infinity faster than any inverse power of E, it can be shown, by means of some simple manipulations, that

$$\frac{\mathcal{P}}{\pi}\int_0^\infty \frac{\delta_1(E')}{E'-E}dE' \sim \frac{\text{const}}{E}, \quad \text{for} \quad E \to +\infty\,.$$

It follows immediately that

$$\left[\frac{1}{|D_1(E)|^2}-1\right]^2 \sim \frac{\text{const}}{E^2}, \quad \text{for} \quad E \to +\infty\,,$$

and this fact ensures the convergence of the integral appearing in Eq. (18).

We have therefore completed the proof of the existence of a local potential yielding the form (13) of the phase shift.

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Spin-Parity Analysis for Boson Resonances*

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A general helicity formalism is developed for the determination of spin and parity of boson resonances of arbitrary spin which have sequential decay modes. The procedure is illustrated with a few simple but, in practice, important decay modes, namely, 1^-+1^-, $1^\pm+0^-$, and 2^++0^-, where $1^\pm$ and 2^+ mesons in turn decay into 2 or 3 pseudoscalar mesons. The method proposed here is independent of the dynamics of the production and decay process.

I. INTRODUCTION

We present in this paper a general helicity formalism[1,2] that enables one to determine the spin and parity of boson resonances with sequential decay modes; we treat as the maximum complexity the case of a boson resonance decaying into two intermediate bosons of arbitrary spin, both of which in turn decay into three pseudoscalar mesons. It is shown that the formalism thus developed can easily be applied to cases when the intermediate bosons decay into two pseudoscalar mesons or one of the intermediate bosons is a pseudoscalar.

Our basic tool for the spin-parity determination is the moments which are experimental averages of the product of three D functions (see Appendix A and Ref. 12). It is shown that these moments are conveniently parametrized in terms of the multipole parameters.[3,4] Our main task in this paper has been to show that there exist linear relations among different moments for certain spin-parity combinations of the parent bosons and that for some of the linear equations the coefficients themselves are known functions of the spin of the parent bosons; this affords a straightforward means of determining the spin and parity of the parent bosons.

A remarkable aspect of this method is that it is independent of the detailed dynamics of the production and decay mechanism of the parent bosons. In addition, our method is independent of the interference among the three decay products of either of the intermediate bosons. Our method does not apply, however, if there exists appreciable interference between the decay products of one of the intermediate bosons with those of the other. It is shown that our method can still be applied, if we limit our analysis to those events for which the interference is minimal. Of course, there is always the problem of interference with background events. However, our method can be used if the interference is not appreciable and if the moments for the background events alone are small, as should be the case when the background events consist mostly of phase-space events.

In Sec. II, we derive the general angular distributions starting with the Lorentz-invariant amplitude for the production and decay of the parent bosons. We introduce in Sec. III the multipole parameters and then the moments and give the symmetry properties satisfied by these moments. In Secs. IV–VI, we illustrate our spin-parity analysis with simple but, in practice, important examples. These include the case of a boson resonance

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[1] For the helicity formalism, the reader is referred to the standard work: M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959). However, we use a slightly different convention for the argument of D functions; instead of their $D_{\mu m}{}^J(\varphi,\theta,-\varphi)$, we use $D_{\mu m}{}^J(\varphi,\theta,0)$.

[2] We give three references for different approaches to spin-parity analysis of bosons: M. Ademollo, R. Gatto, and G. Preparata, Phys. Rev. **139**, B1608 (1965); C. Zemach, *ibid.* **140**, B109 (1965); E. de Rafael, Ann. Inst. Henri Poincaré **5**, 83 (1966).

[3] N. Byers and S. Fenster, Phys. Rev. Letters **11**, 52 (1963).

[4] See the first of two lectures by J. D. Jackson, *High Energy Physics* (Gordon and Breach Science Publishers, New York, 1965).

decaying into a vector and a pseudoscalar meson, which has been treated in our earlier work[5] and by others,[6–8] as well as into a pseudovector and a pseudoscalar meson. In addition, we consider the decay into two vector (or pseudovector) mesons and into a 2^+ meson and a pseudoscalar meson. Finally, our method is further illustrated in Appendix B with the example of a resonance decaying into $\pi+f^0(1250)$; the relevant tests are given directly in terms of the experimentally accessible angles.

It is hoped that our methods provide a timely tool for analyzing the recently reported resonances[9] with decay modes of $\rho+\rho$ or $\pi+f^0(1250)$. It is perhaps inevitable that, should higher-spin boson nonets exist, they would have appreciable decay modes into the 0^-, 1^-, or 2^+ nonets.

II. ANGULAR DISTRIBUTIONS

Let us consider a reaction which produces a boson resonance B with spin J and parity η:

$$a+b\rightarrow B(J^{\eta})+c. \qquad (1)$$

The resonance B then decays into six pseudoscalar mesons via two intermediate bosons B_1 and B_2 with spins S_1 and S_2 and parities η_1 and η_2, respectively:

$$\begin{array}{l} B(J^{\eta})\rightarrow B_1(S_1^{\eta_1})+B_2(S_2^{\eta_2}) \\ \qquad\qquad B_2 \searrow a_6+a_7+a_8 \\ \qquad\qquad B_1 \searrow a_3+a_4+a_5\,, \end{array} \qquad (2)$$

where a_3 through a_8 denote the six pseudoscalar mesons.

The cross section for this chain of reactions is given by

$$d\sigma\sim\frac{1}{P_iE_T}\sum_{\text{spins for }a,b,c}|A|^2\left(\frac{P_f}{E_T}d\cos\theta_0\right)(qdM_Bd\Omega)\times\left(\frac{dM_1}{M_1}d\Omega_1d\gamma_1dM_{34}{}^2dM_{45}{}^2\right)\times\left(\frac{dM_2}{M_2}d\Omega_2d\gamma_2dM_{67}{}^2dM_{78}{}^2\right), \qquad (3)$$

where E_T stands for the total energy in the c.m. system of a and b, and $P_i(P_f)$ is the c.m. momentum of a or b (B or c) and θ_0 is the angle between B and a in the c.m. system. M_B, M_1, M_2, and $M_{k\ell}$ are the effective masses of B_1+B_2, $a_3+a_4+a_5$, $a_6+a_7+a_8$, and a_k+a_ℓ systems, respectively. q and $\Omega(\theta,\varphi)$ describe the magnitude and direction of B_1 momentum in the rest frame of B (BRF), where we choose the z axis parallel to the direction of B in the c.m. system. φ_1, θ_1, and γ_1 (φ_2, θ_2, and γ_2) are the Euler angles for the configuration of a_3, a_4, and a_5 (a_6, a_7, and a_8) in the B_1RF (B_2RF) with $\Omega_1(\theta_1,\varphi_1)$ [$\Omega_2(\theta_2,\varphi_2)$] describing the normal to the decay plane. Again, the coordinate system for angles Ω_1 (Ω_2) has the z axis parallel to the direction of B_1 (B_2) in the BRF.[10]

The Lorentz-invariant amplitude A in (3) is given by

$$A\sim\sum_{\Lambda,\lambda_1,\lambda_2}\langle M_{34}{}^2,M_{45}{}^2,\Omega_1,\gamma_1|\mathfrak{M}_1|S_1\lambda_1\rangle\times\langle M_{67}{}^2,M_{78}{}^2,\Omega_2,\gamma_2|\mathfrak{M}_2|S_2\lambda_2\rangle\langle S_1\lambda_1,S_2\lambda_2,q,\Omega|\mathfrak{M}|J\Lambda\rangle\times\langle c,J\Lambda|T|a,b\rangle\tilde{\delta}(M_1,\Gamma_1)\tilde{\delta}(M_2,\Gamma_2)\tilde{\delta}(M_B,\Gamma_B)\,, \qquad (4)$$

where the first, the second, and the third factor in (4) are the decay amplitudes for B_1, B_2, and B, and the fourth factor is the transition amplitude for the reaction (1). As usual, Λ, λ_1, and λ_2 denote helicities for B, B_1, and B_2, respectively. $\tilde{\delta}(M_B,\Gamma_B)$ is the Breit-Wigner form given by

$$\tilde{\delta}(M_B,\Gamma_B)=1/[M_B{}^2-(M_B{}^0-i\Gamma_B/2)^2]\,, \qquad (5a)$$

where $M_B{}^0$ and Γ_B are the mass and width of the resonance B. Likewise, we have

$$\tilde{\delta}(M_{1,2},\Gamma_{1,2})=1/[M_{1,2}{}^2-(M_{1,2}{}^0-i\Gamma_{1,2}/2)^2]\,, \qquad (5b)$$

where $M_{1,2}{}^0$ and $\Gamma_{1,2}$ are the mass and width of $B_{1,2}$.

