Calculation of the $\Delta(1236)$ Resonant Partial Amplitudes in Pion Photoproduction*

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Using an N/D-type ansatz, a modified dispersion relation for the partial-wave amplitudes is derived. The equations, if applied to the first resonance in pion photoproduction, allow a fairly complete incorporation of our phenomenological knowledge of the pion-nucleon final state together with a systematic treatment and estimate of the terms to be neglected in practice. Various possibilities of including unknown or uncertain high-energy contributions are discussed and applied. Explicit results for the multipoles $E_{1+}^{3/2}$, $M_{1+}^{3/2}$ are presented.

INTRODUCTION

THE uncertainties involved in the present-day evaluation of the partial-wave amplitudes $M_{1+}^{3/2}$ and $E_{1+}^{3/2}$ are of the same order of magnitude as several of the other $J=\frac{1}{2}$, $\frac{3}{2}$ partial-wave amplitudes. The determination of these smaller amplitudes from experimental data is extremely sensitive to the theory used for the first resonance, since a complete partial-amplitude analysis has not been possible up to now.

In spite of several recent attempts to improve the evaluation of the partial-wave amplitudes of the first resonance,[1–6] a systematic treatment—including estimates of the terms necessarily neglected to get a practical result—is still lacking. This is partly due to the fact that in the usual formalism the well-established theoretical and phenomenological knowledge about the first resonance was in no *lucid* way separated from our ignorance or hypothetical assumptions, so that the influence of the latter on the final result was not clear.

In this paper it is therefore the main aim to derive an N/D result for $E_{1+}^{3/2}$ and $M_{1+}^{3/2}$, in which the present knowledge about the first resonance—following mostly from pion-nucleon scattering—is incorporated as completely as possible. Furthermore, the result should be suitable for a systematic study of the neglected terms in practical applications. Unknown important contributions should be summarized in terms of as few parameters as possible. We shall use as basic assumptions (a) analyticity, (b) the knowledge of the phase in a *finite* interval, and (c) asymptotic properties of the amplitudes. The results derived from the assumptions are, of course, not restricted to pion photoproduction. But the following experimental facts about the first resonance will be decisive for the application of our results to the multipoles of pion photoproduction: (1) The possibility of applying the Watson theorem to obtain the phase φ of the partial amplitudes up to the region of the second resonance, although the strict threshold for two-pion photoproduction is already around the first resonance. (2) The fact that $\varphi<\pi$ and very near to π at the end of the interval, where φ is known. (3) The decrease of the ratios $\mathrm{Im}M_{1+}^{3/2}(W)/\mathrm{Im}M_{1+}^{3/2}(W_R)$ and $\mathrm{Im}E_{1+}^{3/2}(W)/\mathrm{Im}E_{1+}^{3/2}(W_R)$ for $W\gg W_R$, where W_R is the resonance energy. Finally, in order to apply our results we need an explicit representation for the partial-amplitude dispersion relations, particularly of the inhomogeneous term. Up to now one has obtained this only by projecting fixed-t dispersion relations.[7] This method yields a result for the inhomogeneous term which is strictly valid only in the region of the first resonance. But at present it has to be applied also at higher energies, so that in this way an arbitrariness of the final results might be introduced, which has to be bypassed. This point will be discussed thoroughly in Sec. V.

Our starting point is the paper of Finkler.[1] His result will be generalized. The question of uniqueness of the solutions for $E_{1+}^{3/2}$ and $M_{1+}^{3/2}$, emphasized in Refs. 2 and 3, will *not* arise.

Instead of the type of ambiguity discussed there, we find—from a practical point of view—that we must introduce at least one free parameter in each partial amplitude because of the unknown high-energy behavior of the inhomogeneous term.

Finally, we mention that in the following we use units such that $\hbar=c=m_\pi=1$ except for the amplitudes. In these units the nucleon mass $M=6.722$ (or 6.952) if the π^+ (or π^0) mass is chosen for m_π. All amplitudes are given in units of $10^{-2}\hbar/(m_\pi c)$, so that, e.g., $\mathrm{Im}M_{1+}^{3/2}$ is

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of order unity around the resonance. If not otherwise stated, $m_\pi = m_{\pi^+}$.

I. OUTLINE OF THE METHOD

A. Notation

For convenience in numerical calculations we shall use the parity-conserving helicity amplitudes $F_\lambda^{J\mp,I}(W)$ introduced in Ref. 8. J denotes the total angular momentum and I the two isovector amplitudes $\frac{1}{2}$, $\frac{3}{2}$. $\lambda=\frac{1}{2}$, $\frac{3}{2}$ is the helicity index and the signatures $\pm$ distinguish the parity for each J (for further explanation, see Ref. 8). For $J=\frac{1}{2}$ and $\frac{3}{2}$ we give the relation to the more familiar multipole notation[9] (with the isospin index I dropped).

$$F_{1/2}^{1/2^-}=\sqrt{2}E_{0+},\quad F_{1/2}^{1/2^+}=-\sqrt{2}M_{1-},\quad F_{3/2}^{1/2^\pm}\equiv 0, \tag{1.1a}$$

$$F_{1/2}^{3/2^-}=\tfrac{1}{2}\sqrt{2}(3E_{1+}+M_{1+}),\quad F_{3/2}^{3/2^-}=(\surd\tfrac{3}{2})(E_{1+}-M_{1+}), \tag{1.1b}$$

$$F_{1/2}^{3/2+}=-\tfrac{1}{2}\sqrt{2}(-3M_{2-}+E_{2-}),\quad F_{3/2}^{3/2+}=-(\surd\tfrac{3}{2})(M_{2-}+E_{2-}). \tag{1.1c}$$

It will be useful to separate from the F's a kinematical function $u_\lambda^{J\pm}(W)$ including the threshold factor

$$F_\lambda^{J\pm,I}(W)=u_\lambda^{J\pm}(W)H_\lambda^{J\pm,I}(W), \tag{1.2}$$

with

$$u_\lambda^{J\pm}(W)=\mp\frac{C(\mp W)(qk)^{J-1/2}}{k^{\lambda-1/2}}. \tag{1.3}$$

In (1.3), q and k are the momenta of the meson and photon in the c.m. system, respectively, and $C(W)$ is a kinematical function. The quantities q, k, and C are, in terms of the total energy W,

$$s=W^2, \tag{1.4a}$$

$$C(W)=\frac{s-M^2}{16\pi s}[(W+M)^2-1]^{1/2}, \tag{1.4b}$$

$$q^2=(1/4s)[s-(M+1)^2][s-(M-1)^2], \tag{1.4c}$$

$$k=(s-M^2)/2W. \tag{1.4d}$$

B. Basic Assumptions

1. Dispersion Relation

The partial amplitudes of the first pion-nucleon resonance $\Delta(1236)$ [denoted by $H_\lambda(W)$] satisfy the dispersion relation

$$\mathrm{Re}H_\lambda(W)=H_{\lambda,\mathrm{inh}}(W)+\frac{1}{\pi}P\int_{M+1}^{\infty}dW'\frac{\mathrm{Im}H_\lambda(W')}{W'-W}, \tag{1.5}$$

where $H_{\lambda,\mathrm{inh}}(W)$ is the inhomogeneous term specified in Sec. II C.

2. Asymptotic Properties

If it is assumed that the total pion-photoproduction cross section is finite for $W\to\infty$ on the positive real axis ($W\to+\infty$), then one has at least

$$W\to+\infty:\quad |E_{1+}(W)|\to\text{const},\quad |M_{1+}(W)|\to\text{const}, \tag{1.6a}$$

and according to (1.1)–(1.3)

$$W\to+\infty:\quad |W^3H_{1/2}|\to\text{const},\quad |W^2H_{3/2}|\to\text{const}. \tag{1.6b}$$

Then, because of (1.6b), one can introduce the functions

$$h_\lambda^n(W)=H_\lambda(W)-\frac{1}{\pi}\frac{1}{W^n}\int_{M+1}^{\infty}dW'\frac{W'^n\,\mathrm{Im}H_\lambda(W')}{W'-W}, \tag{1.7a}$$

with

$$n\leqslant 2\ \text{ for }\ \lambda=\tfrac{1}{2},\qquad \leqslant 1\ \text{ for }\ \lambda=\tfrac{3}{2}. \tag{1.7b}$$

Using unitarity and time-reversal invariance in Compton scattering, one can even show that

$$W\to+\infty:\quad |E_{1+}(W)|\to\text{const}/W,\quad |M_{1+}(W)|\to\text{const}/W, \tag{1.6c}$$

so that the bounds for n in (1.7b) can be raised by one unit. But we shall not use this stronger assumption in the following.

