Correspondence between Unstable Particles and Poles in S-Matrix Theory: The Exponential Decay Law

G. CALUCCI AND G. C. GHIRARDI

Istituto di Fisica Teorica dell'Università, Trieste, Italy and Istituto Nazionale di Fisica Nucleare, Sottosezione di Trieste, Italy

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The discussion on the correspondence between unstable particles and poles of the S matrix, started in a previous paper, is continued with particular attention to the decay law. It is shown that the previous formalism can be modified to obtain an S matrix, satisfying all the usual requirements of analyticity, unitarity, and asymptotic behavior, which exhibits an isolated resonance *with exponential decay* without an accompanying pole in the unphysical sheet. The existence of a local potential which yields the S matrix considered is also proved.

1. INTRODUCTORY CONSIDERATIONS

In a recent paper[1] it has been shown that it is possible to build up S matrices, satisfying all the usual requirements of analyticity, unitarity, and asymptotic behavior in energy, and exhibiting an isolated sharp resonance, without an accompanying pole in the unphysical sheet. This has been obtained by constructing a phase shift which goes rapidly through $\pi/2$ with positive derivative and which possesses the desired analytic properties. Such a phase shift gives rise to a bump in the cross section and to a time delay of the emitted wave packet, and this leads to the usual interpretation of the phenomenon as the production of an unstable system.

Usually, however, when one speaks of an unstable system, one also requires the exponential decay law for a large time interval, and this feature of the process has not been discussed in detail in I.

It can be easily understood, at least qualitatively, that the development of the decaying system depends on the detailed behavior of the phase shift as a function of the energy and is not automatically guaranteed by the rapid variation through $\pi/2$ of the phase shift. In fact, the exponential decay law occurs for times larger than the mean lifetime of the system. Since the time dependence of the process is obtained by means of an integration over the relevant interval of the energy [see Eq. (3) below], for large times we are analyzing the scattering amplitude with an oscillating function of the energy of very short wavelength, so that also the details of the amplitude become important.

To see explicitly how the decay law can depend on the details of the resonating phase shift, we look at a very simple, although unrealistic, example. Let us assume that the scattering phase shift has the following form

$$\begin{aligned}\delta(E)&=0 && \text{for}\quad E<a-\tfrac{1}{4}\pi\Gamma,\\ \delta(E)&=\tfrac{1}{2}\pi+\frac{2}{\Gamma}(E-a) && \text{for}\quad a-\tfrac{1}{4}\pi\Gamma\leqslant E\leqslant a+\tfrac{1}{4}\pi\Gamma,\\ \delta(E)&=\pi && \text{for}\quad E>a+\tfrac{1}{4}\pi\Gamma.\end{aligned}\qquad(1)$$

The evaluation of the integral

$$\mathcal{J}(r,t)=\frac{1}{2i}\int_{a-\Gamma/2}^{a+\Gamma/2}e^{i(kr-Et)}[e^{2i\delta(E)}-1]dE,$$

using the approximation

$$k\equiv(2mE)^{1/2}\simeq(2ma)^{1/2}(1+E/2a+\cdots),$$

leads to the following time dependence:

$$|\mathcal{J}(r,t)|^2=\frac{16}{\Gamma^2(4/\Gamma+wr-t)^2(wr-t)^2}[\sin\tfrac{1}{4}\pi\Gamma(wr-t)]^2,\qquad w=(m/2a)^{1/2}.\qquad(2)$$

We observe that the result (2) is not due to the unphysical presence of edges in the assumed phase shift. In fact, as shown below, given two phase shifts $\delta_1(E)$ and $\delta_2(E)$, the two resulting decay laws differ as little as one wishes for finite times, provided $|\delta_1(E)-\delta_2(E)|$ is made sufficiently small within the interesting range of energies.

For the above reasons, if one wishes to obtain also the exponential decay law, one cannot simply build up a phase shift going rapidly through $\pi/2$, as in I, but one must also take care of the details of the energy dependence in the resonance region. In this paper we shall show how the formalism introduced in I can be modified to obtain an S matrix with all the usual properties, and possessing an isolated sharp resonance *with the exponential decay law*, without an associated pole.