The decay amplitudes in (4) are given by[1,11]

$$\langle S_1\lambda_1,S_2\lambda_2,q,\Omega|\mathfrak{M}|J\Lambda\rangle=F_{\lambda_1\lambda_2}{}^J(M_B,M_1,M_2)D_{\Lambda\lambda_1-\lambda_2}{}^{J*}(\varphi,\theta,0)\,, \qquad (6a)$$

$$\langle M_{34}{}^2,M_{45}{}^2,\Omega_1,\gamma_1|\mathfrak{M}_1|S_1\lambda_1\rangle=\sum_\mu F_\mu{}^1(M_1,M_{34}{}^2M_{45}{}^2)D_{\lambda_1\mu}{}^{S_1*}(\varphi_1,\theta_1,\gamma_1)\,, \qquad (6b)$$

$$\langle M_{67}{}^2,M_{78}{}^2,\Omega_2,\gamma_2|\mathfrak{M}_2|S_2\lambda_2\rangle=\sum_\nu F_\nu{}^2(M_2,M_{67}{}^2,M_{78}{}^2)D_{\lambda_2\nu}{}^{S_2*}(\varphi_2,\theta_2,\gamma_2)\,, \qquad (6c)$$

where $F_{\lambda_1\lambda_2}$, $F_\mu{}^1$, and $F_\nu{}^2$ are the helicity amplitudes for B, B_1, and B_2, respectively, and the $D_{m'm}{}^J$ functions are the $(2J+1)$-dimensional representation of the rotation group as defined in Rose.[12] If B_2 decays into *two* pseudoscalar mesons a_6 and a_7, the decay amplitude is given by

$$\langle q_2,\Omega_2|\mathfrak{M}_2|S_2\lambda_2\rangle=F^2(M_2)D_{\lambda_20}{}^{S_2*}(\varphi_2,\theta_2,0)\,, \qquad (6d)$$

where q_2 and $\Omega_2(\theta_2,\varphi_2)$ are the magnitude and direction of the a_6 momentum in the B_2RF. The parity conserva-

[5] S. U. Chung, Phys. Rev. **138**, B1541 (1965).
[6] M. Ademollo, R. Gatto, and G. Preparata, Phys. Rev. Letters **12**, 462 (1964). See also Ref. 2.
[7] S. M. Berman and M. Jacob, Stanford Linear Accelerator Rept. No. SLAC-43, 1965 (unpublished). See also Ref. 11.
[8] C. Zemach, Nuovo Cimento **32**, 1605 (1964).
[9] A 3π enhancement at around 1660 MeV has been reported at the Heidelberg Conference, September 1967, with appreciable decay mode into $\pi+f^0(1250)$ (unpublished). For a possible $\rho+\rho$ resonance at 1401 MeV, see A. Bettini *et al.*, Nuovo Cimento **42A**, 695 (1966).
[10] For a more detailed definition of angles Ω and Ω_1, see Appendix B. The angle Ω_2 is defined in the same way as the angle Ω_1.
[11] S. M. Berman and M. Jacob, Phys. Rev. **139**, B1023 (1965).
[12] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

tion in the decay process leads to the following symmetry relations[1,4,11]:

$$F_{\lambda_1\lambda_2}{}^J=\epsilon F_{-\lambda_1-\lambda_2}{}^J\,, \tag{7a}$$

$$F_\mu{}^1=\eta_1(-)^{\mu+1}F_\mu{}^1\,, \tag{7b}$$

$$F_\nu{}^2=\eta_2(-)^{\nu+1}F_\nu{}^2\,, \tag{7c}$$

$$F^2=0 \quad \text{if} \quad \eta_2(-)^{S_2}=-1\,, \tag{7d}$$

where ϵ is defined to be

$$\epsilon=\eta\eta_1\eta_2(-)^{J-S_1-S_2}. \tag{8}$$

If B_1 and B_2 are identical particles, we have the additional relation[1]

$$F_{\lambda_1\lambda_2}{}^J(M_B,M_1,M_2)=(-)^JF_{\lambda_2\lambda_1}{}^J(M_B,M_2,M_1)\,. \tag{9a}$$

If a_3 and a_5 are identical particles, we also have[11]

$$F_\mu{}^1(M_1,M_{34}{}^2,M_{45}{}^2)=(-)^{S_1}F_{-\mu}{}^1(M_1,M_{45}{}^2,M_{34}{}^2)\,. \tag{9b}$$

Similarly, if a_6 and a_8 are identical, we get

$$F_\nu{}^2(M_2,M_{67}{}^2,M_{78}{}^2)=(-)^{S_2}F_{-\nu}{}^2(M_2,M_{78}{}^2,M_{67}{}^2)\,. \tag{9c}$$

Now, we define the density matrix for the B resonance as follows:

$$\rho_{\Lambda\Lambda'}{}^J(M_B)\sim\int d\cos\theta_0 \sum_{\text{spins for } a,b,c} \langle c,J\Lambda|T|a,b\rangle \times\langle a,b|T^\dagger|c,J\Lambda'\rangle\,. \tag{10}$$

By this definition, the density matrix is in general a function of M_B. The limits on the $\cos\theta_0$ integration is meant to correspond to the experimental cuts used in case of the peripheral production of B. The differential cross section can be written in terms of the density matrix ($\lambda\equiv\lambda_1-\lambda_2$, $\lambda'\equiv\lambda_1'-\lambda_2'$):

$$\frac{d\sigma}{dM_Bd\Omega d\Omega_1d\Omega_2}\sim\rho_{\Lambda\Lambda'}{}^J(M_B)g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}(M_B) \times D_{\Lambda\lambda}{}^{J*}(\varphi,\theta,0)D_{\Lambda'\lambda'}{}^J(\varphi,\theta,0)D_{\lambda_1\mu}{}^{S_1*}(\varphi_1,\theta_1,0) \times D_{\lambda_1'\mu}{}^{S_1}(\varphi_1,\theta_1,0)D_{\lambda_2\nu}{}^{S_2*}(\varphi_2,\theta_2,0)D_{\lambda_2'\nu}{}^{S_2}(\varphi_2,\theta_2,0)\,, \tag{11}$$

where summation is implied over repeated indices, and we have integrated over $d\gamma_1$ and $d\gamma_2$ and introduced a new parameter $g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}(M_B)$ which depends on the helicity amplitudes:

$$g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}(M_B)=\int dM_1dM_2K(M_B,M_1,M_2)F_{\lambda_1\lambda_2}{}^J \times F_{\lambda_1'\lambda_2'}{}^{J*}\int dM_{34}{}^2dM_{45}{}^2|F_\mu{}^1|^2 \times\int dM_{67}{}^2dM_{78}{}^2|F_\nu{}^2|^2\,, \tag{12}$$

where $K(M_B,M_1,M_2)$ contains all the kinematic factors as given in the following:

$$K(M_B,M_1,M_2)=\frac{1}{E_T{}^2}(p_f/p_i)[q/(M_1M_2)] \times|\bar\delta(M_B,\Gamma_B)\bar\delta(M_1,\Gamma_1)\bar\delta(M_2,\Gamma_2)|^2. \tag{13}$$

Owing to the symmetry relations for the F's as given in (7) and the definition of g as given in (12), we obtain the following properties for g:

$$g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}=\epsilon g_{-\lambda_1-\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}=\epsilon g_{\lambda_1\lambda_2-\lambda_1'-\lambda_2'}{}^{\mu\nu} =g_{-\lambda_1-\lambda_2-\lambda_1'-\lambda_2'}{}^{\mu\nu}\,, \tag{14a}$$

$$g^{\mu\nu}=\eta_1(-)^{\mu+1}g^{\mu\nu}=\eta_2(-)^{\nu+1}g^{\mu\nu}\,, \tag{14b}$$

$$g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}=g_{\lambda_1'\lambda_2'\lambda_1\lambda_2}{}^{\mu\nu*}. \tag{14c}$$