We assume sufficient smoothness conditions (like a Hölder condition; see, e.g., the discussion in Ref. 10, Sec. 2.II), so that

$$W\to+\infty:\quad P\int_{M+1}^{\infty}dW'\frac{W'^n\,\mathrm{Im}H_\lambda(W')}{W'-W}\to\text{const}/W^\beta, \tag{1.8}$$

with $\beta>0$. Because of (1.6) and (1.8),

$$W\to+\infty:\quad |W^nh_\lambda^n(W)|\to\text{const}/W^\beta. \tag{1.9}$$

Now using the identity

$$\frac{1}{W'-W}=-\frac{1}{W}\left[1+\frac{W'}{W}+\left(\frac{W'}{W}\right)^2+\cdots+\left(\frac{W'}{W}\right)^{l-1}\right]+\left(\frac{W'}{W}\right)^l\frac{1}{W'-W}, \tag{1.10}$$

with $l>0$, one derives from the dispersion relation

[8] W. Schmidt and G. Schwiderski, Fortschr. Physik **15**, 393 (1967).

[9] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[10] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 373 (1963).

(1.5) the relation

$$H_{\lambda,\mathrm{inh}}(W)=\frac{1}{W}\left[g_\lambda+\frac{g_{\lambda,1}}{W}+\cdots+\frac{g_{\lambda,n-1}}{W^{n-1}}\right]+h_\lambda{}^n(W)\,, \tag{1.11a}$$

with

$$g_{\lambda,i}=\frac{1}{\pi}\int_{M+1}^{\infty}dW\,W^i\,ImH_\lambda(W),\quad g_{\lambda,0}\equiv g_\lambda. \tag{1.11b}$$

Equation (1.11), together with (1.9), shows explicitly the leading asymptotic terms of the inhomogeneous term $H_{\lambda,\mathrm{inh}}$ for W on the positive real axis.

3. *Phase $\varphi_\lambda(W)$ of the Partial Amplitudes*

Finally, we assume that the phase $\varphi_\lambda(W)$ of $H_\lambda(W)$,

$$H_\lambda(W)=|H_\lambda(W)|e^{i\varphi_\lambda(W)}\,, \tag{1.12}$$

is known for W on the physical cut, up to the point $W=W_\lambda$. $\varphi_\lambda(W)$ is independent of λ and given by the (experimental) values for the pion-nucleon phase shift α_{33} of the $\Delta(1236)$ resonance (Watson theorem) as long as inelastic effects can be neglected. According to the phase-shift analyses[11] this should be possible at least up to the second resonance.

4. *D Function*

With the assumption *c*, the D function

$$D_\lambda(W)=\exp\left(-\frac{1}{\pi}\int_{M+1}^{W_\lambda}dW'\frac{\varphi_\lambda(W')}{W'-W}\right) \tag{1.13}$$

is completely known in the W-plane cut along $M+1\leq W\leq W_\lambda$. In the following we shall assume

$$\varphi_\lambda(W_\lambda)<\pi\,, \tag{1.14}$$

so that

$$\lim_{W\to W_\lambda}[(W-W_\lambda)D_\lambda(W)]=0\,. \tag{1.15}$$

The D function (1.13) obeys the dispersion relation

$$D_\lambda(W)=1+\frac{1}{\pi}\int_{W+1}^{W_\lambda}dW'\frac{\mathrm{Im}D_\lambda(W')}{W'-W}. \tag{1.16}$$

The same applies for $D_\lambda{}^{-1}(W)$:

$$D_\lambda{}^{-1}(W)=1+\frac{1}{\pi}\int_{M+1}^{W_\lambda}dW'\frac{\mathrm{Im}D_\lambda{}^{-1}(W')}{W'-W}. \tag{1.16'}$$

C. Derivation of the Main Result

1. *Basic Formula*

In this section it is our aim to derive a new representation for $H_\lambda(W)$ starting from the dispersion relation (1.5) and using the other assumptions specified in Sec. I B.

We introduce the N function

$$N_\lambda(W)=H_\lambda(W)D_\lambda(W)\,, \tag{1.17}$$

having all singularities of $H_\lambda(W)$ except the low-energy part of the physical cut L_1: $(M+1)\leq W\leq W_\lambda$. $N_\lambda(W)$ has the same asymptotic behavior as $H_\lambda(W)$ since $D_\lambda(W)\to 1$ for $W\to\infty$. Let us also define the modified N function $\bar{N}_\lambda{}^u(W)$ by

$$\bar{N}_\lambda{}^u(W)=N_\lambda(W)-\bar{N}_\lambda{}^h(W)\,, \tag{1.18}$$

with

$$\bar{N}_\lambda{}^h(W)=\frac{1}{\pi}\int_{W_\lambda}^{\infty}dW'\frac{D_\lambda(W')\,\mathrm{Im}H_\lambda(W')}{W'-W}. \tag{1.19}$$

$\bar{N}_\lambda{}^u(W)$ is regular on the total physical cut, so that the application of Cauchy's theorem yields the representation

$$\begin{aligned}\bar{N}_\lambda{}^u(W)&=\frac{1}{\pi}\int_{L_\mu}dW'\frac{D_\lambda(W')\,\mathrm{disc}H_\lambda(W')}{W'-W}\\&=\frac{1}{2\pi i}\int_{L_\mu}dW'\frac{D_\lambda(W')}{W'-W}\\&\qquad\times[H_{\lambda,\mathrm{inh}}{}^+(W')-H_{\lambda,\mathrm{inh}}{}^-(W')]\\&=\frac{1}{2\pi i}\int_C dW'\frac{D_\lambda(W')H_{\lambda,\mathrm{inh}}(W')}{W'-W}.\end{aligned} \tag{1.20}$$

In (1.20), $H_{\lambda,\mathrm{inh}}{}^+(W)$ [or $H_{\lambda,\mathrm{inh}}{}^-(W)$] denotes the definite limit of the inhomogeneous term $H_{\lambda,\mathrm{inh}}(W)$ if W approaches the unphysical cuts L_u along any path, which remains, however, on the left (or the right) of L_u. The contour C is a closed infinite contour surrounding the unphysical singularities in a counterclockwise sense. The vanishing of the contributions of the infinite circle is a consequence of the asymptotic assumptions already made implicitly in the representation (1.5). For any W far from a cut, one can easily contract the contour C, which now surrounds the point W and the low-energy part L_1 of the physical cut. One therefore arrives at the result

$$\bar{N}_\lambda{}^u(W)=H_{\lambda,\mathrm{inh}}(W)D_\lambda(W')-\frac{1}{\pi}\int_{M+1}^{W_\lambda}dW'\frac{H_{\lambda,\mathrm{inh}}(W')\,\mathrm{Im}D_\lambda(W')}{W'-W}. \tag{1.21a}$$

Approaching L_1 from the left or right, one obtains from (1.21a) for $W\in L_1$

$$\bar{N}_\lambda{}^u(W)=H_{\lambda,\mathrm{inh}}(W)\,\mathrm{Re}D_\lambda(W)-\frac{1}{\pi}P\int_{M+1}^{W_\lambda}dW'\frac{H_{\lambda,\mathrm{inh}}(W')\,\mathrm{Im}D_\lambda(W')}{W'-W}. \tag{1.21b}$$

For numerical calculations it is suitable to rewrite

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[11] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965); P. Auvil, A. Donnachie, A. T. Lea, and C. Lovelace, Phys. Letters **12**, 76 (1964); P. Bareyre, C. Bricman, A. V. Stirling, and G. Villet, *ibid.* **18**, 243 (1965).

(1.21) by means of (1.16) into the form

$$\bar{N}_\lambda^u(W)=H_{\lambda,\text{inh}}(W_\lambda)+D_\lambda(W)\times[H_{\lambda,\text{inh}}(W)-H_{\lambda,\text{inh}}(W_\lambda)]-\frac{1}{\pi}\int_{M+1}^{W_\lambda}dW'\times\frac{H_{\lambda,\text{inh}}(W')-H_{\lambda,\text{inh}}(W_\lambda)}{W'-W}\,\text{Im}D_\lambda(W'), \quad (1.21\text{a}')$$

and correspondingly also (1.21b).

From (1.21a′) the transition to Finkler's result[1] is easy. He uses, where allowed, the sharp-resonance approximation for the phase φ_λ:

$$\varphi_\lambda=0 \quad \text{for} \quad W<W_R, \qquad =\pi \quad \text{for} \quad W>W_R, \quad (1.22)$$

which yields the following approximate form for $D_\lambda(W)$:

$$D_\lambda(W)\approx(W-W_R)/(W-W_\lambda). \quad (1.23)$$

$[W_R\approx 8.87(m_\pi=m_{\pi^+})$, resonance energy$]$. In this approximation, the last integral in (1.21a′) is zero, so that

$$N_\lambda(W)\approx H_{\lambda,\text{inh}}(W_\lambda)+\frac{W-W_R}{W-W_\lambda}[H_{\lambda,\text{inh}}(W)-H_{\lambda,\text{inh}}(W_\lambda)] \quad (1.24)$$

if one neglects, as did Finkler, the high-energy contribution $N_\lambda^h(W)$.

