greater than the K mass in order to give the observed sign of the mass difference.

III. SATURATION OF THE ADLER SUM RULE

Adler[1] has derived a pion-pion scattering sum rule which is poorly satisfied by including only $J=I=1$ (ρ) and $J=2, I=0$ (f^0) contributions. The sum rule may be written as

$$\frac{2}{g_A^2}=1.43=\frac{4M_N^2}{g_r^2}\frac{1}{2\pi}\int_{4m_\pi^2}^\infty\frac{ds}{s-m_\pi^2}\times\Bigg\{\sum_{\substack{J=0\\(J\ \mathrm{even})}}^\infty\left[\frac{s-m_\pi^2}{s(s-4m_\pi^2)}\right]^J\tfrac{2}{3}\left[\sigma_{\pi\pi}^{0,J}(s)-\sigma_{\pi\pi}^{2,J}(s)\right]+\sum_{\substack{J=1\\(J\ \mathrm{odd})}}^\infty\left[\frac{(s-m_\pi^2)^2}{s(s-4m_\pi^2)}\right]^J\sigma_{\pi\pi}^{1,J}(s)\Bigg\}. \tag{11}$$

In Eq. (11), g_A is the ratio of the weak axial-vector coupling constant to the vector coupling constant, g_r is the renormalized strong interaction coupling constant, M_N is the nucleon mass, $\sigma_{\pi\pi}^{(I,J)}(s)$ is the π-π cross section in the isospin I and angular momentum J channel at center-of-mass energy $\sqrt{s}$.

The pion-pion cross sections may be parametrized as follows:

$$\sigma_{\pi\pi}{}^{1,1}(s)=\frac{3\pi\gamma_\rho{}^2(s-4m_\pi{}^2)^2/s}{(m_\rho{}^2-s)^2+\gamma_\rho{}^2(s-4m_\pi{}^2)^3/16s}, \quad (12a)$$

$$\sigma_{\pi\pi}{}^{0,2}(s)=\frac{5\pi\gamma_f{}^2(s-4m_\pi{}^2)^4/16s}{(m_f{}^2-s)^2+\gamma_f{}^2(s-4m_\pi{}^2)^5/256s}, \quad (12b)$$

where

$$\gamma_\rho{}^2=16m_\rho{}^4\Gamma_\rho{}^2/(m_\rho{}^2-4m_\pi{}^2)^3, \quad (13a)$$

$$\gamma_f{}^2=256m_f{}^4\Gamma_f{}^2/(m_f{}^2-4m_\pi{}^2)^5, \quad (13b)$$

and m_ρ (m_f) is the mass of the ρ (f^0) meson.

Adler obtains a ρ contribution of 0.42 and an f^0 contribution of 0.11, which together constitute only 37% of the total of 1.43 required by the sum rule. He concludes: "Thus, the pion-pion sum rule can be satisfied only if there is a large low-energy $I=0$, s-wave pion-pion scattering cross section."

Using the ϵ meson to parametrize the s-wave scattering, we write

$$\sigma_{\pi\pi}{}^{0,0}(s)=\frac{16\pi\gamma_\epsilon{}^2/s}{(m_\epsilon{}^2-s)^2+\gamma_\epsilon{}^2(s-4m_\pi{}^2)/s} \quad (14)$$

with γ_ϵ defined by Eq. (10). Requiring the ϵ together with the ρ and f^0 to saturate the Adler sum rule, we again get a relationship between the mass and width of the ϵ. This is shown by curve (c) in Fig. 2.

Demanding now that the ϵ simultaneously give the observed $K_1{}^0$–$K_2{}^0$ mass difference, and together with the ρ and f^0 saturate the Adler sum rule, we obtain the parameters of the ϵ: $m_\epsilon\sim700$ MeV and $\Gamma_\epsilon\sim400$ MeV.