In case there are identical particles in the problem, we have from (9a)

$$g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}=(-)^Jg_{\lambda_2\lambda_1\lambda_1'\lambda_2'}{}^{\mu\nu}=(-)^Jg_{\lambda_1\lambda_2\lambda_2'\lambda_1'}{}^{\mu\nu} =g_{\lambda_2\lambda_1\lambda_2'\lambda_1'}{}^{\mu\nu} \tag{15a}$$

and, from (9b) and (9c),

$$g^{\mu\nu}=g^{-\mu\nu}=g^{\mu-\nu}=g^{-\mu-\nu}. \tag{15b}$$

Now, we define $I(M_B,\Omega,\Omega_1,\Omega_2)$ to be the angular distribution for a given effective mass M_B,

$$I(M_B,\Omega,\Omega_1,\Omega_2)\sim\frac{d\sigma}{dM_Bd\Omega d\Omega_1d\Omega_2}$$

which is normalized to 1, i.e.,

$$\int dM_Bd\Omega d\Omega_1d\Omega_2I(M_B,\Omega,\Omega_1,\Omega_2)=1\,. \tag{16}$$

It is easy to see from (A3) that the normalized angular distribution can be written[13]

$$I(M_B,\Omega,\Omega_1,\Omega_2)=\left(\frac{2J+1}{4\pi}\right)\left(\frac{2S_1+1}{4\pi}\right)\left(\frac{2S_2+1}{4\pi}\right) \times\rho_{\Lambda\Lambda'}{}^J(M_B)g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}(M_B)D_{\Lambda\lambda}{}^{J*}(\Omega)D_{\Lambda'\lambda'}{}^J(\Omega) \times D_{\lambda_1\mu}{}^{S_1*}(\Omega_1)D_{\lambda_1'\mu}{}^{S_1}(\Omega_1)D_{\lambda_2\nu}{}^{S_2*}(\Omega_2)D_{\lambda_2'\nu}{}^{S_2}(\Omega_2) \tag{17}$$

with the condition[14]

$$\int dM_B\sum_\Lambda\rho_{\Lambda\Lambda}{}^J(M_B)\sum_{\mu\nu\lambda_1\lambda_2}g_{\lambda_1\lambda_2\lambda_1\lambda_2}{}^{\mu\nu}(M_B)=1\,. \tag{18}$$

The angular distribution as given in (17) is not valid if there is interference between the decay products of B_1 and B_2. In order to apply (17) to such a situation, we have to restrict our analysis to those events outside the

[13] We use the shorthand notation for D functions: $D_{\mu m}{}^J(\Omega)\equiv D_{\mu m}{}^J(\varphi,\theta,0)$. See Appendix A.

[14] Note that we *do not* require that $\sum_\Lambda\rho_{\Lambda\Lambda}{}^J(M_B)=1$.

interference region. This restriction, in our formalism, corresponds to restricting limits of integrations over dM_1 and dM_2, which simply means that the value of $g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}$ has changed. This can be done in such a way that all the symmetry relations of (14) and (15) are still valid. Since we have chosen the normal to the decay plane of B_1 (or B_2) as the analyzer,[11] the angular distribution (17) is valid even if there is interference among the three decay particles of B_1 (or B_2). In fact, if two of the three decay products are identical, we simply have the additional symmetry as given in (15b).

It is easy to apply (17) to a simpler decay mode of B. Consider, for example, the decay of B into B_1 and B_2, each of which in turn decays into *two* pseudoscalar mesons, i.e.,

$$B(J^\eta)\to B_1(S_1{}^{\eta_1})+B_2(S_2{}^{\eta_2}).\quad B_1\searrow a_3+a_4,\quad B_2\searrow a_6+a_7 \tag{19}$$

It is clear from (6d) that we merely need to set $\mu=\nu=0$ in (17):

$$I(M_B,\Omega,\Omega_1,\Omega_2)=\left(\frac{2J+1}{4\pi}\right)\left(\frac{2S_1+1}{4\pi}\right)\left(\frac{2S_2+1}{4\pi}\right)\times\rho_{\Lambda\Lambda'}{}^J(M_B)g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}(M_B)D_{\Lambda\lambda}{}^{J*}(\Omega)D_{\Lambda'\lambda'}{}^J(\Omega)\times D_{\lambda_10}{}^{S_1*}(\Omega_1)D_{\lambda_1'0}{}^{S_1}(\Omega_1)D_{\lambda_20}{}^{S_2*}(\Omega_2)D_{\lambda_2'0}{}^{S_2}(\Omega_2). \tag{20}$$

Of course, the kinematic factor $K(M_B,M_1,M_2)$ which enters in the definition of $g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}$ [see (12)] will have a different form owing to the change in the number of final states.

We shall consider, as another example, a decay mode of B into B_1 and B_2, where B_2 is a pseudoscalar meson:

$$B(J^\eta)\to B_1(S^{\eta_1})+B_2(0^-).\quad B_1\searrow a_3+a_4+a_5 \tag{21}$$

Then, the angular distribution simplifies to

$$I(M_B,\Omega,\Omega_1)=\left(\frac{2J+1}{4\pi}\right)\left(\frac{2S+1}{4\pi}\right)\rho_{\Lambda\Lambda'}{}^J(M_B)g_{\lambda\lambda'}{}^\mu(M_B)\times D_{\Lambda\lambda}{}^{J*}(\Omega)D_{\Lambda'\lambda'}{}^J(\Omega)D_{\lambda\mu}{}^{S*}(\Omega_1)D_{\lambda'\mu}{}^S(\Omega_1), \tag{22}$$

where we have dropped the subscript 1 from the spin and helicity of B_1. Consider as a final example, the following decay mode of B:

$$B(J^\eta)\to B_1(S^{\mu_1})+B_2(0^-).\quad B_1\searrow a_3+a_4 \tag{23}$$

The corresponding angular distribution is obtained by setting $\mu=0$ in (22):

$$I(M_B,\Omega,\Omega_1)=\left(\frac{2J+1}{4\pi}\right)\left(\frac{2S+1}{4\pi}\right)\rho_{\Lambda\Lambda'}{}^J(M_B)g_{\lambda\lambda'}(M_B)\times D_{\Lambda\lambda}{}^{J*}(\Omega)D_{\Lambda'\lambda'}{}^J(\Omega)D_{\lambda0}{}^{S*}(\Omega_1)D_{\lambda'0}{}^S(\Omega_1). \tag{24}$$

III. MOMENT ANALYSIS

We now define the multipole parameter $t_L{}^M$ by[3,4]

$$t_L{}^{M*}(M_B)=\sum_{\Lambda\Lambda'}\rho_{\Lambda\Lambda'}{}^J(M_B)(J\Lambda'LM|J\Lambda), \tag{25}$$

where we use the notation $(j_1m_1j_2m_2|j_3m_3)$ for the usual Clebsch-Gordan coefficients. Because of our definition of $\rho_{\Lambda\Lambda'}{}^J(M_B)$ which depends on M_B, the multipole parameter $t_L{}^M$ also depends on M_B. Note that $t_L{}^M=0$ if $L>2J$ and $t_0{}^0=\sum_\Lambda\rho_{\Lambda\Lambda}{}^J$. We can express $\rho_{\Lambda\Lambda'}{}^J$ in terms of $t_L{}^M$ by inverting (25):

$$\rho_{\Lambda\Lambda'}{}^J(M_B)=\sum_{LM}\left(\frac{2L+1}{2J+1}\right)t_L{}^{M*}(M_B)(J\Lambda'LM|J\Lambda). \tag{26}$$

If the initial particles a and b in reaction (1) are not polarized, the parity conservation in the production process leads to the condition[11]

$$\rho_{\Lambda\Lambda'}{}^J=(-)^{\Lambda-\Lambda'}\rho_{-\Lambda-\Lambda'}{}^J, \tag{27}$$

which in turn leads to the following symmetry on $t_L{}^M$:

$$t_L{}^M=(-)^{L+M}t_L{}^{-M}. \tag{28a}$$

Hermiticity of the density matrix implies the following additional symmetry:

$$t_L{}^M=(-)^Mt_L{}^{-M*}. \tag{28b}$$

By combining (28a) and (28b), we obtain

$$t_L{}^{M*}=(-)^Lt_L{}^M \tag{28c}$$

so that $t_L{}^M$ is purely real (imaginary) if L is even (odd).