2. *Inclusion of Asymptotic Properties*

To incorporate more explicitly the asymptotic properties (1.6b), we proceed in the following way: Let $P_{\lambda,n}(W)$ be a polynomial of degree n,

$$P_{\lambda,n}(W)=(W-W_1)(W-W_2)\cdots(W-W_n), \quad (1.25)$$

with arbitrary constants W_i, which will be specified below, and with n chosen in accordance with (1.7b). Let $a_{\lambda,l}$, $b_{\lambda,l}$ be the constants

$$a_{\lambda,l}=\frac{1}{\pi}\int_{W_\lambda}^{\infty}dW'W'^lD_\lambda(W')\,\text{Im}H_\lambda(W'), \quad (1.26)$$

$$b_{\lambda,l}=\frac{1}{\pi}\int_{M+1}^{W_\lambda}dW'W'^lH_{\lambda,\text{inh}}(W')\,\text{Im}D_\lambda(W'). \quad (1.27)$$

Then the representation

$$N_\lambda(W)=H_{\lambda,\text{inh}}(W)D_\lambda(W)+\frac{1}{P_{\lambda,n}(W)}\sum_{l=0}^{n-1}c_\lambda^{l,n}(W)\times(a_{\lambda,l}-b_{\lambda,l})-\frac{1}{P_{\lambda,n}(W)}\left\{\frac{1}{\pi}\int_{M+1}^{W_\lambda}dW'\frac{P_{\lambda,n}(W')}{W'-W}\times H_{\lambda,\text{inh}}(W')\,\text{Im}D_\lambda(W')-\frac{1}{\pi}\int_{W_\lambda}^{\infty}dW'\frac{P_{\lambda,n}(W')}{W'-W}\times D_\lambda(W')\,\text{Im}H_\lambda(W')\right\} \quad (1.28)$$

is identical to (1.18), (1.19), and (1.21a) because of the identity

$$\frac{P_{\lambda,n}(W)}{W'-W}=\frac{P_{\lambda,n}(W')}{W'-W}+\sum_{l=0}^{n-1}c_\lambda^{l,n}(W)W'^l, \quad (1.29)$$

where $c_\lambda^{l,n}$ is a polynomial of degree $(n-1-l)$. From (1.28) and (1.8), it follows that the asymptotic properties (1.6) are fulfilled if the subsidiary conditions

$$a_{\lambda,l}-b_{\lambda,l}=e_{\lambda,l},$$

with

$$e_{\lambda,l}=\sum_{i=0}^{l}d_{\lambda,i}g_{\lambda,l-i}, \qquad l=0,\,1,\,\cdots(n-1) \quad (1.30)$$

are applied. The constants e_l are defined by the asymptotic expansion

$$H_{\lambda,\text{inh}}(W)D_\lambda(W)=\frac{1}{W}\left(e_{\lambda,0}+\frac{e_{\lambda,1}}{W}+\cdots+\frac{e_{\lambda,k}}{W^k}\right)+V_\lambda(W), \quad (1.31)$$

following from (1.11a), with

$$W^{k+1}V_\lambda(W)\to 0 \quad \text{for} \quad W\to+\infty. \quad (1.32)$$

The constants $d_{\lambda,i}$ are defined by

$$D_\lambda(W)=1+\frac{d_{\lambda,1}}{W}+\frac{d_{\lambda,2}}{W^2}+\cdots \quad (1.33)$$

and $g_{\lambda,i}$ is taken from (1.11b). In the sharp-resonance approximation (1.22) and (1.23), one has

$$g_{\lambda,l}\approx W_R{}^lg_\lambda, \quad g_\lambda=\frac{1}{\pi}\int_{M+1}^{\infty}dW'\,\text{Im}H_\lambda(W') \quad (1.34)$$

and

$$d_{\lambda,0}\approx 1, \quad d_{\lambda,1}\approx W_\lambda-W_R, \quad \cdots, \quad d_{\lambda,k}\approx(W_\lambda-W_R)W_\lambda{}^{k-1}, \quad \cdots, \quad (1.35)$$

so that

$$e_{\lambda,l}=\sum_{i=0}^{l}d_{\lambda,i}g_{\lambda,l-i}\approx g_\lambda\Big(W_R{}^l+(W_\lambda-W_R)\sum_{i=1}^{l}W_\lambda{}^{i-1}W_R{}^{l-i}\Big)=g_\lambda W_\lambda{}^l. \quad (1.36)$$

Therefore, one obtains in this limit

$$\sum_{l=0}^{n-1}c_\lambda^{l,n}(W)e_{\lambda,l}\approx g_\lambda\sum_{l=0}^{n-1}c_\lambda^{l,n}(W)W_\lambda{}^l=\frac{g_\lambda}{W_\lambda-W}[P_{\lambda,n}(W)-P_{\lambda,n}(W_\lambda)], \quad (1.37)$$

where the definition (1.29) for the c's has been used. One should note that the result (1.37) is exact for $n=1$.

Using (1.37), we write the expression (1.28) in the form corresponding to (1.18) and (1.21).

$$N_\lambda(W)=N_\lambda{}^u(W)+N_\lambda{}^h(W)\,, \qquad (1.38)$$

$$N_\lambda{}^h(W)=\frac{1}{P_{\lambda,n}(W)}\frac{1}{\pi}\int_{W_\lambda}^{\infty}dW'\frac{P_{\lambda,n}(W')}{W'-W}\times D_\lambda(W')\,\mathrm{Im}H_\lambda(W')\,, \qquad (1.39)$$

$$N_\lambda{}^u(W)=H_{\lambda,\mathrm{inh}}(W)D_\lambda(W)+\frac{g_\lambda}{W_\lambda-W}\left(1-\frac{P_{\lambda,n}(W_\lambda)}{P_{\lambda,n}(W)}\right)-\frac{1}{P_{\lambda,n}(W)}\frac{1}{\pi}\int_{M+1}^{W_\lambda}dW'\frac{\mathrm{Im}D_\lambda(W')}{W'-W}\times P_{\lambda,n}(W')H_{\lambda,\mathrm{inh}}(W')\,. \qquad (1.40)$$

The result (1.38)–(1.40) is a direct consequence of the dispersion relation (1.5), the further assumptions in Sec. II B, the Cauchy theorem, and the approximation (1.37). With (1.38)–(1.40) we are able to formulate approximations in practical applications which fulfill the asymptotic properties (1.6b). Since the application of the Cauchy theorem always yields a unique result, there is also no ambiguity in the representation (1.38)–(1.40). Its usefulness stems from the fact that it is adapted to a systematic study on the basis of a few pieces of information about the first resonance, which we must have from other sources, e.g., experiment.

One should note that $H_\lambda(W)$ following from (1.13), (1.17), and (1.38)–(1.40) is by construction independent of the parameters W_λ and W_i [of (1.25)], which we have left free up to now. Any dependence on these parameters would imply that some of the approximations to be made are unjustified. Now in practice, when one evaluates $N_\lambda(W)$ at low energies W, one would like to neglect the high-energy contribution $N_\lambda{}^h(W)$ (1.39) or $\bar{N}_\lambda{}^h(W)$ (1.19). Then the parameter W_λ is fixed by the need for the contributions (1.19) or (1.39) to be negligible at the energies considered. In our case, W_λ has to lie at least above the second resonance $W_\lambda>11$, as will be discussed in the following section.

If $W_\lambda>11$, the phases $\varphi_\lambda(W)$ are needed up to rather high energies in order to calculate $D_\lambda(W)$. At these energies one has to expect a serious deviation from the Watson theorem in its simple form. Instead of

$$\varphi_\lambda(W)=\alpha_{33}(W)\,, \qquad (1.41)$$

one then has to write

$$\varphi_\lambda(W)=\alpha_{33}(W)+\Delta\varphi_\lambda(W)\,, \qquad (1.41')$$

where $\Delta\varphi_\lambda(W)$ arises from inelastic processes. To these, $\Delta\varphi_\lambda$ is related by a generalization of the Watson theorem. Then $\alpha_{33}(W)$ is the real part of the phase shift for the pion-nucleon scattering amplitude belonging to the first resonance.

Finally, let us consider the difference

$$\Delta_\lambda(W)=N_\lambda{}^u(W)-\bar{N}_\lambda{}^u(W)=\bar{N}_\lambda{}^h(W)-N_\lambda{}^h(W)=\frac{1}{P_{\lambda,n}(W)}\sum_{l=0}^{n-1}c_\lambda{}^{l,n}(W)a_{\lambda,l}\,. \qquad (1.42)$$

If at least one of the zeros W_i of $P_{\lambda,n}(W)$ goes to infinity, then this difference goes to zero. There is some evidence that an optimal $\Delta_\lambda(W)$ is obtained if $P_{\lambda,n}(W_\lambda)=0$. Because of the strong singularity of $D_\lambda(W)$ at $W=W_\lambda$ (see Sec. II A), the largest contributions to the integrals (1.19) and (1.26) for $\bar{N}_\lambda{}^h(W)$ and $a_{\lambda,l}$ come from a very narrow region at the lower end $W'\approx W_\lambda$ of the integration interval. Therefore, one has the *approximate* relationship

$$\frac{a_{\lambda,l}}{\bar{N}_\lambda{}^h(W)}\approx(W_\lambda-W)W_\lambda{}^l \quad \text{for} \quad W\ll W_\lambda\,, \qquad (1.43)$$

which we need only for $l=0$ and 1. Using (1.29), (1.42), and (1.43), one obtains

$$\frac{\Delta_\lambda(W)}{\bar{N}_\lambda{}^h(W)}\approx1-\frac{P_{\lambda,n}(W_\lambda)}{P_{\lambda,n}(W)}\,, \qquad (1.44)$$

from which follows the condition $P_{\lambda,n}(W_\lambda)=0$ for the best $\Delta_\lambda(W)$.