2. COMPARISON BETWEEN TWO DECAY LAWS

We compare here the decay laws obtained from two phase shifts $\delta_1(E)$ and $\delta_2(E)$ which differ very little in the relevant energy interval.

The decay law resulting from a given phase shift is governed by the modulus of the integral[2]

$$\mathcal{L}(r,t)=\int_A^\infty g(r,E)e^{-iEt}[(e^{2i\delta(E)}-1)/2i]dE,\qquad(3)$$

[1] G. Calucci, L. Fonda, and G. C. Ghirardi, Phys. Rev. 166, 1719 (1968). This paper will be indicated as I in what follows.

[2] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Co., New York, 1966), Chap. 19.

where $g(r,E)=f(E)e^{ikr}$, and $f(E)$ is the energy form factor of the incoming wave packet. Now let

$$\delta_2(E)=\delta_1(E)+\epsilon(E);$$

then

$$|\mathcal{L}_2-\mathcal{L}_1|=\frac{1}{2}\left|\int_A^\infty g(r,E)e^{-iEt}e^{2i\delta_1(E)}(e^{2i\epsilon(E)}-1)dE\right| \leqslant \int_A^\infty |f(E)|\,|\sin\epsilon(E)|dE. \quad (4)$$

For any reasonable impinging wave packet the integral $\int_A^\infty |f(E)|dE$ is convergent. Since, moreover, $f(E)$ is chosen to be appreciably different from zero only in a finite interval around the resonance energy, it follows that $|\mathcal{L}_1-\mathcal{L}_1|$ can be made arbitrarily small by making $\epsilon(E)$ sufficiently small in this region. Therefore, if $\delta_1(E)$ gives rise to a pure exponential decay law for a certain time interval, $\delta_2(E)$ will give rise to the same decay law in the time interval considered, with a superimposed disturbance which can, however, be made very small. The result of Eq. (4) shows also that the actual value of the derivative at the resonance point is not of crucial relevance for the time development of the state.

The unusual time dependence of Eq. (2) can now be simply understood. Let us consider the phase shift corresponding to a Breit-Wigner resonance

$$\delta_{\rm BW}(E)=\arctan[-\Gamma/2(E-a)] \quad (5)$$

and the phase shift of Eq. (1), which has the same derivative at $E=a$. Even though, for $\Gamma\to 0$, both phase shifts approach the step function $\pi\vartheta(E-a)$, the difference between them at the points $a\pm\pi\Gamma/4$ is equal to $\arctan(2/\pi)$ which does not depend on Γ. Therefore the above argument cannot be applied to this case.

3. APPROXIMATION OF A BREIT-WIGNER PHASE SHIFT

We start now by considering the phase shift of Eq. (5), which can be written as

$$\delta(E)=\tfrac{1}{2}\pi+2\Gamma\int_0^{E-a}\frac{dx}{4x^2+\Gamma^2} \quad (5')$$

and we try to express this function as a series of polynomials which converges in an interval of the real axis $a-\Delta\leqslant E\leqslant a+\Delta$ with $\Delta\gg\Gamma$, $\Delta<a$. We write

$$\delta(E)=\tfrac{1}{2}\pi+\frac{2\Gamma}{\Gamma^2+2\Delta^2}\times\int_0^{E-a}\left[1+\frac{2\Delta^2}{\Gamma^2+2\Delta^2}\left(\frac{2}{\Delta^2}x^2-1\right)\right]^{-1}dx \quad (6)$$

and we observe that in the region considered $x^2\leqslant(E-a)^2\leqslant\Delta^2$ the integrand can be expanded in a geometrical series which is absolutely and uniformly convergent. The substitution $x=v\Delta/\sqrt{2}$ leads to

$$\delta(E)=\tfrac{1}{2}\pi+\frac{\sqrt{2}\Gamma\Delta}{\Gamma^2+2\Delta^2}\sum_{j=0}^{\infty}\left(\frac{-2\Delta^2}{\Gamma^2+2\Delta^2}\right)^j\times\int_0^{(E-a)\sqrt{2}/\Delta}(v^2-1)^j dv. \quad (7)$$