Next, we introduce what we shall call the experimental "moments," which are the experimental averages of the product of three D functions. We refer the reader to other works[15,16] for the statistical analysis involved in dealing with such averages. We define the moments by

$$H(l_1m_1l_2m_2LM)=\langle D_{Mm}{}^L(\Omega)D_{m_10}{}^{l_1}(\Omega_1)D_{m_20}{}^{l_2}(\Omega_2)\rangle, \tag{29a}$$

where $m=m_1-m_2$. We can express the moments in terms of the angular distributions given in Sec. II by

$$H(l_1m_1l_2m_2LM)=\int dM_B\int d\Omega d\Omega_1d\Omega_2I(M_B,\Omega,\Omega_1,\Omega_2)\times D_{Mm}{}^L(\Omega)D_{m_10}{}^{l_1}(\Omega_1)D_{m_20}{}^{l_2}(\Omega_2), \tag{29b}$$

where the limits for the integration over dM_B are meant to correspond to the experimental cuts used to delineate the B-resonance sample.

[15] See Appendix, J. Button-Shafer and D. W. Merrill, Lawrence Radiation Laboratory Report No. UCRL-11884, 1964 (unpublished).

[16] See Appendix B, N. Byers, CERN Rept. No. 67-20 (Theoretical Study Division), 1967 (unpublished).

Using (17), (25), (29), and (A4), we obtain

$$H(l_1m_1l_2m_2LM)=\int dM_B t_L{}^{M*}(M_B)\sum_{\lambda_1\lambda_2\lambda_1'\lambda_2'}(J\lambda'Lm|J\lambda)\times(S_1\lambda_1'l_1m_1|S_1\lambda_1)(S_2\lambda_2'l_2m_2|S_2\lambda_2)\sum_{\mu\nu}g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}(M_B)\times(S_1\mu l_10|S_1\mu)(S_2\nu l_20|S_2\nu)\,, \quad (30a)$$

where $m=m_1-m_2$, $\lambda=\lambda_1-\lambda_2$, and $\lambda'=\lambda_1'-\lambda_2'$. Owing to (18), we have the following normalization for the moments:

$$H(000000)=1\,. \quad (30b)$$

We shall henceforth omit the integration sign over dM_B in (30a), along with the argument M_B for $t_L{}^M$ and $g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}$: *Whenever* $t_L{}^M$ *and* $g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}$ *appear together, integration over* dM_B *is implied.*

We now list a number of properties of the moments. First, it is obvious from (30) that we have

$$H(l_1m_1l_2m_2LM)=0 \quad \text{if} \quad L>2J$$

or

$$l_1>2S_1 \quad \text{or} \quad l_2>2S_2\,. \quad (31a)$$

By using (14a), we obtain

$$H(l_1-m_1l_2-m_2LM)=(-)^{l_1+l_2+L}H(l_1m_1l_2m_2LM) \quad (31b)$$

and, from (28a),

$$H(l_1m_1l_2m_2L-M)=(-)^{L+M}H(l_1m_1l_2m_2LM) \quad (31c)$$

and, from (14a), (14c), and (28c),

$$H^*(l_1m_1l_2m_2LM)=(-)^{l_1+l_2}H(l_1m_1l_2m_2LM)\,. \quad (31d)$$

So, we see that it is not necessary to consider all the values of m_1 and m_2 and, especially, we need not evaluate the moments for negative values of M. Furthermore, the moments H are pure real (imaginary) if l_1+l_2 is even (odd). From (31b), we have

$$H(l_10l_20LM)=0 \quad \text{if} \quad l_1+l_2+L=\text{odd}\,, \quad (32a)$$

also, from (31c),

$$H(l_1m_1l_2m_2L0)=0 \quad \text{if} \quad L=\text{odd}\,. \quad (32b)$$

If the particles B_1 and B_2 are identical, we have from (15a),

$$H(l_1m_1l_2m_2LM)=(-)^L H(l_2m_2l_1m_1LM)\,. \quad (33)$$

As is clear from the discussion in Sec. II, the relation (30) is meaningful so long as there is no interference between the decay products of B_1 and those of B_2. If the interference is appreciable, we should limit our moment analysis to those events for which the interference is minimal. This restriction of events simply means that the value of $g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}$ will have changed due to the corresponding change in the limits of integrations appearing in the defining equation for g [see (12)].

The essential idea involved in determining the spin and parity of the parent resonance B is that we can evaulate a larger number of different moments than the unknowns $t_L{}^M$ and $g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}$. This means that we can find linear relations among different moments, the coefficients of which involve the unknown spin J. This affords a straightforward means of determining the spin. Similarly, the parity can be determined due to the fact that for certain spin-parity combinations we have the additional condition that certain g's are identically zero. This leads to further linear relations among different moments, which can be checked experimentally. We emphasize that our spin-parity analysis is independent of the production process of B; the multipole parameters $t_L{}^M$ which contain all the production information are treated as unknowns in our analysis, except for certain symmetry relations implied by the parity conservation.

We list below the special cases of the relation (30). If B_1 and B_2 decay into two pseudoscalar mesons, we obtain by using (20),

$$H(l_1m_1l_2m_2LM)=t_L{}^{M*}(S_10l_10|S_10)(S_20l_20|S_20)\times\sum_{\lambda_1\lambda_2\lambda_1'\lambda_2'}g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}(J\lambda'Lm|J\lambda)(S_1\lambda_1'l_1m_1|S_1\lambda_1)\times(S_2\lambda_2'l_2m_2|S_2\lambda_2)\,. \quad (34)$$

If B_2 is a pseudoscalar meson, we get from (22)

$$H(lmLM)=t_L{}^{M*}\sum_{\lambda\lambda'}(J\lambda'Lm|J\lambda)(S\lambda'lm|S\lambda)\times\sum_{\mu}g_{\lambda\lambda'}{}^{\mu}(S\mu l0|S\mu) \quad (35a)$$

and $H(lmLM)$ is the experimental average given by

$$H(lmLM)=\langle D_{Mm}{}^L(\Omega)D_{m0}{}^l(\Omega_1)\rangle\,, \quad (35b)$$

where we have dropped subscripts 1 from S_1, λ_1, l_1, and m_1. Of course, all the symmetry relations given in (31) and (32) are also satisfied by $H(lmLM)$ with $l_2=m_2=0$. If, in addition, B_1 decays into *two* pseudoscalar mesons, we get from (24)

$$H(lmLM)=t_L{}^{M*}(S0l0|S0)\sum_{\lambda\lambda'}g_{\lambda\lambda'}(J\lambda'Lm|J\lambda)\times(S\lambda'lm|S\lambda)\,, \quad (36)$$

where $H(lmLM)$ is to be measured as shown in (35b).

We shall illustrate our spin-parity analysis with a few simple but rather important examples in the following sections. In Sec. IV, we consider the decay of B into two vector (or pseudovector) mesons. We have devised our test in such a way that they can be applied with equal facility to cases when the two vector (or pseudovector) mesons are identical, e.g., $B(J^\eta)\to\omega+\omega$: This has been accomplished by considering always the combinations $[H(l_1m_1l_2m_2LM)+H(l_2m_2l_1m_1LM)]$ for even L [see (33)]. In addition, we use only those moments with $l_1+l_2=$even, so that *all* the moments $H(l_1m_1l_2m_2LM)$ used are pure real. In other words, the imaginary part of $H(l_1m_1l_2m_2LM)$ ought to be identically zero [see (31d)]. In Secs. V and VI, we treat the case when B_2

is a pseudoscalar. Again, we have used for our tests only those moments $H(lmLM)$ with $l=$even; the imaginary part of all $H(lmLM)$ used should be zero identically.