By this consideration, one of the zeros of $P_{\lambda,n}(W)$ should be practically always fixed. But the condition $P_{\lambda,n}(W_\lambda)=0$ has also the additional advantage that it suppresses the contributions at the end of the integration interval for $N_\lambda{}^u(W)$ (1.40) by giving a larger weight to the low-energy region, where $H_{\lambda,\mathrm{inh}}(W)$ is better known.

II. NUMERICAL APPROXIMATIONS

Before we turn in Sec. III to the evaluation of the partial amplitudes $H_\lambda(W)$ starting from Eq. (1.40), we discuss the numerical results for the D function, the approximation of the inhomogeneous term $H_{\lambda,\mathrm{inh}}(W)$, and consider the neglect of the high-energy contribution $N_\lambda{}^h(W)$ (1.39).

A. Numerical Results for $D_\lambda(W)$

Near the branch point $W=W_\lambda$ of the D function (1.13), the modulus of $D_\lambda(W)$ is infinite and may be represented in the form

$$|D_\lambda(W)|=|W-W_\lambda|^{-\varphi(W_\lambda)/\pi}f(W)\,, \qquad (2.1)$$

where $f(W)$ is finite at $W=W_\lambda$. In our applications the function $(W-W_\lambda)D_\lambda(W)$ usually appears, which is finite at $W=W_\lambda$ because of assumption (1.14). Numerical results for $(W-W_\lambda)D_\lambda(W)$ are plotted in Fig. 1(a) for three values of $W_\lambda=11.022$, 12.022, and 13.012, and with the assumption that the phase $\varphi_\lambda(W)$ is given by (1.41) [Fig. 1(b)]. From Fig. 1(a) it follows

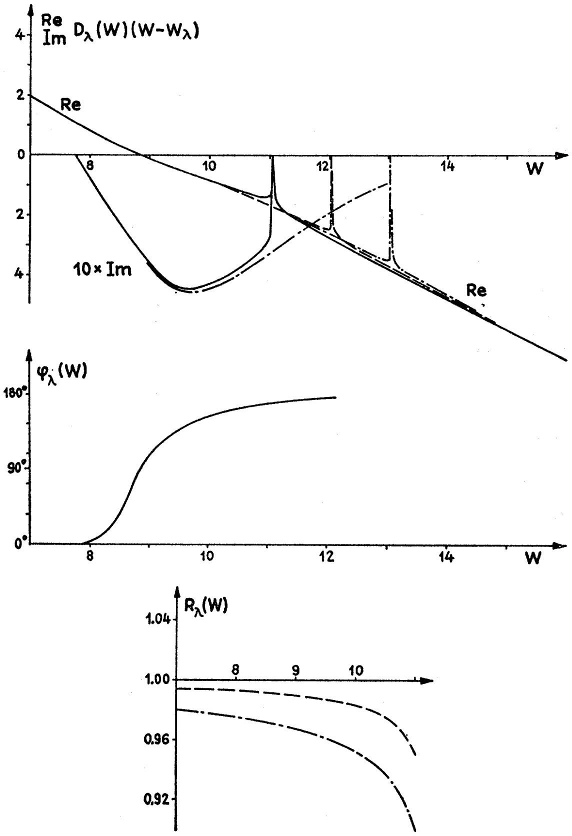

FIG. 1. (a) $\mathrm{Re}D_\lambda(W)(W-W_\lambda)$, $10\times\mathrm{Im}D_\lambda(W)(W-W_\lambda)$ for $W_\lambda=11.022$, 12.022, 13.022; (b) phase $\varphi_\lambda(W)=\alpha_{33}(W)$; (c) the ratio $R_\lambda(W)$ (2.3).

that apart from a very narrow region at $W=W_\lambda$, the function $(W-W_\lambda)\,\mathrm{Re}D_\lambda(W)$ follows roughly a straight line as is suggested by the narrow-width approximation (1.23).

Important for applications is an estimate of the uncertainty in $(W-W_\lambda)D_\lambda(W)$ caused by a failure of assumption (1.41), which is (as already mentioned) to be expected above the second pion–nucleon resonance $N^*(1518)$ according to Ref. 11. We tentatively, therefore, replaced Eq. (1.41) by the assumption (1.41′), with

$$\Delta\varphi_\lambda(W)=\tfrac{1}{9}\pi\frac{[q(W)-q(11.0)]}{[q(13.022)-q(11.0)]}\Theta(W-11.0)\,,\quad(2.2)$$

where $\Theta(W)$ is the unit step function. Equation (2.2)—according to which $\varphi_\lambda(W)$ would deviate from (1.41) above the second resonance—is simply an assumption to study possible effects on $D_\lambda(W)$. The ratio $R_\lambda(W)$ of the modulus of the new D function to the old one,

$$R_\lambda(W)=\exp\left(-\frac{1}{\pi}\int_{M+1}^{W_\lambda}dW'\frac{\Delta\varphi_\lambda(W')}{W'-W}\right),\quad(2.3)$$

is shown in Fig. 1(c). It varies at $W=W_R$ from $1.0\leqslant R_\lambda(W_R)\leqslant 0.97$, if W_λ varies between $11.022\leq W_\lambda\leq 13.022$ and if $\Delta\varphi_\lambda(W)$ is taken according to (2.2). Since at $W=13.022$ a violation of (1.41) by $\Delta\varphi_\lambda(W)=20°$ [as in (2.2)] is at least to be expected, the present ignorance of $\Delta\varphi_\lambda(W)$ gives an upper bound for W_λ, which seems to lie around $W\approx 12$.

B. High-Energy Contribution $N_\lambda{}^h(W)$

To estimate the high-energy contribution $N_\lambda{}^h(W)$ (1.39) for $W\ll W_\lambda$, we write

$$|N_\lambda{}^h(W)|\leq\left|\frac{1}{P_{\lambda,n}(W)}\right|\frac{1}{\pi}\int_{W_\lambda}^{\infty}dW'$$
$$\times\left|\frac{P_{\lambda,n}(W')}{W'-W}D_\lambda(W')\,\mathrm{Im}H_\lambda(W')\right|$$
$$=\left|\frac{\mathrm{Im}F_\lambda{}^{3/2-}(\bar{W})}{P_{\lambda,n}(W)}\right|\frac{1}{\pi}\int_{W_\lambda}^{\infty}dW'\left|\frac{P_{\lambda,n}(W')}{u_\lambda{}^{3/2-}(W')}\right|\left|\frac{D_\lambda(W')}{W'-W}\right|$$
$$=x_\lambda n_\lambda(W,W_i)\,,\quad(2.4)$$

with

$$x_\lambda=\left|\frac{\mathrm{Im}F_\lambda{}^{3/2-}(\bar{W})}{\mathrm{Im}F_\lambda{}^{3/2-}(W_R)}\right|\quad(2.5)$$

and

$$n_\lambda(W,W_i)=\left|\frac{\mathrm{Im}F_\lambda{}^{3/2-}(W_R)}{P_{\lambda,n}(W)}\right|\frac{1}{\pi}$$
$$\times\int_{W_\lambda}^{\infty}dW'\left|\frac{P_{\lambda,n}(W')}{u_\lambda{}^{3/2-}(W')}\right|\left|\frac{D_\lambda(W')}{W'-W}\right|.\quad(2.6)$$

In the third line of (2.4) we applied the mean-value theorem to take out from the integral the unknown part $\mathrm{Im}F_\lambda{}^{3/2-}(W)$ ($\bar{W}$ denotes the mean-value parameter). In Fig. 2 the ratio $n_\lambda(W,W_i)/|H_{\lambda,\mathrm{inh}}(W_\lambda)|$ is plotted against W_1 for $W_\lambda=12.022$ and different polynomials $P_{\lambda,n}(W)$ with

$$H_{1/2,\mathrm{inh}}(12.022)=0.51,\quad H_{3/2,\mathrm{inh}}(12.022)=-2.40,$$
$$\mathrm{Im}F_{1/2}{}^{3/2-}(W_R)=2.48,\quad \mathrm{Im}F_{3/2}{}^{3/2-}(W_R)=-4.29.\quad(2.7)$$

According to (1.24), one obtains in the sharp-resonance approximation

$$N_\lambda(W_R)\approx H_{\lambda,\mathrm{inh}}(W_\lambda)\,,\quad(2.8)$$

so that

$$|N_\lambda{}^h(W)/N_\lambda(W_R)|\leq x_\lambda n_\lambda(W,W_i)\,|H_{\lambda,\mathrm{inh}}(W_\lambda)|\,.\quad(2.9)$$