The absolute value of the integral at the right-hand side is certainly less than $\sqrt{2}$; therefore, writing

$$\delta(E)=\delta_n(E)+R_n(E), \quad (8)$$

with

$$R_n(E)=\frac{\sqrt{2}\Gamma\Delta}{\Gamma^2+2\Delta^2}\sum_{j=n}^{\infty}\left(\frac{-2\Delta^2}{\Gamma^2+2\Delta^2}\right)^j\times\int_0^{(E-a)\sqrt{2}/\Delta}(v^2-1)^j dv, \quad (9)$$

we have

$$|R_n(E)|\leqslant\frac{2\Gamma\Delta}{\Gamma^2+4\Delta^2}\left(\frac{2\Delta^2}{\Gamma^2+2\Delta^2}\right)^n, \quad (10)$$

which shows that $|R_n(E)|$ can be made arbitrarily small by increasing n. It is clear that $\delta_n(E)$ is a polynomial in E of degree $2n-1$, approximating the Breit-Wigner phase shift in the energy interval considered.

The so-obtained function $\delta_n(E)$ cannot be chosen as a physical phase shift, because it does not possess the correct asymptotic behavior at zero and infinite energies.

An acceptable phase shift will be obtained by multiplying $\delta_n(E)$ by two factors governing, respectively, the zero and infinite energy behavior. Care must be taken, however, that the resulting phase shift remain as near as one wishes to the Breit-Wigner phase shift in the relevant energy interval.

As regards the zero-energy limit, we confine ourselves, for simplicity, to an S-wave scattering and we consider the identity

$$1\equiv\left(\frac{E}{a}\right)^{1/2}\left(1+\frac{E-a}{a}\right)^{-1/2}=\left(\frac{E}{a}\right)^{1/2}\sum_{k=0}\frac{(2k-1)!!}{(2k)!!}\left(\frac{a-E}{a}\right)^k \quad (11)$$

valid for $|E-a|<a$, which is certainly verified when $|E-a|<\Delta$, being $\Delta<a$. The absolute value of the mth remainder of the above series is less than

$$|1-E/a|^{m+1}[1-|E-a|/a]^{-1}(E/a)^{1/2}$$

so that the function

$$Z_m(E)=\left(\frac{E}{a}\right)^{1/2}\sum_{k=0}^{m}\frac{(2k-1)!!}{(2k)!!}\left(\frac{a-E}{a}\right)^k, \quad (12)$$

besides having the correct zero-energy behavior, can be made arbitrarily close to 1, in the interval of interest $|E-a|<\Delta$, by increasing m. With a slight modification of the above argument one could deal with the higher angular momenta.

As regards the infinite-energy limit, we consider the function

$$F(E,c)=\exp\{-c\ln^2[1+(E-a)/(a+b)]\};\quad b,\,c>0$$

which goes to zero at infinity faster than any inverse power of E. It is easily seen that by properly choosing the parameter c, this function can be made arbitrarily close to 1 in the usual energy interval.

Finally, we consider the function

$$\delta_{n,m}(E,c)=Z_m(E)\delta_n(E)F(E,c).\tag{13}$$

This function has the correct zero- and infinite-energy behavior and approaches, as well as one wishes, a Breit-Wigner phase shift in the interval $|E-a|<\Delta$. We remark that the above procedure does not require Γ to be very small.

The analytic continuation $\delta_{n,m}(z,c)$ of the above function to complex energies does not possess any singularity apart from the kinematical cut from 0 to $+\infty$, and the dynamical cut from $-\infty$ to $-b$, and is zero at the branch point $z=-b$. These conclusions hold also for the other sheets of the Riemann energy surface. The corresponding S matrix $e^{2i\delta(z)}$ has no singularities, apart from the above-mentioned cuts, in any sheet of its Riemann surface, and goes to 1 when $|z|\to+\infty$.