IV. s-WAVE PION-PION PHASE SHIFTS

The s-wave $I=0$ pion-pion phase shifts at low energies are given by the assumption of resonance dominance to be

$$\delta_{0,0}=\tan^{-1}\left[\frac{2k\Gamma_\epsilon m_\epsilon}{(m_\epsilon{}^2-s)s^{1/2}(m_\epsilon{}^2-4m_\pi{}^2)^{1/2}}\right], \quad (15)$$

where $k=\frac{1}{2}(s-4m_\pi{}^2)^{1/2}$. In this section we shall compare the predicted phase shifts with other data on pion-pion phase shifts.

Malamud and Schlein[2] have used the data for pion production in pion-nucleon collisions to extract the pion-pion elastic phase shifts. They have three solutions for $\delta_{0,0}$ which they call "up-up," "up-down," and "down-up." All of these solutions are consistent with a broad resonance at about 700 MeV. Their preferred "up-up" solution is shown in Fig. 3. The curve shown in Fig. 3 is our prediction for $\delta_{0,0}$ obtained from Eq. (4). The agreement is quite good.

Pion-pion phase shifts also manifest themselves in the backward pion-nucleon dispersion relations through the t-channel cut. This problem has been analyzed in detail by Lovelace[9] and many possible solutions for the pion-pion phase shifts has been given by him. It is observed that one of his best fits corresponds to $\delta_{0,0}$ passing through $\frac{1}{2}\pi$ at about 700 MeV with a slope which corresponds to $\Gamma\sim400$ MeV. For this fit, $\delta_{0,0}+\delta_{0,2}$ passes through $\frac{3}{2}\pi$ at about 1300 MeV. Using the ϵ model (with the parameters $m_\epsilon\sim700$ MeV and $\Gamma_\epsilon\sim400$ MeV) dominating the $I=0$, $J=0$ channel and with the f^0 dominating the $I=0$, $J=2$ channel, this is precisely the nature of the pion-pion phase shifts.

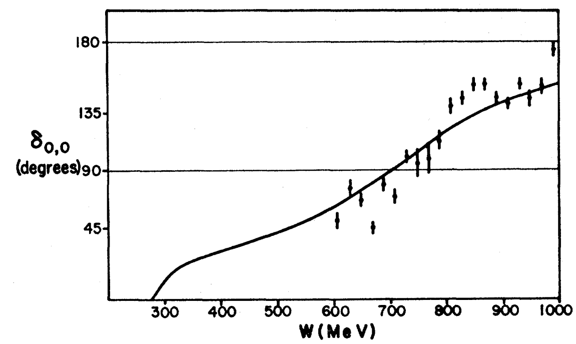

FIG. 3. Pion-pion phase shifts $I=J=0$. The solid curve is the prediction of the ϵ model given by Eq. (14). The "experimental" points are taken from Ref. 2.

From the analysis of the two-pion decays of neutral K mesons it has been inferred[10] that

$$|\delta_{0,0}(s=m_K{}^2)-\delta_{2,0}(s=m_K{}^2)|=66\pm13°. \quad (16)$$

In the present model

$$\delta_{0,0}(m_K{}^2)=\tan^{-1}\left[\gamma_\epsilon\left(\frac{m_K{}^2-4m_\pi{}^2}{m_\epsilon{}^2}\right)^{1/2}\Big/(m_K{}^2-m_\epsilon{}^2)\right]\approx45°. \quad (17)$$

If we take the phase shift in the $I=2$, $J=0$ channel to be negative at low energies (corresponding to a repulsion) and of the order of $-10°$ or $-20°$ as some studies seem to indicate,[11] then we see that the present model is consistent with the constraint on pion-pion s-wave phase shifts at the K mass obtained from the K decays.