For $P_{\lambda,n=0}(W)\equiv 1$, the main contribution to the integral (2.4) comes from a small region at the lower end of the integration interval because of the singularity in $D_\lambda(W)$ at $W=W_\lambda$, so that in this case the mean-value parameter $\bar{W}\approx W_\lambda$. For $n=1$, $n_\lambda(W)$ has a sharp minimum, if the polynomial $P_{\lambda,n}(W)$ has a zero at $W_1=W_\lambda$. Since for $W_1=W_\lambda$ the mean-value parameter $\bar{W}\gg W_\lambda$, one expects that generally also the ratio x_λ (2.5) decreases. For $n=2$ no minimum appears, if one of the zeros, say, W_1

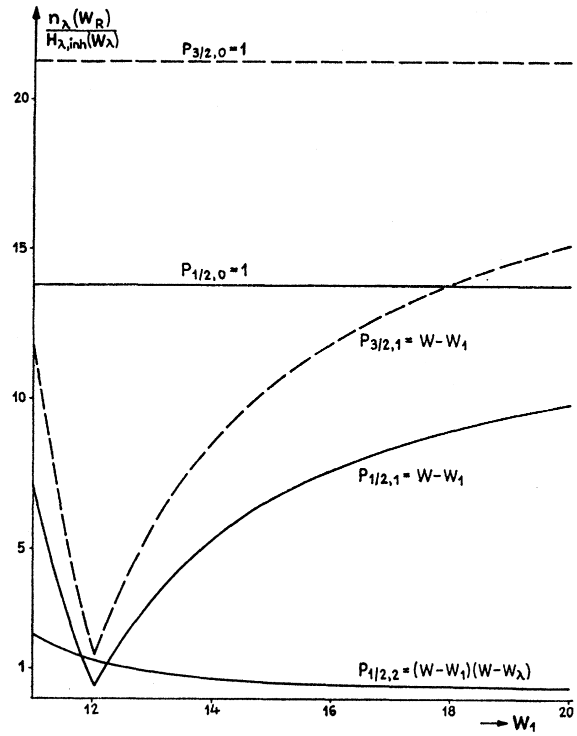

FIG. 2. The ratio $n_\lambda(W_R)/H_{\lambda,\text{inh}}(W_\lambda)$ for $W_\lambda=12.022$ and different degrees $n=0$, 1, 2, of the polynomials $P_{\lambda,n}(W)$.

$=W_\lambda$. The function $n_\lambda(W)$ decreases with W_1 and, for $W_1 \gg W_\lambda$, is of the same order as $n_\lambda(W)$ at the minimum for $n=1$. Therefore, the influence of the high-energy contribution should be damped for $W \ll W_\lambda$, (a) if the degree n of the polynomials $P_{\lambda,n}(W)$ is larger than zero and (b) if one of the zeros $W_i=W_\lambda$. If $x_\lambda<10^{-2}$, the ratio (2.9) is also $<10^{-2}$ for $n=1$ and $W_1=W_\lambda$ as one realizes from Fig. 2. For $W_\lambda>12$, it is reasonable to assume that $x_\lambda<10^{-2}$.

C. Results for $H_{\lambda,\text{inh}}(W)$

From fixed-t dispersion relations one derives an explicit result for $H_{\lambda,\text{inh}}(W)$ in terms of a partial-amplitude expansion, which converges in the region of the first resonance;

$$H_{\lambda,\text{inh}}(W)=H_\lambda(W)_{\text{p.t.c.}}+\Delta H_{\lambda,\text{inh}}(W)\,, \quad (2.10)$$

where $H_\lambda(W)_{\text{p.t.c.}}$ denotes the pole-term contribution,[8] and

$$\Delta H_{\lambda,\text{inh}}(W)=\frac{1}{\pi}\int_{M+1}^{\infty} dW' \sum_{L'} [\text{Im}H^{L'-}(W')K_\lambda{}^{L'}(W,W') - \text{Im}H^{L'+}(W')K_\lambda{}^{L'}(W,-W')]\,, \quad (2.11)$$

$$L'=\{2J',2I',2^{L'}\}\,.$$

Some numerical results for the exact kernels $K_\lambda{}^{L'}$ (taken from Ref. 8) are shown in Fig. 3 for $W=10$.

Typical for low values of J' is the smooth behavior in W', so that one can approximate (2.11) by

$$\Delta H_{\lambda,\text{inh}}(W)\approx \sum_{L';\,2J'\leq 5} [g^{L'-}K_\lambda{}^{L'}(W,W_\lambda{}^{L'-}) - g^{L'+}K_\lambda{}^{L'}(W,-W_\lambda{}^{L'+})]\,, \quad (2.12)$$

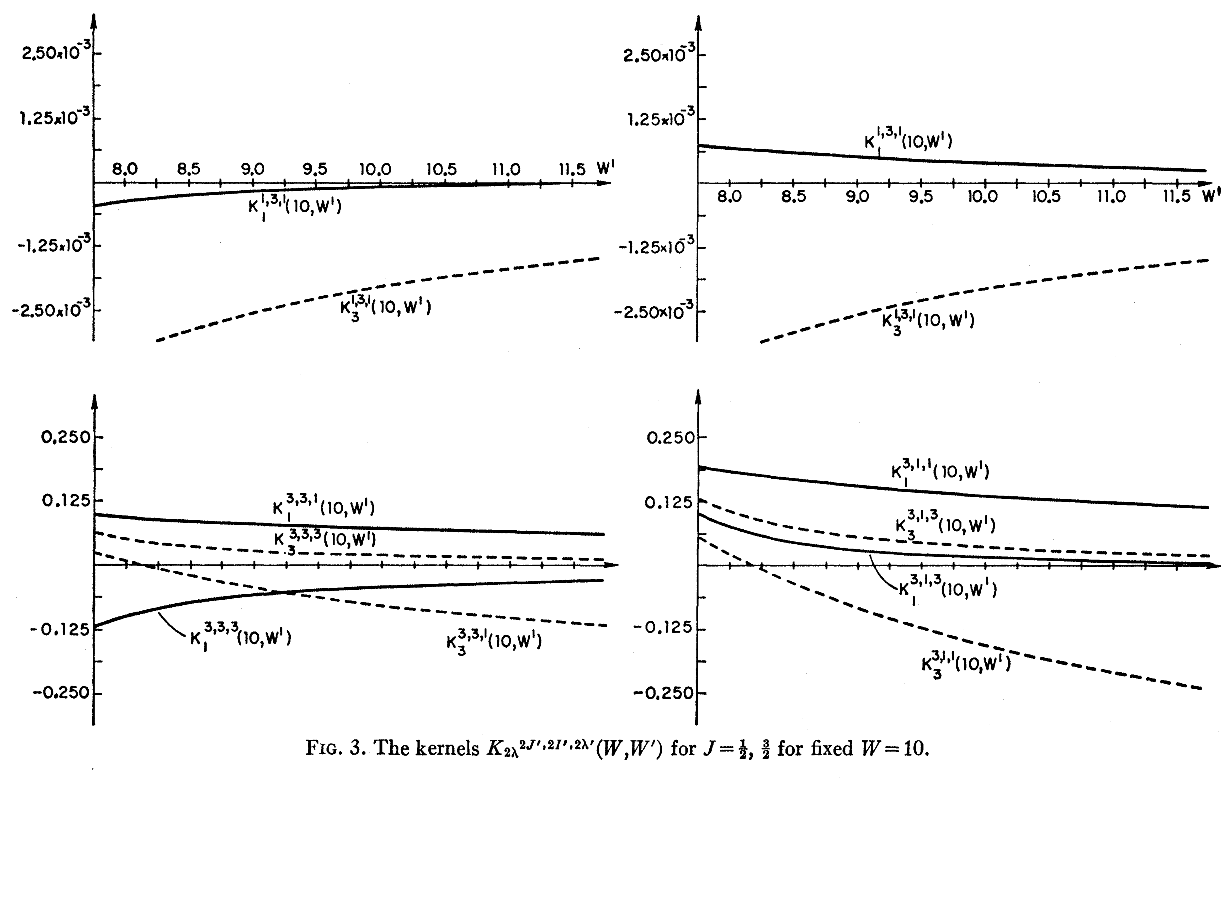

FIG. 3. The kernels $K_{2\lambda}{}^{2J',2I',2\lambda'}(W,W')$ for $J=\frac{1}{2}$, $\frac{3}{2}$ for fixed $W=10$.

TABLE I. Estimate of the different contributions (up to $J=\frac{5}{2}$) to the inhomogeneous term $H_{\lambda,\text{inh}}(W)$ at $W=W_R=8.73$ according to the respresentation (2.12).