We have therefore completed our task of building up an S matrix giving rise to a resonance, which has the exponential decay law apart from the usual deviations at very small and large times, without having any singularity in the unphysical sheets which can be associated to the unstable particle.

4. INVERSION PROBLEM

We shall discuss in this section the following problem: Does there exist a local potential yielding the phase shift of Eq. (13)? Obviously, we are not interested in obtaining the explicit form of the potential, but simply in showing that it exists.

According to the formalism of the inversion problem, the potential $\mathcal{V}_1$, which gives rise to a given phase shift, is obtained through the formula

$$\mathcal{V}_1(r)=\mathcal{V}_0(r)-2\frac{d}{dr}K(r,r),\tag{14}$$

where $K(r,r')$ is the solution of the linear integral equation of Gel'fand and Levitan[3,4]

$$K(r,r')=g(r,r')-\int_0^r dr''K(r,r'')g(r'',r).\tag{15}$$

[3] Reference 2, Chap. 20.

[4] I. M. Gel'fand and B. M. Levitan, Dokl. Akad. Nauk SSSR **77**, 557 (1951); Izv. Akad. Nauk SSSR Ser. Fiz. **15**, 309 (1951).

The function $g(r,r')$ appearing in this equation is obtained, in the case of s-wave scattering and under the hypothesis that both $\mathcal{V}_1$ and $\mathcal{V}_0$ do not possess s-wave bound states, through the formula

$$g(r,r')=\int_0^\infty dh(E)\varphi_0(E,r)\varphi_0(E,r').\tag{16}$$

In Eq. (16), $\varphi_0(E,r)$ is the solution of the s-wave radial Schrödinger equation for the potential $\mathcal{V}_0$:

$$-\frac{d^2}{dr^2}\varphi_0(E,r)+\mathcal{V}_0(r)\varphi_0(E,r)=k^2\varphi_0(E,r),\quad k^2=2mE$$

subject to the boundary condition

$$\lim_{r\to0}\varphi_0(E,r)/r=1;$$

here

$$dh(E)=d\rho_1(E)-d\rho_0(E),$$

with

$$\frac{d\rho_i(E)}{dE}=\frac{2mk}{\pi|D_i(E)|^2},\quad i=0,\,1$$

and $D_i(E)$ is the s-wave Jost function associated with $\mathcal{V}_i(r)$. We choose $\mathcal{V}_0(r)=0$ so that

$$\varphi_0(r)=(1/k)\sin kr,\quad D_0(E)=1,$$

and we remember that $D(E)$ is given in terms of the phase shift of Eq. (13) by the formula

$$D(E)=\exp\left[-\frac{\mathcal{P}}{\pi}\int_0^\infty\frac{\delta(E')}{E'-E}dE'\right].\tag{17}$$

The existence of the potential $\mathcal{V}_1(r)$ is automatically guaranteed[3] if the kernel of Eq. (15)

$$\langle r'|\mathbf{G}_r|r''\rangle=g(r',r'')\vartheta(r-r'')$$

is of the Hilbert-Schmidt type, that is, if the integral

$$\begin{aligned}\mathrm{Tr}\mathbf{G}_r\mathbf{G}_r^\dagger&=\int_0^\infty dE\left\{\frac{2mk}{\pi}\left[\frac{1}{|D_1(E)|^2}-1\right]^2\times\int_0^r dr'\frac{1}{k^2}\sin^2kr'\right\}\\&=\int_0^\infty dE\left\{\frac{m}{\pi k}\left[\frac{1}{|D_1(E)|^2}-1\right]^2\times\left(r-\frac{1}{2k}\sin2kr\right)\right\}\end{aligned}\tag{18}$$

is convergent. The existence of a potential which gives rise to a given phase shift depends therefore in a crucial manner on the way in which $|D_1(E)|$ tends to 1 when E tends to infinity.