The pion-pion scattering length in this model is given by

$$\frac{1}{a}=\lim_{k\to0}k\cot\delta=\frac{m_\epsilon{}^2-4m_\pi{}^2}{\gamma_\epsilon/m_\pi} \quad (18)$$

or

$$a=\frac{m_\epsilon{}^2\Gamma_\epsilon}{m_\pi(m_\epsilon{}^2-4m_\pi{}^2)^{3/2}}\approx0.7\lambda_\pi. \quad (19)$$

This result is to be contrasted with the current-algebra

[9] C. Lovelace, in Proceedings of the Heidelberg Conference on High-Energy Physics, 1967 (unpublished).
[10] T. D. Lee and C. S. Wu, Ann. Rev. Nucl. Sci. **16**, 511 (1966).
[11] W. D. Walker *et al.*, Phys. Rev. Letters **18**, 630 (1967).

prediction by Weinberg,[12] who obtains a small scattering length. However, Iliopoulos[13] and Donnachie[14] have shown that the incorporation of unitarity into the Weinberg method of obtaining the scattering length makes Weinberg's result ambiguous and could lead to larger scattering lengths.

V. OTHER K-DECAY PHENOMENA

The pion-pion interaction may also manifest itself through final-state interactions in the decays $K\to 3\pi$ and K_{e4}. Some previous work on these decays have emphasized the role of an $I=0$, $J=0$ σ-meson resonance at $m_\sigma\sim 400$ MeV with $\Gamma_\sigma\sim 100$ MeV.[15] This model has the feature that the low mass of the σ gives sufficient variation in the matrix-element needed to produce the correct slope in the spectrum of the odd pion in $K\to 3\pi$ decay. But the σ model yields too large a contribution to the K_{e4} rate[16,17] and also contradicts the present data on the dipion spectrum in K_{e4} decay.[18] Besides, it is difficult to understand why a resonance not particularly broad should have evaded detection for so long.

The diagrams that contribute to $K\to 3\pi$ and K_{e4} in the present model are shown in Figs. 4 and 5. Since in both these decays the two pions are at very low energies, we may neglect the f^0 contribution. The spectrum and the rates are determined by the contributions of the s- and p-wave channels, namely, through the coupling of ϵ and ρ to the $K\pi$ and K-lepton vertices.

Consider first the $K_2{}^0\to\pi^+ +\pi^- +\pi^0$ decay. In Fig. 4 there is no ρ^0 contribution if CP is conserved (we neglect small CP-violating effects) since the final state must have $I=1$ or 3 but $\pi^0-\rho^0$ cannot couple to $I=1$ or 3.

The matrix element for $K_2{}^0\to\pi^+\pi^-\pi^0$ is then given by

$$\mathfrak{M}=\frac{g_{K\epsilon\pi^0}g_{\epsilon\pi^+\pi^-}}{(q_+ + q_-)^2-m_\epsilon{}^2}+\frac{g_{K\rho\pi}g_{\rho\pi\pi}}{m_K{}^2}\times\left[\frac{(P+q_-)\cdot(q_+-q_0)}{(q_+ + q_0)^2-m_\rho{}^2}+\frac{(P+q_+)\cdot(q_- - q_0)}{(q_- + q_0)^2-m_\rho{}^2}\right],\qquad(20)$$

where p, q_+, q_-, and q_0 are the four-momenta of the K, π^+, π^-, and π^0, respectively. The g's are the respective three-particle coupling constants. In order to determine the slope of the reduced spectrum of the odd pion, we expand the propagator in powers of momentum, keeping only lowest-order terms. Let

$$\zeta_\epsilon = g_{K\epsilon\pi^0}g_{\epsilon\pi^+\pi^-}/(s_0-m_\epsilon{}^2)\,,\qquad(21)$$

$$\zeta_\rho = g_{K\rho\pi^0}g_{\rho\pi\pi}/(s_0-m_\rho{}^2)\,.\qquad(22)$$

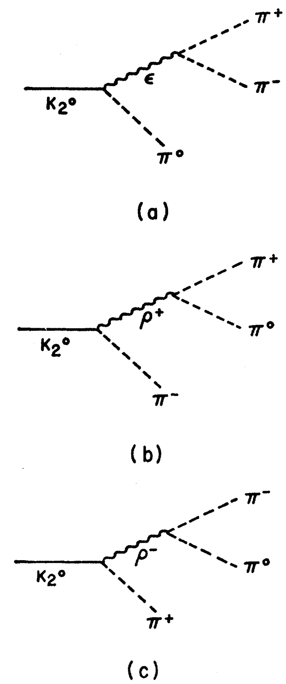

FIG. 4. Diagrams for $K\to 3\pi$ decays.