	$2I'$	$2J'$	$2\lambda'$	P'	$g^{L'}$	$\bar{W}$	$K_\lambda{}^{L'}(8.73, \pm\bar{W})$ $\lambda=\frac{1}{2}$	$\lambda=\frac{3}{2}$	$g^{L'}K_\lambda{}^{L'}/H_\lambda(8.73)_{\text{p.t.c.}}$ $\lambda=\frac{1}{2}$	$\lambda=\frac{3}{2}$
S_1	1	1	1	−1	2.0	10.23	0.0002	−0.0046	0.001	0.0024
S_3	3	1	1	−1	6.1	10.23	0.0001	−0.0023	0.001	0.0036
P_{11}	1	1	1	+1	4.1	10.23	0.0014	−0.0046	0.012	0.0049
P_{31}	3	1	1	+1	−4.8	10.23	0.0007	−0.0023	−0.007	−0.0029
P_{33}	3	3	1	+1	1.60	8.73	0.11	−0.08	0.185	0.017
	3	3	3	+1	−6.25	8.73	−0.12	0.02	0.769	0.016
P_{13}	1	3	1	+1	-1.3×10^{-3}	10.23	0.19	−0.26	−0.001	$\sim-10^{-4}$
	1	3	3	+1	-3.0×10^{-1}	10.23	−0.02	0.02	0.010	0.002
D_{13}	1	3	1	−1	0.0	10.73	−0.01	−0.10	0.000	0.000
	1	3	3	−1	−2.7	10.73	−0.01	0.02	0.074	0.015
D_{33}	3	3	1	−1	-2.2×10^{-2}	10.23	−0.06	−0.04	0.003	$\sim-10^{-4}$
	3	3	3	−1	1.8×10^{-1}	10.23	0.00	−0.04	$\sim10^{-4}$	0.002
D_{15}	1	5	1	−1	1.0×10^{-3}	10.73	3.9	−5.4	0.011	0.002
	1	5	3	−1	4.3×10^{-3}	10.73	−0.5	0.9	−0.006	0.001
D_{35}	3	5	1	−1	1.1×10^{-3}	10.73	18.4	−2.7	0.057	0.001
	3	5	3	−1	$1.1\diagdown10^{-3}$	10.73	−1.0	10.4	−0.003	−0.004
F_{15}	1	5	1	+1	9.1×10^{-4}	10.73	−0.8	−1.4	−0.002	$\sim5.10^{-5}$
	1	5	3	+1	-1.1×10^{-2}	10.73	0.0	0.4	0.001	0.001
F_{35}	3	5	1	+1	-2.0×10^{-4}	10.73	−1.3	−0.7	0.001	$\sim5.10^{-5}$
	3	5	3	+1	3.7×10^{-3}	10.73	0.0	−0.4	$\sim10^{-4}$	$\sim5.10^{-4}$

with

$$g^{L'\pm}=\frac{1}{\pi}\int_{M+1}^{\infty} dW' \,\text{Im}H^{L'\pm}(W'). \tag{2.13}$$

The advantage of the approximation (2.12) is that details about the imaginary parts $\text{Im}H^{L'}(W)$ do not necessarily have to be known. To determine the influence of $\text{Im}H^{L'\pm}$ for $J'\leq\frac{5}{2}$ on the solution of the first resonance only some estimates of the "coupling constants" $g^{L'\pm}$ and the mean-value parameters $\bar{W}_\lambda{}^{L'\pm}$ are needed. According to Fig. 3, the dependence on the parameter $\bar{W}_\lambda{}^{L'\pm}$, which also depends of course on W, should not be critical in the cases considered.

Because of the threshold factor $(q'k')^{l'}$ in the kinematical factor (1.3), the high-energy region in (2.13) is strongly suppressed for $l'=(J'\pm\frac{1}{2})>0$. The quantity $q'k'$ becomes 10 around the second resonance, exactly at $W'=10.75$. Therefore, an estimate of the order of magnitude of $g^{L'\pm}$ should be possible using only the low-energy data for $\text{Im}H^{L'\pm}$.

In Table I, results for the contributions $g^{L'\pm}K_\lambda{}^{L'}$ to $\Delta H_{\lambda,\text{inh}}(W=W_R)$ are gathered. One observes, for $\lambda=\frac{1}{2}$, a strong influence of the D_{13} resonance and of some of the $J=\frac{5}{2}$ final states, apart from the first resonance itself. On the other hand, all $J=\frac{1}{2}$ contributions seem to be small, and partly cancel for the P waves. The coupling constants $g^{L'\pm}$, estimated as in Ref. 12, are to be considered only as very rough guesses apart from the first and second resonance. The total result for $H_{\lambda,\text{inh}}(W)$ is shown in Fig. 4 with the $J=\frac{1}{2}$ coupling constants $g^{L'}$ taken from Table I. The $g^{L'}$'s for the first resonance correspond to the later solutions 2, 4, 5, 6, and 8 in Fig. 5; those of the second resonance are again taken from Table I. All other couplings are neglected. Note that according to (2.12), $H_{\lambda,\text{inh}}(W)$ becomes flat above the second resonance. But at these energies one has to consider the results in Fig. 4 with the utmost care, since the partial-amplitude expansion (2.11) does not converge above $E=500$ MeV according to the postulates of the Mandelstam representation.[7]

[12] J. Engels, W. Schmidt, and G. Schwiderski, External Report No. 3/67-1, Gesellschaft für Kernforschung, Karlsruhe, 1967 (unpublished).

III. RESULTS FOR THE MULTIPOLES $M_{1+}{}^{3/2}$ AND $E_{1+}{}^{3/2}$

In the case of the first resonance it is more suitable to discuss *numerical* results in terms of the multipoles $M_{1+}{}^{3/2}$ and $E_{1+}{}^{3/2}$ (1.1) than in terms of helicity amplitudes $F_\lambda{}^{3/2-}$. The reason for this is that the enhancement due to the resonance is very pronounced in $M_{1+}{}^{3/2}$, and is not so strong in $E_{1+}{}^{3/2}$ because of a cancellation in the pole-term contribution. $E_{1+}{}^{3/2}$ is therefore particularly sensitive to some of the approximations. The N functions corresponding to E_{1+} and M_{1+} are

$$N_E(W)=\frac{1}{2\sqrt{2}}\left(\frac{D_{3/2}(W)}{D_{1/2}(W)}N_{1/2}(W)+N_{3/2}(W)\frac{1}{k\sqrt{3}}\right), \tag{3.1a}$$

$$N_M(W)=\frac{1}{2\sqrt{2}}\left(\frac{D_{3/2}(W)}{D_{1/2}(W)}N_{1/2}(W)-N_{3/2}(W)\frac{\sqrt{3}}{k}\right), \tag{3.1b}$$

so that

$$E_{1+}(W)=qkC(W)\frac{N_E(W)}{D_{3/2}(W)},$$

$$M_{1+}(W)=qkC(W)\frac{N_M(W)}{D_{3/2}(W)}, \tag{3.2}$$

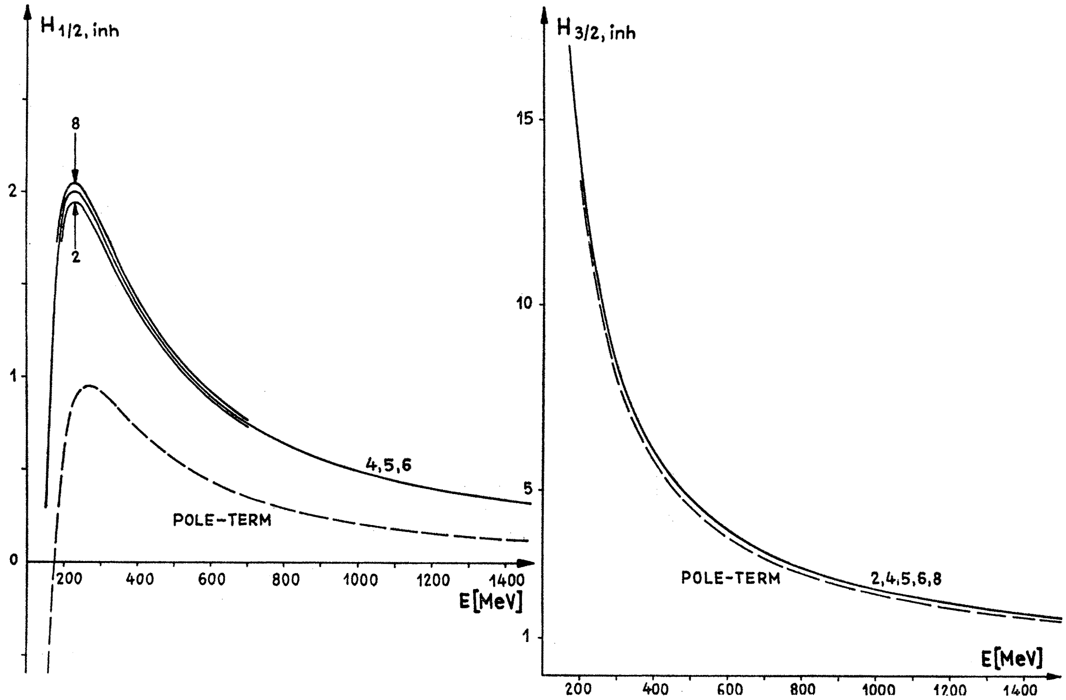

FIG. 4. $H_{\lambda,\mathrm{inh}}(W)$ according to (2.12) for the solutions 2, 4, 5, 6, 8; $m_\pi = m_{\pi^0}$.

according to (1.1), (1.2), and (1.7). We shall assume in the following that

$$D_{1/2}(W) = D_{3/2}(W)\,, \tag{3.3}$$

which is justified (a) if one puts

$$W_{1/2} = W_{3/2} = \max(\tilde{W}_{1/2}, \tilde{W}_{3/2})\,, \tag{3.4}$$

where $\tilde{W}_{1/2}$, $\tilde{W}_{3/2}$ are lower bounds for W_λ following from the considerations in Sec. II, and (b) if in the interval $M+1 \leq W \leq W_{1/2} = W_{3/2}$ the Watson theorem (1.41) is valid, so that $\varphi_{1/2} = \varphi_{3/2} = \alpha_{33}$.