IV. $B(J^\eta)\to B_1(1^\pm)+B_2(1^\pm)$

The analysis presented here applies to the cases when $B_{1,2}(1^-)$ decays into two or three pseudoscalar mesons or when $B_{1,2}(1^+)$ decays into two pseudoscalars. The fundamental relation we use is obtained from (34):

$$H(l_1m_1l_2m_2LM)=t_L^{M*}(10l_10|10)(10l_20|10)\sum_{\lambda_1\lambda_2\lambda_1'\lambda_2'}g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}(J\lambda'Lm|J\lambda)(1\lambda_1'l_1m_1|1\lambda_1)\times(1\lambda_2'l_2m_2|1\lambda_2). \quad (37)$$

That (37) can also be applied to the three-particle decay mode of $B_1(1^-)$ or $B_2(1^-)$ is seen as follows: Since the spin is one, μ and ν of $g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}{}^{\mu\nu}$ can have 0, 1, or -1. But the symmetry of (14b) restricts μ and ν to only one value, i.e., $\mu=\nu=0$, so that applying this condition to (30), we simply get (37) if we set $g_{\lambda_1\lambda_2\lambda_1'\lambda_2'}\equiv g_{\lambda_1\lambda_2\lambda_1'\lambda_2}{}^{00}$. We have devised our tests so that they are independent of whether B_1 and B_2 are identical particles or not.

From (37) we see that

$$H(l_1m_1l_2m_2LM)=0 \quad \text{if} \quad l_1 \quad \text{or} \quad l_2=\text{odd}. \quad (38a)$$

In addition, because of (32a), we have

$$H(l_10l_20LM)=0 \quad \text{if} \quad L=\text{odd}. \quad (38b)$$

Now, by writing down explicitly the moments (37) for different values of l_1, m_1, l_2, and m_2, one can prove the following relations for even L:

$$9t_L^{M*}g_{0000}(J0L0|J0)=H(0000LM)+5[H(2000LM)+H(0020LM)]+25H(2020LM), \quad (39a)$$

$$18t_L^{M*}[g_{1111}(J0L0|J0)+g_{1-11-1}(J2L0|J2)]=4H(0000LM)-10[H(2000LM)+H(0020LM)]+25H(2020LM), \quad (39b)$$

$$18t_L^{M*}[g_{1010}+g_{0101})(J1L0|J1)=4H(0000LM)+5[H(2000LM)+H(0020LM)]-50H(2020LM), \quad (39c)$$

$$-\epsilon(\tfrac{3}{5}\surd 6)t_L^{M*}(g_{1010}+g_{0101})(J-1L2|J1)=H(2200LM)+H(0022LM)+5[H(2220LM)+H(2022LM)], \quad (39d)$$

$$\epsilon(6/25)t_L^{M*}g_{1111}(J0L0|J0)=H(2222LM), \quad (39e)$$

where the parameter ϵ has the form $\epsilon=\eta(-)^J$ [see (8)].

Suppose B has spin 0, i.e., $J=0$. Then, we must have

$$H(l_1m_1l_2m_2LM)=0 \quad \text{for all} \quad L\geq 1. \quad (40a)$$

In addition, by inspection of (39c) and (39d), we obtain the following relations for $L=M=0$:

$$4+5[H(200000)+H(002000)]-50H(202000)=0, \quad (40b)$$

$$H(220000)+H(002200)+5[H(222000)+H(202200)]=0. \quad (40c)$$

If the parity of B is negative, i.e., $\epsilon=\eta=-1$, we have from (14a) the condition $g_{0000}=0$, so that we obtain the additional relation from (39a)

$$1+5[H(200000)+H(002000)]+25H(202000)=0. \quad (41a)$$

This affords a means of determining the parity of B, if g_{0000} is not also zero for $\epsilon=\eta=+1$. By combining (39b) and (39c), we obtain the following formula involving ϵ:

$$75\epsilon H(222200)=4-10[H(200000)+H(002000)]+25H(202000). \quad (41b)$$

This gives an additional test for the parity of B.

Next, we consider the case $J\geq 1$. If the spin and parity of B are such that ϵ is -1, g_{0000} is identically zero [see (14a)], and we have the following condition from (39a) for even L:

$$H(0000LM)+5[H(2000LM)+H(0020LM)]+25H(2020LM)=0. \quad (42)$$

Note that, since J is equal or greater than 1, we can use (42) for at least two values of L, i.e., $L=0$ or 2. In order to use (42) for higher values of L, we must first know the value of J, since $H(l_1m_1l_2m_2LM)=0$ if $L>2J$. Once the value of ϵ is known, we may determine the spin itself as follows: If the factor

$$\int dM_Bt_L^{M*}(M_B)[g_{1010}(M_B)+g_{0101}(M_B)]$$

is different from zero, we may take the ratio of (39c) and (39d) and obtain, by using (A5) and (A6),

$$\left[\frac{2L(L+1)}{3(L-1)(L+2)}\right]^{1/2}R_1(LM)=5\epsilon\left[2-\frac{L(L+1)}{J(J+1)}\right] \quad (\text{even } L\geq 2), \quad (43)$$

where $R_1(LM)$ is an experimentally measurable quantity given by

$$R_1(LM)=\frac{4H(0000LM)+5[H(2000LM)+H(0020LM)]-50H(2020LM)}{H(2200LM)+H(0022LM)+5[H(2220LM)+H(2022LM)]}. \quad (44)$$

The relation (43) is most useful for $L=2$ because of its applicability for all values of $J\geq 1$. For higher values of L, the condition $H(l_1m_1l_2m_2LM)=0$ for $L>2J$ could make $R_1(LM)$ indeterminate. However, once the value of J has been determined, one could use (43) for higher values of L as a consistency check. It is possible to devise other relations involving ϵ and J. For example, if ϵ is -1, the ratio

$$R_2(LM)=\frac{2H(2121LM)}{H(212-1LM)+H(2-121LM)} \quad (\text{even } L\geq 2) \quad (45)$$

is related to the spin $J(\geq 1)$ by

$$\left[\frac{L(L+1)}{(L-1)(L+2)}\right]^{1/2} R_2(LM)=1-\frac{L(L+1)}{2J(J+1)} \quad (\text{even } L\geq 2). \quad (46)$$

This can be used as an additional test of spin J if ϵ is -1. Finally, we note that the parity of B can be obtained by the relation $\epsilon=\eta(-)^J$, once ϵ and J has been determined.

V. $B(J^\eta)\to B_1(1^\pm)+B_2(0^-)$

As pointed out in Sec. IV, we can treat on an equal footing the case of $B_1(1^-)$ decaying into 2 or 3 psuedoscalars and that of $B_1(1^+)$ decaying into two pseudoscalars. These decay modes of B have been considered by others[6–8] and by this author in an earlier work.[5] We simply list in Sec. V A the results again, since we have used in this paper somewhat different conventions and notations.[17] In Sec. V B, we take up the case of $B_1(1^+)$ decaying into 3 pseudoscalars.

A. $B_1(1^-)\to 2$ or 3 Pseudoscalar Mesons

The starting relation for this case is obtained from (36)

$$H(lmLM)=t_L^{M*}(10l0|10)\sum_{\lambda\lambda'} g_{\lambda\lambda'}(J\lambda'Lm|J\lambda) \times(1\lambda'lm|1\lambda). \quad (47)$$

Using this equation, we easily obtain the following formulas for even L:

$$3t_L^{M*}g_{00}(J0L0|J0)=H(00LM)+5H(20LM), \quad (48a)$$

$$6t_L^{M*}g_{11}(J1L0|J1)=2H(00LM)-5H(20LM), \quad (48b)$$

$$-\epsilon(\tfrac{1}{5}\sqrt{6})t_L^{M*}g_{11}(J-1L2|J1)=H(22LM). \quad (48c)$$

[17] The moments $H(lmLM)$ are related to $G^*(lmLM)$ defined in Ref. 5, but there is difference in the argument of D functions; in this paper we use $D_{\mu m}{}^J(\varphi,\theta,0)$ instead of $D_{\mu m}{}^J(\varphi,\theta,-\varphi)$ in Ref. 5. We also used a different coordinate system for the decay of B in Ref. 5; there we used the normal to the production plane as the z axis.

If the spin of B is zero, the parity is simply ± 1 for $\eta_1=\pm 1$. Therefore, we have $\epsilon=+1$ for both $\eta_1=+1$ and $\eta_1=-1$. Just as in Sec. IV, *all* $H(lmLM)$ should be zero for $L\geq 1$. In addition, we should have $H(0000)=1$ and $H(2000)=\frac{2}{5}$.