Then

$$\mathfrak{M}=\zeta_\epsilon+\zeta_\rho(m_K{}^2+3m_\pi{}^2-3s)\,,\qquad(23)$$

where

$$s=(q_+ + q_-)^2=2m_\pi{}^2+2q_+\cdot q_-\,.$$

If we introduce the Dalitz variable[19] $y=(T_0-\frac{1}{3}Q)/\frac{1}{3}Q$ and normalize the reduced spectrum, we obtain

$$|\mathfrak{M}|^2\approx 1+\alpha\frac{m_K Q}{m_\pi{}^2}y\,,\qquad(24)$$

where

$$Q=m_K-3m_\pi$$

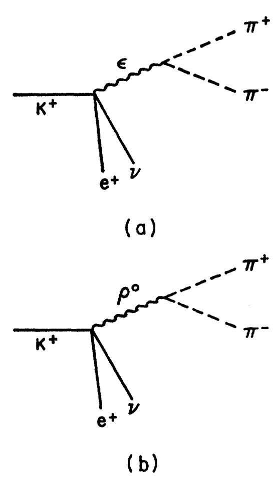

FIG. 5. Diagrams for $K\to 2\pi e\nu$ decays.

[12] S. Weinberg, Phys. Rev. Letters 17, 616 (1966).
[13] J. Iliopoulos, CERN Report No. TH 775, 1967 (unpublished).
[14] A. Donnachie, CERN Report No. TH 804, 1967 (unpublished).
[15] L. M. Brown and P. Singer, Phys. Rev. 133, B812 (1964).
[16] C. Kacser, P. Singer, and T. N. Truong, Phys. Rev. 137, B1605 (1965); 139, AB5(E) (1965).
[17] J. Iliopoulos, Nuovo Cimento 38, 907 (1965).
[18] F. A. Berends, A. Donnachie, and G. C. Oades, CERN Report No. TH 792, 1967 (unpublished).

[19] R. H. Dalitz, Phil. Mag. 44, 1068 (1953).

and

$$\alpha=4m_\pi^2[1/(s_0-m_\epsilon^2)+(\zeta_\rho/\zeta_\epsilon)]. \quad (25)$$

Experimentally,[20] $\alpha=-0.32$.

This, then, determines the relative magnitudes of the coupling constants of the ϵ and ρ to the K-π vertices.

We note that our model cannot give the value of these coupling constants independently. However, we may use the model to also calculate the quadratic terms of the spectrum. The quadratic terms are uniquely determined and are very small. Thus our model is at least consistent with the $K_2^0\to\pi^+\pi^-\pi^0$ decay data, although it cannot predict the slope *a priori*.

If we assume the $|\Delta I|=\frac{1}{2}$ rule, the slope for the other 3π decays of K mesons may also be determined.[21]

We also note that our model maintains the correct predictions for the branching ratios of the different decay modes as obtained, for example, in the σ model. Since there is no interference in the total rate between the s- and p-wave contributions, the p-wave contribution is small and may be neglected. The branching ratios are then determined essentially by isotopic spin considerations.

The matrix element for K_{e4} decay is obtained from the diagrams in Fig. 6(b). We write the matrix element in the form

$$\mathfrak{M}=\frac{G}{\sqrt{2}}\left[\frac{f_+}{m_K}(q_++q_-)_\mu+\frac{f_-}{m_K}(q_+-q_-)_\mu\right]j_\mu, \quad (26)$$

where f_+ and f_- are the K_{e4} axial-vector form factors, and j_μ is the lepton current

$$j_\mu=\bar{\psi}_l\gamma_\mu(1+\gamma_5)\psi_\nu. \quad (27)$$

(The vector form factors have been omitted, since they give small contributions to the quantities we shall consider.)