In the following we use as reference for the comparison of $M_{1+}^{3/2}$ the old result of Chew, Goldberger, Low,

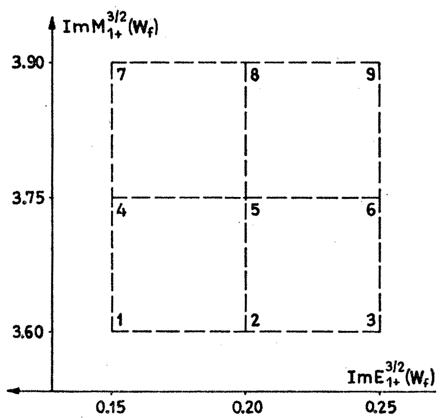

FIG. 5. The present range of uncertainty of $\mathrm{Im}M_{1+}^{3/2}(W_f)$ and $\mathrm{Im}E_{1+}^{3/2}(W_f)$. $W_f = 9.01$ ($E_f = 320$ MeV) and $m_\pi = m_{\pi^0}$.

and Nambu (CGLN)[9]:

$$M_{1+}^{3/2}(W)_{\mathrm{CGLN}} = M_{1+,\mu}^{3/2} + M_{1+,e}^{3/2}\,, \tag{3.5a}$$

$$M_{1+,\mu}^{3/2} = \frac{\mu_v}{f}\frac{k}{q}\frac{e^{i\alpha_{33}}\sin\alpha_{33}}{q}\,, \tag{3.5b}$$

$$M_{1+,e}^{3/2} = \tfrac{1}{3}efkqe^{i\alpha_{33}}\cos\alpha_{33}F_M(W)\,, \tag{3.5c}$$

with

$$\mu_v = \tfrac{1}{2}(g_P' + 1 - g_N)e/2M\,, \tag{3.5d}$$

$$F_M(W) = \frac{3}{4q^2}\left[1 + \frac{1-v^2}{2v}\ln\frac{1-v}{1+v}\right]\frac{M}{W}\,, \tag{3.5e}$$

$$v = q/(1+q^2)^{1/2}\,, \tag{3.5f}$$

$$e^2 = 1/137.0388, \quad f^2 = 0.080, \quad g_P' = 1.7928, \tag{3.5g}$$
$$g_N = -1.9131.$$

We shall usually consider the ratios

$$R_M(W) = M_{1+}^{3/2}(W)/M_{1+}^{3/2}(W)_{\mathrm{CGLN}} \tag{3.6a}$$

and

$$R_E(W) = E_{1+}^{3/2}(W)/M_{1+}^{3/2}(W)\,. \tag{3.6b}$$

A. Results Following From $\bar{N}_\lambda^u(W)$ [Eq. (1.21)]

From the approximate relation (1.24) follows the relationship

$$N_\lambda(W_R) \approx H_{\lambda,\mathrm{inh}}(W_\lambda)\,. \tag{3.7}$$

Equation (3.7) demonstrates the critical dependence on the high-energy behavior of $H_{\lambda,\mathrm{inh}}(W)$, if one calcu-

TABLE II. The parameters g_λ, $\bar{H}_{\lambda,\mathrm{inh}}$, $g_\lambda/(W_\lambda-W_R)$, and $H_{\lambda,\mathrm{inh}}(W_\lambda)$ for the solutions 1, 2, $\cdots$, 9. $W_\lambda=12.25$, $W_R=9.17$, and $m_\pi=m_{\pi^0}$.

Solution	$g_{1/2}$	$\bar{H}_{1/2,\mathrm{inh}}$	$g_{1/2}/(W_{1/2}-W_R)$	$H_{1/2,\mathrm{inh}}(W_{1/2})$	$g_{3/2}$	$\bar{H}_{3/2,\mathrm{inh}}$	$g_{3/2}/(W_{3/2}-W_R)$	$H_{3/2,\mathrm{inh}}(W_{3/2})$
1	1.59	0.53	0.52	0.49	−6.02	−1.99	−1.95	−2.36
2	1.50	0.50	0.49	0.49	−6.11	−2.03	−1.98	−2.37
3	1.41	0.47	0.46	0.49	−6.21	−2.06	−2.02	−2.37
4	1.67	0.55	0.54	0.50	−6.30	−2.09	−2.05	−2.37
5	1.58	0.52	0.51	0.50	−6.39	−2.12	−2.07	−2.37
6	1.49	0.49	0.48	0.50	−6.49	−2.16	−2.11	−2.38
7	1.75	0.58	0.57	0.52	−6.58	−2.18	−2.14	−2.38
8	1.66	0.55	0.54	0.52	−6.67	−2.22	−2.17	−2.38
9	1.58	0.52	0.51	0.52	−6.76	−2.25	−2.19	−2.39

lates the N function $\bar{N}_\lambda{}^u(W)$ in the region of the first resonance. We mentioned at the end of Sec. III C that the result (2.11) for $H_{\lambda,\mathrm{inh}}(W)$ becomes doubtful for energies W above the second resonance. In view of this difficulty we used therefore the following semiphenomenological ansatz for $\bar{N}_\lambda{}^u(W)$:

$$\bar{N}_\lambda{}^u(W)=\bar{H}_{\lambda,\mathrm{inh}}+\mathrm{Re}D_\lambda(W)[H_{\lambda,\mathrm{inh}}(W)-\bar{H}_{\lambda,\mathrm{inh}}] -\frac{1}{\pi}\int_{M+1}^{W_\lambda}dW'\frac{H_{\lambda,\mathrm{inh}}(W')-H_{\lambda,\mathrm{inh}}(W_\lambda)}{W'-W}\times\mathrm{Im}D_\lambda(W'). \quad (3.8)$$

Equation (3.8) is identical with (1.21a′) on the physical cut if $\bar{H}_{\lambda,\mathrm{inh}}=H_{\lambda,\mathrm{inh}}(W_\lambda)$. In (3.8) the difference

$$H_{\lambda,\mathrm{inh}}(W)-H_{\lambda,\mathrm{inh}}(W_\lambda) \quad (3.9)$$

appears only under the integral, which gives a small contribution at our energies compared to that of $\bar{H}_{\lambda,\mathrm{inh}}$ and which is not very sensitive to the high-energy behavior of $H_{\lambda,\mathrm{inh}}(W)$.

We shall calculate the difference (3.9) up to the energy W_λ using the approximation (2.12) for $H_{\lambda,\mathrm{inh}}(W)$. The values for the coupling constants $g^{L'\pm}$ are taken from Table I apart from those of the first resonance. But the dependence on the first group of constants is not critical. To determine the coupling constants $g^{33\lambda'-}=g_{\lambda'}$ of the first resonance, one establishes a linear relation between the constants $\bar{H}_{\lambda,\mathrm{inh}}$ and g_λ. This relationship is obtained by expressing in the definition (1.11b) for g_λ the integrand $\mathrm{Im}H_\lambda(W)$ by means of (1.13), (1.17), and (3.8). In Eq. (1.11b) we have chosen as cutoff the energy $W_c=11.0$ ($E_c\approx 800$ MeV). The parameters $\bar{H}_{\lambda,\mathrm{inh}}$ are then fixed in such a way that at one point $W_f=9.01$ ($E_f=320$ MeV), which is near to the resonance, $\mathrm{Im}M_{1+}{}^{3/2}(W_f)$ and $\mathrm{Im}E_{1+}{}^{3/2}(W_f)$ lie within certain limits (Fig. 5). These are given by the present status of the phenomenological interpretation of the π^0 photoproduction data.[13]

In Table II results for $\bar{H}_{\lambda,\mathrm{inh}}$ are shown and compared with $H_{\lambda,\mathrm{inh}}(W_\lambda)$ for solutions 1–9 which correspond to the points 1–9 indicated on the $\mathrm{Im}M_{1+}{}^{3/2}(W_f)-\mathrm{Im}E_{1+}{}^{3/2}(W_f)$ plane (Fig. 5). In all cases the difference