Now consider the case when J is greater than zero. The first information on the spin and parity of B comes from the condition $g_{\lambda\lambda'}=0$ for λ or $\lambda'=0$ when $\epsilon=-1$ [see (14a)]. In this case, it is easy to show that

$$H(00LM)+5H(20LM)=0 \quad (\text{even } L), \quad (49a)$$

$$H(21LM)=0 \quad (\text{all } L\geq 1). \quad (49b)$$

These conditions are not expected to be satisfied in general if ϵ is $+1$, so that (49) yields information on ϵ. If ϵ is known and if the factor

$$\int dM_B t_L^{M*}(M_B)g_{11}(M_B)$$

is different from zero, we can take the ratio of (48b) and (48c) to get

$$\left[\frac{2L(L+1)}{3(L-1)(L+2)}\right]^{1/2} R_3(LM)=5\epsilon\left[2-\frac{L(L+1)}{J(J+1)}\right] \quad (\text{even } L\geq 2), \quad (50)$$

where we have used (A5) and (A6) and $R_3(LM)$ is a quantity to be determined experimentally:

$$R_3(LM)=\frac{2H(00LM)-5H(20LM)}{H(22LM)}. \quad (51)$$

We note that Eq. (50) is essentially the same as that given in our earlier work.[18] In practice, the above spin formula should be used for $L=2$. If J turns out to be greater than one, one may use higher values of L as a check. The parity is obtained through the relation $\epsilon=\pm\eta(-)^J$ for $\eta_1=\pm 1$.

B. $B_1(1^+)\to 3$ Pseudoscalar Mesons

We start with Eq. (35a). From the symmetry relation (14b), we have $g_{\lambda\lambda'}{}^\mu=0$ if μ is even, so that we have only two values of μ, i.e. $\mu=\pm 1$. Therefore, from (35a) we have

$$H(lmLM)=t_L^{M*}(11l0|11)\sum_{\lambda\lambda'}[g_{\lambda\lambda'}{}^{+1}+(-)^l g_{\lambda\lambda'}{}^{-1}] \times(J\lambda'Lm|J\lambda)(1\lambda'lm|1\lambda). \quad (52)$$

It is easy to show the following formulas for even L:

$$3t_L^{M*}(g_{00}{}^{+1}+g_{00}{}^{-1})(J0L0|J0) =H(00LM)-10H(20LM), \quad (53a)$$

[18] Compare with Eq. (23), Ref. 5.

$$3t_L{}^{M*}(g_{11}{}^{+1}+g_{11}{}^{-1})(J1L0|J1)=H(00LM)+5H(20LM),\quad(53b)$$

$$\epsilon(\tfrac{1}{5}\sqrt{\tfrac{3}{2}})t_L{}^{M*}(g_{11}{}^{+1}+g_{11}{}^{-1})(J-1L2|J1)=H(22LM).\quad(53c)$$

If the spin of B is zero, we again have the condition $H(lmLM)=0$ for all $L\geq 1$. The parity of B is $+1$ so that $\epsilon=\eta=+1$. Furthermore, the following relations are true for $L=M=0$, i.e., $H(0000)=1$ and $H(2000)=-\frac{1}{5}$.

If J is greater than zero, we first determine the value of ϵ by the condition that, if ϵ is -1, we should have

$$H(00LM)-10H(20LM)=0\quad(\text{even } L),\quad(54a)$$

$$H(21LM)=0\quad(\text{all } L\geq 1).\quad(54b)$$

The spin J itself can be determined by taking the ratio of (53b) and (53c), if the factor

$$\int dM_B t_L{}^{M*}(M_B)[g_{11}{}^{+1}(M_B)+g_{11}{}^{-1}(M_B)]$$

is different from zero:

$$\left[\frac{2L(L+1)}{3(L-1)(L+2)}\right]^{1/2}R_4(LM)=-5\epsilon\left[2-\frac{L(L+1)}{J(J+1)}\right]\quad(\text{even } L\geq 2),\quad(55)$$

where $R_4(LM)$ is to be determined by

$$R_4(LM)=\frac{H(00LM)+5H(20LM)}{H(22LM)}.\quad(56)$$

Again, one would want to use (55) for $L=2$ and, as a check, for higher values of L. The parity η of B is then determined by the relation $\epsilon=\eta(-)^J$. Finally, we note that (54) and (55) can be used even if two of the three decay products of B_1 are identical. This has been achieved by considering only the combination $g_{\lambda\lambda'}{}^{+1}+g_{\lambda\lambda'}{}^{-1}$ [see (15b)].

VI. $B(J^\eta)\to B_1(2)+B_2(0^-)$

We shall consider two decay modes of B_1; in Sec. VI A we treat the decay of B_1 into two pseudoscalars and in Sec. VI B the decay of B_1 into three pseudoscalars.

A. $B_1(2^+)\to 2$ Pseudoscalar Mesons

We have from (36)

$$H(lmLM)=t_L{}^{M*}(20l0|20)\sum_{\lambda\lambda'}g_{\lambda\lambda'}(J\lambda'Lm|J\lambda)\times(2\lambda'lm|2\lambda).\quad(57)$$

Using this, it is straightforward to prove the following formulas for even L:

$$5t_L{}^{M*}g_{00}(J0L0|J0)=H(00LM)+5H(20LM)+9H(40LM),\quad(58a)$$

$$10t_L{}^{M*}g_{11}(J1L0|J1)=2H(00LM)+5H(20LM)-12H(40LM),\quad(58b)$$

$$10t_L{}^{M*}g_{22}(J2L0|J2)=2H(00LM)-10H(20LM)+3H(30LM),\quad(58c)$$

$$-\epsilon(\sqrt{\tfrac{2}{3}})t_L{}^{M*}g_{11}(J-1L2|J1)=H(22LM)+(2\sqrt{\tfrac{3}{5}})H(42LM),\quad(58d)$$

$$\epsilon[\tfrac{1}{3}\sqrt{(10/7)}]t_L{}^{M*}g_{22}(J-2L4|J1)=H(44LM).\quad(58e)$$

First, we want to determine the value of $\epsilon=\eta(-)^{J+1}$ [see (8)]. If ϵ is -1, we have from (14a) the condition $g_{\lambda\lambda'}=0$ for λ or $\lambda'=0$. Therefore, if ϵ is -1, we obtain the following conditions:

$$H(00LM)+5H(20LM)+9H(40LM)=0\quad(\text{even } L,\ J\geq 1),\quad(59a)$$

$$H(41LM)=H(21LM)=0\quad(L=1\text{ or }2,\ J=1),\quad(59b)$$

$$H(22LM)-(\tfrac{3}{2}\sqrt{\tfrac{3}{5}})H(42LM)=0\quad(\text{even } L\geq 2,\ J\geq 2),\quad(59c)$$

$$H(21LM)+[3\sqrt{(6/5)}]H(41LM)=0\quad(\text{all } L\geq 1,\ J\geq 2),\quad(59d)$$

where (59a) follows immediately from (58a), and the rest of the equations are shown easily by writing down (57) explicitly for indicated values of l and m. Of course, if J is zero, η is -1 so that $\epsilon=+1$.

Once ϵ is known, we proceed to determine J in the following manner: First consider the case $J=0$. Then, as before, we must have $H(lmLM)=0$ for all $L\geq 1$. Also, we must have $H(0000)=1$ and $H(2000)=H(4000)=2/7$. Next, we consider the case $J=1$. Then, it is clear that $H(lmLM)=0$ for all $L\geq 3$. In addition, we have from (58c),

$$2H(00LM)-10H(20LM)+3H(40LM)=0\quad(L=0,\ 2).\quad(60a)$$

Also, by writing down (57) explicitly for $J=1$, we obtain the following formulas:

$$H(22LM)-(\tfrac{3}{2}\sqrt{\tfrac{3}{5}})H(42LM)=0\quad(L=2),\quad(60b)$$

$$H(21LM)-(\sqrt{\tfrac{3}{10}})H(41LM)=0\quad(L=1,\ 2).\quad(60c)$$

Note that (59c) and (60b) are exactly the same for $L=2$. Therefore, if this condition is met experimentally, it means either that J is one or that ϵ is -1 for $J\geq 2$.