In order to determine f_+ and f_- we make use of the usual assumption of the partially conserved axial-vector current for the $\Delta S=1$ part of the weak current

$$\partial_\mu A_\mu^{\Delta S=1}=m_K^2 g_{K\mu\nu}\phi_K, \quad (28)$$

where $g_{K\mu\nu}$ is the coupling constant for $K\to\mu+\nu$ and ϕ_K is the kaon field. $g_{K\mu\nu}$ is determined by the rate for $K\to\mu+\nu$:

$$\Gamma(K\to\mu+\nu)=\frac{g_{K\mu\nu}^2}{4\pi}m_K m_\mu^2\left(1-\frac{m_\mu^2}{m_K^2}\right)^2. \quad (29)$$

We also assume that the lepton center-of-mass energy

[20] G. H. Trilling, in Proceedings of the Argonne International Conference on Weak Interactions, Agronne National Laboratory Report No. ANL-7130, 1965 (unpublished). H. W. K. Hopkins, T. C. Bacon, and F. R. Eisler, Phys. Rev. Letters **19**, 185 (1967).

[21] T. Heutter *et al.*, Phys. Rev. **140**, B355 (1965), and references contained therein.

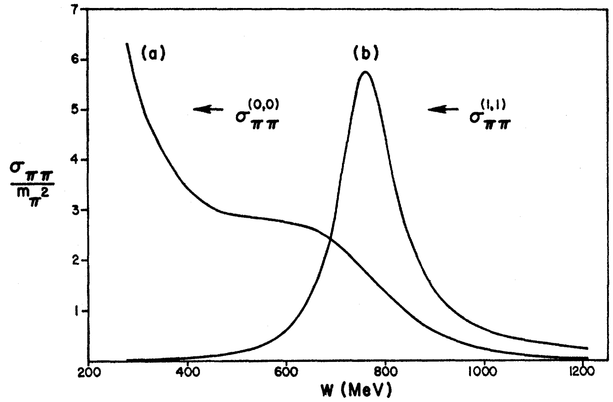

FIG. 6. Pion-pion scattering cross sections. (a) $I=J=0$ channel (ϵ). (b) $I=J=1$ channel (ρ).

is small compared to the K mass and obtain the result

$$\frac{G}{\sqrt{2}}\frac{f_+}{m_K}(p-q_+-q_-)\cdot(q_++q_-)\approx\frac{g_{K\mu\nu}g_{\epsilon K\bar{K}}g_{\epsilon\pi\pi}}{(q_++q_-)^2-m_\epsilon^2}, \quad (30)$$

$$\frac{G}{\sqrt{2}}\frac{f_-}{m_K}\approx\frac{g_{K\mu\nu}g_{\rho K\bar{K}}g_{\rho\pi\pi}}{(q_++q_-)^2-m_\rho^2}. \quad (31)$$

We use $g_{\rho\pi\pi}^2/4\pi=2.4$ and from SU_3 (or the universal coupling of the ρ to the isovector current) we have $g_{\rho K\bar{K}}^2/4\pi=1.2$. Then, $f_-\simeq1$. We also note that f_- is approximately constant. This is in agreement with earlier determinations of the K_{e4} form factors.[17]

From Eq. (13) we also note that f_+ does not vary rapidly in the region of interest. This behavior is to be contrasted with the σ model which gives rise to significant variations due to the low mass of the σ in contradiction to experimental evidence.[18]

In order to determine the value of f_+ it is necessary to evaluate $g_{\epsilon K\bar{K}}$. This is an unknown parameter since our model does not make any predictions regarding this coupling constant. However, if we assume that the ϵ has SU_3 properties, we may use this assumption to evaluate this coupling constant. If the ϵ is an SU_3 singlet, $g_{\epsilon K\bar{K}}=g_{\epsilon\pi\pi}$. In this case, $f_+\simeq2.7$. On the other hand, if ϵ is a member of an SU_3 octet, then $g_{\epsilon K\bar{K}}=\frac{1}{2}g_{\epsilon\pi\pi}$ and $f_+\simeq1.3$. This latter value is roughly in agreement with previous analyses[18] of the experimental data.