In our case, by recalling that the phase shift of Eq. (13) is a continuous function of E which decreases at infinity faster than any inverse power of E, it can be shown, by means of some simple manipulations, that

$$\frac{\mathcal{P}}{\pi}\int_0^\infty \frac{\delta_1(E')}{E'-E}dE' \sim \frac{\text{const}}{E}, \quad \text{for} \quad E\to+\infty.$$

It follows immediately that

$$\left[\frac{1}{|D_1(E)|^2}-1\right]^2 \sim \frac{\text{const}}{E^2}, \quad \text{for} \quad E\to+\infty,$$

and this fact ensures the convergence of the integral appearing in Eq. (18).

We have therefore completed the proof of the existence of a local potential yielding the form (13) of the phase shift.

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Spin-Parity Analysis for Boson Resonances*

SUH URK CHUNG
Brookhaven National Laboratory, Upton, New York
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A general helicity formalism is developed for the determination of spin and parity of boson resonances of arbitrary spin which have sequential decay modes. The procedure is illustrated with a few simple but, in practice, important decay modes, namely, 1^-+1^-, $1^\pm+0^-$, and 2^++0^-, where $1^\pm$ and 2^+ mesons in turn decay into 2 or 3 pseudoscalar mesons. The method proposed here is independent of the dynamics of the production and decay process.

I. INTRODUCTION

We present in this paper a general helicity formalism[1,2] that enables one to determine the spin and parity of boson resonances with sequential decay modes; we treat as the maximum complexity the case of a boson resonance decaying into two intermediate bosons of arbitrary spin, both of which in turn decay into three pseudoscalar mesons. It is shown that the formalism thus developed can easily be applied to cases when the intermediate bosons decay into two pseudoscalar mesons or one of the intermediate bosons is a pseudoscalar.

Our basic tool for the spin-parity determination is the moments which are experimental averages of the product of three D functions (see Appendix A and Ref. 12). It is shown that these moments are conveniently parametrized in terms of the multipole parameters.[3,4] Our main task in this paper has been to show that there exist linear relations among different moments for certain spin-parity combinations of the parent bosons and that for some of the linear equations the coefficients themselves are known functions of the spin of the parent bosons; this affords a straightforward means of determining the spin and parity of the parent bosons.

A remarkable aspect of this method is that it is independent of the detailed dynamics of the production and decay mechanism of the parent bosons. In addition, our method is independent of the interference among the three decay products of either of the intermediate bosons. Our method does not apply, however, if there exists appreciable interference between the decay products of one of the intermediate bosons with those of the other. It is shown that our method can still be applied, if we limit our analysis to those events for which the interference is minimal. Of course, there is always the problem of interference with background events. However, our method can be used if the interference is not appreciable and if the moments for the background events alone are small, as should be the case when the background events consist mostly of phase-space events.

In Sec. II, we derive the general angular distributions starting with the Lorentz-invariant amplitude for the production and decay of the parent bosons. We introduce in Sec. III the multipole parameters and then the moments and give the symmetry properties satisfied by these moments. In Secs. IV–VI, we illustrate our spin-parity analysis with simple but, in practice, important examples. These include the case of a boson resonance

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[1] For the helicity formalism, the reader is referred to the standard work: M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959). However, we use a slightly different convention for the argument of D functions; instead of their $D_{\mu m}{}^J(\varphi,\theta,-\varphi)$, we use $D_{\mu m}{}^J(\varphi,\theta,0)$.

[2] We give three references for different approaches to spin-parity analysis of bosons: M. Ademollo, R. Gatto, and G. Preparata, Phys. Rev. **139**, B1608 (1965); C. Zemach, *ibid.* **140**, B109 (1965); E. de Rafael, Ann. Inst. Henri Poincaré **5**, 83 (1966).

[3] N. Byers and S. Fenster, Phys. Rev. Letters **11**, 52 (1963).

[4] See the first of two lectures by J. D. Jackson, *High Energy Physics* (Gordon and Breach Science Publishers, New York, 1965).