Now, we derive the spin formula valid for all J greater than zero. If the factor

$$\int dM_B t_L{}^{M*}(M_B)g_{11}(M_B)$$

is different from zero, we can take the ratio of (58b) and

(58d) and obtain

$$\left[\frac{2L(L+1)}{3(L-1)(L+2)}\right]^{1/2} R_5(LM)=5\epsilon\left[2-\frac{L(L+1)}{J(J+1)}\right] \quad (\text{even } L\geq 2), \quad (61)$$

where $R_5(LM)$ is to be determined by

$$R_5(LM)=\frac{2H(00LM)+5H(20LM)-12H(40LM)}{H(22LM)+(2\sqrt{\frac{3}{5}})H(42LM)}. \quad (62)$$

In practice, one may use (61) for $L=2$. Should the resulting J be higher than one, one could use higher values of L in (61) as a consistency check. We note in passing that it is possible to obtain another spin formula applicable for $J\geq 2$ by taking the ratio of (58b) and (58e). However, the resulting relation can be used only for even L equal or greater than 4. As before, we can determine the parity of B through the relation $\epsilon=\eta(-)^{J+1}$. A more detailed version of this particular example is given in Appendix B.

B. $B_1(2^+)\rightarrow 3$ Pseudoscalar Mesons

From (35) we can get the following formula:

$$H(lmLM)=t_L^{M*}(21l0|21)\sum_{\lambda\lambda'}[g_{\lambda\lambda'}^{+1}+(-)^l g_{\lambda\lambda'}^{-1}] \times(J\lambda' Lm|J\lambda)(2\lambda' lm|2\lambda), \quad (63)$$

where we have used the condition $g_{\lambda\lambda'}^{\mu}=0$ if μ is even. In order that our analysis be applicable even if two of the decay products of B_1 are identical, we consider only the even values of l, so that only the combinations $(g_{\lambda\lambda'}^{+1}+g_{\lambda\lambda'}^{-1})$ enter in our formula. Note that the factor $(g_{\lambda\lambda'}^{+1}-g_{\lambda\lambda'}^{-1})$ is identically zero, if there are identical particles [see (15b)].

Using (63) we can prove in a straightforward way the following formulas for even L:

$$10t_L^{M*}(g_{00}^{+1}+g_{00}^{-1})(J0L0|J0)=2H(00LM)+20H(20LM)-27H(40LM), \quad (64a)$$

$$5t_L^{M*}(g_{11}^{+1}+g_{11}^{-1})(J1L0|J1)=H(00LM)+5H(20LM)+9H(40LM), \quad (64b)$$

$$20t_L^{M*}(g_{22}^{+1}+g_{22}^{-1})(J2L0|J2)=4H(00LM)-40H(20LM)-9H(40LM), \quad (64c)$$

$$-(\epsilon/\sqrt{6})t_L^{M*}(g_{11}^{+1}+g_{11}^{-1})(J1L0|J1)=H(22LM)-(\tfrac{3}{2}\sqrt{\tfrac{3}{5}})H(42LM), \quad (64d)$$

$$-\epsilon[(2/9)\sqrt{(10/7)}]t_L^{M*}(g_{22}^{+1}+g_{22}^{-1})\times(J-2L4|J2)=H(44LM). \quad (64e)$$

Just as in Sec. VI A, our first task is to determine ϵ for all values of J. If J is zero, we have $\eta=-1$ so that $\epsilon=+1$. For higher values of J, we have the following conditions if ϵ is -1:

$$2H(00LM)+20H(20LM)-27H(40LM)=0 \quad (\text{even } L, J\geq 1), \quad (65a)$$

$$H(41LM)=H(21LM)=0 \quad (L=1 \text{ or } 2, J=1), \quad (65b)$$

$$H(22LM)+(9/8)(\sqrt{\tfrac{3}{5}})H(42LM)=0 \quad (\text{even } L\geq 2, J\geq 2), \quad (65c)$$

$$H(21LM)-[(9/4)\sqrt{(6/5)}]H(41LM)=0 \quad (\text{all } L\geq 1, J\geq 2), \quad (65d)$$

where (65a) follows simply from (64a), and the rest of the relations can be shown by writing down (63) explicitly for indicated values of l and m.

Next we determine the spin J. Suppose $J=0$. Then $H(lmLM)=0$ for all $L\geq 1$. In addition, we have $H(0000)=1$, $H(2000)=1/7$ and $H(4000)=-4/21$. If J is one, we must have $H(lmLM)=0$ for all $L\geq 3$. Furthermore, from (64c) we get

$$4H(00LM)-40H(20LM)-9H(40LM)=0 \quad (L=0, 2). \quad (66a)$$

As in Sec. VI A, we can also show that

$$H(22LM)+(9/8)\sqrt{\tfrac{3}{5}}H(42LM)=0 \quad (L=2), \quad (66b)$$

$$H(21LM)+\tfrac{3}{4}\sqrt{\tfrac{3}{10}}H(41LM)=0 \quad (L=1, 2). \quad (66c)$$

Again, (65c) and (66b) are exactly the same. Therefore, this relation implies either that J is one or that ϵ is -1 if J is greater than one.

Next, we derive the spin formula applicable if J is greater than zero. So long as the factor

$$\int dM_B t_L^M(M_B)[g_{11}^{+1}(M_B)+g_{11}^{-1}(M_B)]$$

is different from zero, we can take the ratio of (64b) and (64d) and obtain

$$\left[\frac{2L(L+1)}{3(L-1)(L+2)}\right]^{1/2} R_6(LM)=5\epsilon\left[2-\frac{L(L+1)}{J(J+1)}\right] \quad (\text{even } L\geq 2), \quad (67)$$

where $R_6(LM)$ is given by

$$R_6(LM)=\frac{H(00LM)+5H(20LM)+9H(40LM)}{H(22LM)-\frac{3}{2}(\sqrt{\frac{3}{5}})H(42LM)}. \quad (68)$$

By taking the ratio of (64c) and (64e), we can obtain an additional spin formula (valid for $J\geq 2$) independent of (67). However, we do not give the explicit formula here. Again, note that (67) is to be used for $L=2$. As before, if J turns out to be greater than 1, we can use higher values of L in (67) as a consistency check. Once ϵ and J have been determined, the parity η of B is given by the relation $\epsilon=\eta(-)^{J+1}$.

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APPENDIX A

For easy reference, we collect here the formulas used in the paper. The D functions used here are the same as those defined in Rose,[12] i.e.,

$$D_{\mu m}{}^{J}(\alpha,\beta,\gamma)=e^{-i\mu\alpha}d_{\mu m}{}^{J}(\beta)e^{-im\gamma}. \quad \text{(A1)}$$

The explicit forms of $d_{\mu m}{}^{J}(\beta)$ for J's up to 3 are given in Berman and Jacob.[11] In this paper, we adopt the following shorthand notation:

$$D_{\mu m}{}^{J}(\Omega)\equiv D_{\mu m}{}^{J}(\varphi,\theta,0), \quad \text{(A2)}$$

where $\Omega=(\theta,\varphi)$ as usual. The integrals involving the product of D functions are

$$\int d\Omega D_{\mu_1 m}{}^{j_1 *}(\Omega)D_{\mu_2 m}{}^{j_2}(\Omega)=\frac{4\pi}{2j_1+1}\delta_{\mu_1\mu_2}\delta_{j_1 j_2} \quad \text{(A3)}$$

and if $m_3=m_1+m_2$,

$$\int d\Omega D_{\mu_3 m_3}{}^{j_3 *}(\Omega)D_{\mu_2 m_2}{}^{j_2}(\Omega)D_{\mu_1 m_1}{}^{j_1}(\Omega)=\frac{4\pi}{2j_3+1} \times(j_1\mu_1 j_2\mu_2|j_3\mu_3)(j_1m_1j_2m_2|j_3m_3). \quad \text{(A4)}$$

We have used, frequently, the following two formulas involving Clebsch-Gordan coefficients[19]:

$$(J1L0|J1)=-\left[\frac{L(L+1)}{2J(J+1)}-1\right](J0L0|J0) \quad (\text{even } L), \quad \text{(A5)}$$

$$(J-1L2|J1)=-\left[\frac{L(L+1)}{(L-1)(L+2)}\right]^{1/2}(J0L0|J0) \quad (\text{even } L). \quad \text{(A6)}$$

APPENDIX B

In order to further illustrate how one applies the spin-parity analysis given in Secs. IV–VI, we take for our detailed analysis the example given in Sec. VI A, namely, that of the following decay chain of B: $B(J^\eta)\rightarrow B_1(2^+)+B_2(0^-)$, $B_1(2^+)\rightarrow 0^-+0^-$. Therefore, the method outlined here can be directly applied to the possible resonance[9] A_3 decaying into π and $f^0(1250)$.