VI. CONCLUSIONS

We have considered a simple model for the low-energy pion-pion interaction. This model, which assumes that the $I=J=0$ channel is dominated by an ϵ meson of mass $m_\epsilon\simeq700$ MeV and $\Gamma_\epsilon\simeq400$ MeV is consistent with a large amount of data including available experimental information on K decays, pion-nucleon scattering, and the K_1^0-K_2^0 mass difference.

We have also assumed that the $I=J=1$ channel is dominated by the ρ meson and the $I=0, J=2$ channel is dominated by the f^0 meson. It is also of interest to examine the pion-pion cross sections predicted by our model. In Fig. 6 we show the total cross sections in the $I=J=0$ and $I=J=1$ channels. It is of special interest to note that for such a broad s-wave resonance, although the phase shift passes through 90° at the ϵ mass, there is no "peak" in the cross section, but only a "shoulder." Thus it is not possible to observe the ϵ by merely looking at the pion-pion center-of-mass energy distribution. Hence we must rely on indirect, model-dependent determinations of the pion-pion phase shifts until direct pion-pion scattering "experiments" are possible.[22]

While the ϵ will not manifest itself as a typical resonance "peak," we have seen that it is very useful to introduce the ϵ parametrization for the low-energy pion-pion interaction to describe a variety of experimental data.

It would be extremely useful to have accurate measurements of quantities from which one could infer pion-pion phase shifts in order to better understand the pion-pion interaction.

[22] P. L. Csonka, CERN Report No. TH 836, 1967 (unpublished).

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Solutions of the Faddeev Equation for Short-Range Local Potentials*

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A systematic method for solving the Faddeev equation for three bodies interacting through two-body local potentials is presented. This method is then applied to the problem of three identical particles interacting through a Yukawa potential, and the convergence of the method is studied numerically. Solutions are obtained for one particle scattering off a bound state of the other two, as well as for the three-particle bound-state case.

THE application of the Faddeev equation to nonrelativistic three-body problems has been of considerable interest, as seen in the literature.[1–8] In the present work we address ourselves to the question of how one would solve the equations systematically once the two-body potentials are given, local or otherwise.

The angular momentum decomposition of the Faddeev equations was first treated by Ahmadzadeh and Tjon,[6] resulting in a set of coupled integral equations in two variables. If the two-body potentials are taken to be separable (nonlocal) as was considered by a number of authors,[2–6] then the integral equations reduce to one variable and can be solved by ordinary numerical methods. It was suggested by Zambotti and one of us (DYW)[7] that even if the potentials were local, it would still be convenient to expand the two-body T matrix as a sum of separable terms. A generalized effective-range-type expansion was introduced and applied to the Yukawa-potential problem.[7] Although the method proved to be useful, the treatment was not entirely systematic. In the present paper, we suggest a systematic way of expanding the two-body T matrix for the solution of the Faddeev equation below the three-body threshold and show that the expansion converges fairly rapidly for potentials characteristic of strong interactions. This method is applicable to the calculations of bound-state energies and wave functions as well as the scattering of a particle by a bound state of two other particles. The problem of extending beyond the three-particle threshold is discussed at the end. The treatment of Coulomb potentials is reported in a separate paper.[9]

Although the method outlined below is applicable to any angular momentum state of the three-body system, we consider, for convenience, only states corresponding to zero total angular momentum and no spin. For this three-body state, the Faddeev equation can be written

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