First, we define the angles $\Omega(\theta,\varphi)$ and $\Omega_1(\theta_1,\varphi_1)$ more precisely (see Sec. II). In the B rest frame (BRF), the z axis is chosen to be along the direction of B (in the c.m. system) and the y axis may be taken to be along the normal to the production plane. The angles (θ,φ), which describe the direction of B_1 in the BRF, are measured with respect to this coordinate system. The angles (θ_1,φ_1), which are defined in the B_1RF, describe the direction of one of the pseudoscalars from B_1. The coordinate system for these angles are set up as follows: the z_1 axis is along the direction of B_1 (in the BRF) and y_1 axis is along $\mathbf{Z}\times\mathbf{B}_1$ (both defined in the BRF), which is invariant under the pure timelike Lorentz transformation from the BRF to the B_1RF.

In terms of the angles Ω and Ω_1 defined above, we can then evaluate experimentally the relevant moments from (35b)

$$H(lmLM)=\frac{1}{N}\sum_{i=1}^{N}D_{Mm}{}^{L}(\Omega^i)D_{m0}{}^{l}(\Omega_1{}^i), \quad \text{(B1)}$$

where Ω^i and $\Omega_1{}^i$ are the angles for the ith event in the B resonance sample and N is the total number of events in the sample. [See Appendix A for the definition of D functions in (B1).] In general, if one takes an experimental average of any real function f, the average $\bar{f}$ and its error $\delta\bar{f}$ may be evaluated by

$$\bar{f}=\frac{1}{N}\sum_{i=1}^{N}f_i \quad \text{(B2a)}$$

and

$$\delta\bar{f}=\frac{1}{\sqrt{N}}\left[\frac{1}{N}\sum_{i=1}^{N}f_i{}^2-\left(\frac{1}{N}\sum_{i=1}^{N}f_i\right)^2\right]^{1/2}. \quad \text{(B2b)}$$

So, if $H(lmLM)$ given in (B1) is real, one may use (B2b) to estimate its error.

Note that, because of (31b) and (31c), it is not necessary to evaluate $H(lmLM)$ for negative values of m or M. In addition, the relation (31d) implies that the imaginary part of $H(lmLM)$ for even l should be zero identically. Although (B1) can be used to evaluate directly the moments needed in relations (59), (60), and (62), it is instructive to write down the relations explicitly in terms of the angles (θ,φ) and (θ_1,φ_1); this we have done for a few simple cases of the relations (59), (60), and (62).

First, we shall use (59) in order to determine the parameter $\epsilon=\eta(-)^{J+1}$ for $J\geq 1$ ($\epsilon=+1$ if $J=0$). By setting $L=M=0$ in (59a), we obtain the following condition:

$$\langle 21\cos^4\theta_1-14\cos^2\theta_1+1\rangle=0. \quad \text{(B3)}$$

So, if (B3) is satisfied experimentally, one may conclude that $\epsilon=-1$. Of course, as a consistency check, one can consider other forms of (59a) by setting L greater than zero. In addition, (59b) yields information on ϵ if J is one and (59c) and (59d) can be used to determine ϵ for the case $J\geq 2$.

After having determined ϵ, we proceed to determine the spin itself using the formulas (60) and (61). Sup-

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[19] M. Rotenberg, R. Bivins, N. Metropolis, and J. K. Wooten, Jr., *The 3-j and 6-j Symbols* (The Technology Press, Massachusetts Institute of Technology, Cambridge, Mass., 1959); see the recursion relation (1.49).

pose $J=0$. Then, all $H(lmLM)$ is zero for $L\geq 1$. Furthermore, we should have

$$\langle\cos^2\theta_1\rangle=11/21\,, \qquad (B4)$$

$$\langle\cos^4\theta_1\rangle=3/7\,. \qquad (B5)$$

Suppose $J=1$. Then, of course, all $H(lmLM)$ should be zero for $L\geq 3$. In addition, we have from (60a) for $L=M=0$

$$\langle 21\cos^4\theta_1-42\cos^2\theta_1+13\rangle=0\,. \qquad (B6)$$

Again, one can use (60a) for $L=2$, as well as (60b) and (60c) as a consistency check.

Next, we give the spin formula valid for $J\geq 1$. We have from (61) with $L=2$ and $M=0$.

$$R_5(20)=10\epsilon\{1-3/[J(J+1)]\}\,,$$

where $R_5(20)$ can be calculated easily from (62), so that we obtain the following spin formula:

$$\frac{\langle(3\cos^2\theta-1)(21\cos^2\theta_1\sin^2\theta_1-2)\rangle}{\langle\sin^2\theta\cos^2\theta_1\sin^2\theta_1\cos 2\varphi_1\rangle}=21\epsilon\left[1-\frac{3}{J(J+1)}\right]. \qquad (B7)$$

Of course, we can obtain different spin formulas from (61) by using other allowed values of L and M.

If the numerator and denominator on the left-hand side of (B7) are consistent with zero experimentally, it implies that the factor

$$\int dM_B l_2^{0*}(M_B)g_{11}(M_B)$$

is either zero or too small for the given statistics and, of course, (B7) cannot be applied; we must then try (61) for other values of L and M. The spin-dependent factor on the right-hand side of (B7) is given below explicitly for different values of J:

$J=$	1	2	3	4	5	$\cdots$	∞
$\{1-3/[J(J+1)]\}=$	-0.5	0.5	0.75	0.85	0.90	$\cdots$	1

Thus, in general, a larger number of events is required to determine the spin, if its value is high.

In using (B7), one must evaluate the ratio of two experimental averages, which does not have Gaussian distribution in general even though the averages themselves may be Gaussian. The reader is referred to other works[15,16] for the statistical treatment involved in dealing with such a case.

Finally, we show how the formulas given here may be checked analytically with a few simple expressions for the angular distribution. First, assume $J^\eta=0^-$. Then, from (24) we obtain the *normalized* angular distribution

$$I(\Omega,\Omega_1)\equiv\int dM_B I(M_B,\Omega,\Omega_1)=\tfrac{5}{8}(3\cos^2\theta_1-1)^2\,. \qquad (B8)$$

Using this, it is easy to see that (B4) and (B5) are satisfied.

Next, assume $J^\eta=1^-$ so that $\epsilon=-1$. In order to obtain the simplest angular distribution consistent with symmetry requirements, we set all $\rho_{\lambda\lambda'}{}^J$ to zero except $\rho_{00}{}^J$ and assume that only g_{11} is not zero, i.e. $g_{11}=g_{-1-1}=\epsilon g_{1-1}=\epsilon g_{-11}\neq 0$. Then the following angular distribution is easily obtained from (24):

$$I(\Omega,\Omega_1)\sim\sin^2\theta\sin^2\theta_1\cos^2\theta_1(1-\cos 2\varphi_1)\,. \qquad (B9)$$

Again, it is simple to show, using the angular distribution given above, that (B3), (B6), and (B7) are satisfied. Suppose $J^\eta=1^+$ so that $\epsilon=+1$. As before, we assume $\rho_{00}{}^J\neq 0$ and $g_{11}\neq 0$. In addition, let us assume $g_{00}\neq 0$. Then, from (24), we obtain

$$I(\Omega,\Omega_1)\sim\alpha\cos^2\theta(3\cos^2\theta_1-1)^2+\sin^2\theta\sin^2\theta_1\cos^2\theta_1(1+\cos 2\varphi_1)\,, \qquad (B10)$$

where α is an arbitrary constant. It is straightforward to show that (B10) satisfies (B6) and (B7).

It is rather remarkable to see that (B7) is satisfied by (B10) independent of the arbitrary constant α. Note that the constant α reflects the admixture of two orbital angular momenta, allowed (i.e., 1 and 3) in the decay process $1^+\to 2^++0^-$. This is but a simple example of what we have set out to do in this paper: We have devised spin-parity tests that are *independent* of both the production and the decay dynamics of boson resonances with sequential decay modes.