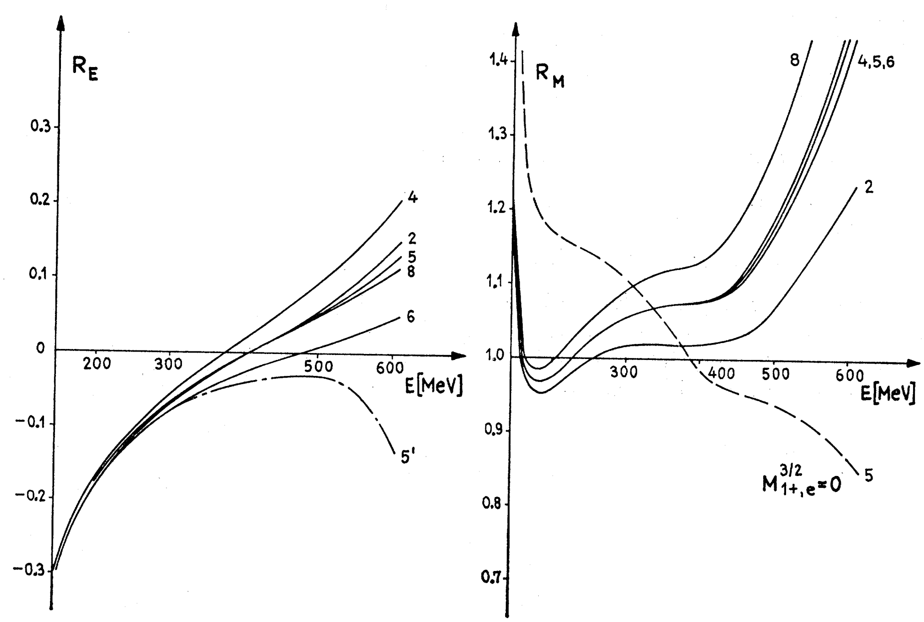

FIG. 6. Results for R_E, R_M according to Sec. III A; solutions 2, 4, 5, 6, 8; dashed line, solution 5 with $M_{1+,e}{}^{3/2}\equiv 0$; dash-dot line, see Sec. III B, case (b); $m_\pi=m_{\pi^0}$.

[13] J. Engels, A. Müllensiefen, and W. Schmidt (to be published).

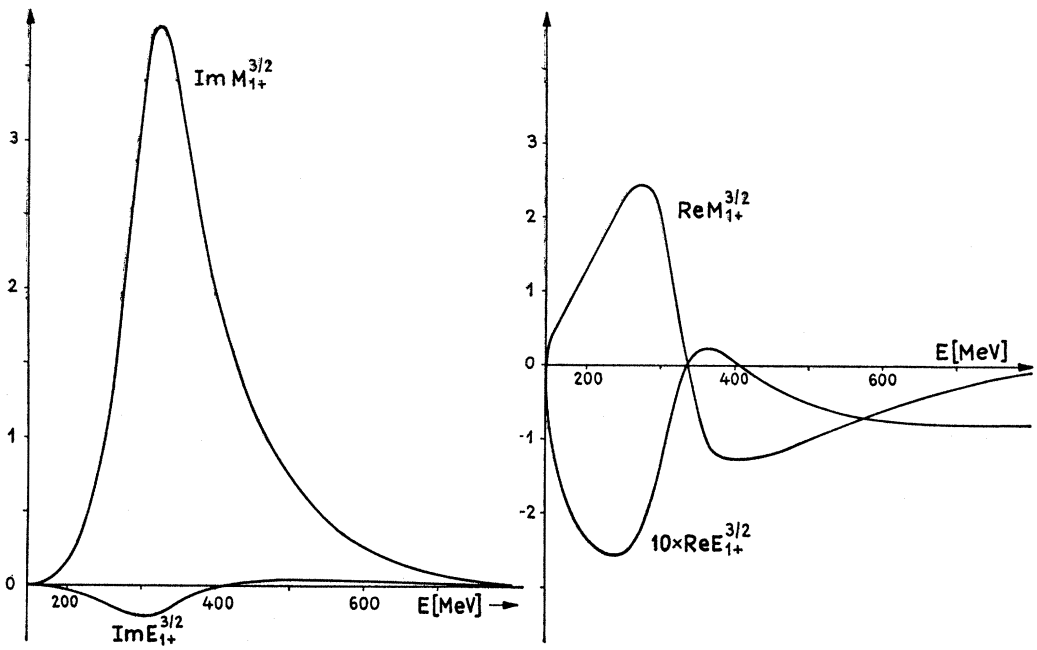

FIG. 7. Results for $M_{1+}{}^{3/2}$ and $E_{1+}{}^{3/2}$ according to solution 5; $m_\pi = m_{\pi^0}$.

between the parameters $\bar{H}_{\lambda,\text{inh}}$ and $H_{\lambda,\text{inh}}(W_\lambda)$ is to be noted. There are several possible reasons for this discrepancy: (a) failure of the approximation (2.12) for $H_{\lambda,\text{inh}}(W)$, (b) a systematic error in $D_\lambda(W)$ because of the uncertainty of the phase φ_λ above $W_\lambda > 11$, or finally, (c) a too optimistic estimate of the high-energy contribution $\bar{N}_\lambda{}^h(W)$ (1.19), which has been neglected. In Fig. 6 we show the results for R_E and R_M corresponding to solutions 2, 4, 5, 6, and 8. At threshold the differences between these solutions are negligible. The relationship to the old result of CGLN is in no case a simple functional behavior. In Fig. 7 the actual result for $E_{1+}{}^{3/2}$ and $M_{1+}{}^{3/2}$ is shown.

B. Results Following From $N_\lambda{}^u(W)$ [Eq. (1.40)]

For $P_{\lambda,n}(W=W_\lambda)=0$ one obtains from (1.40) the relationship

$$N_\lambda{}^u(W_R) \approx g_\lambda/(W_\lambda - W_R). \tag{3.10}$$

This relation now replaces Eq. (3.7), since we shall always assume that one of the zeros (say, W_1) of $P_{\lambda,n}(W)$ is equal to W_λ. This choice for the zero W_1 is in accordance with the discussion on the neglect of the

TABLE III. The ratios $N_M{}^u(E)/\bar{N}_M{}^u(E)$ and $N_E{}^u(E)/\bar{N}_E{}^u(E)$ for $E=500$, 600 MeV, $P_{\lambda,n}=W-W_\lambda$, $W_\lambda=12.25$, and $m_\pi=m_{\pi^0}$.

Solution	2	4	5	6	8	E (MeV)
$N_M{}^u/\bar{N}_M{}^u$	0.94	0.95	0.94	0.95	0.95	500
	0.84	0.87	0.87	0.86	0.89	600
$N_E{}^u/\bar{N}_E{}^u$	1.04	1.02	1.03	1.18	1.02	500
	1.06	1.04	1.04	1.06	1.04	600

high-energy contribution in Sec. II B. For $W_1 = W_\lambda$ the integral in (1.40) does not depend strongly on the high-energy behavior of the inhomogeneous term $H_{\lambda,\text{inh}}(W)$ evaluated in the region of the first resonance. Therefore, a good prediction should be possible in the case that the coupling constants g_λ [or generally $g_{\lambda,n}$, if (1.37) is not used] are known. In this connection one should perhaps stress that the g_λ's (or, generally, $g_{\lambda,n}$) are genuine free parameters, which have to be fixed from other sources. It makes no sense to introduce $N_\lambda{}^u(W)$ into (1.11b), since it is satisfied by the strict result (1.28).

We shall now calculate $N_\lambda{}^u(W)$ with the coupling constants g_λ taken from Table II and we shall compare with the corresponding solutions $\bar{N}_\lambda{}^u(W)$. We consider two possible choices of $P_{\lambda,n}(W)$:

(a) $\lambda=\frac{1}{2},\frac{3}{2};\ n=1:\ P_{1/2,1}(W)=P_{3/2,1}(W)=W-W_\lambda,$

(b) $\lambda=\frac{1}{2};\ n=2:\ P_{1/2,2}(W)=(W-W_\lambda)(W-W_\lambda),$

$\lambda=\frac{3}{2};\ n=1:\ P_{1/2,1}(W)=(W-W_\lambda).$

Case (a). Since there is agreement within 1–2% between $\bar{H}_{\lambda,\text{inh}}(W)$ and the corresponding $g_\lambda/(W_\lambda-W_R)$ in Table II, one cannot expect large differences between $N_\lambda{}^u(W)$ and $\bar{N}_\lambda{}^u(W)$ at the resonance because of the relation (3.10). It turns out that noticeable differences (>1%) exist only for higher energies $E>450$ MeV as is indicated by the results in Table III, which is to be expected. But at these energies it seems at the moment impossible to decide empirically between the two types of approaches to $N_\lambda(W)$. Theoretically we would expect that predictions based on the result (1.40) for $N_\lambda{}^u(W)$ are superior for $n \geq 1$ to those based on (1.21) for $\bar{N}_\lambda{}^u(W)$, since for $W_1 \approx W_\lambda$ the neglect of the high-energy

TABLE IV. Effect of the variation of the zero W_2 for N_E^u and N_M^u calculated with $P_{1/2,2}(W)=(W-W_\lambda)(W-W_2)$ and $P_{3/2,1}(W)=W-W_\lambda$; $W_\lambda=12.25$ and $m_\pi=m_{\pi^0}$.

	W_2 \ E(MeV)	300	400	500		W_2 \ E(MeV)	300	400	500
$N_M^u/\bar{N}_M^u$	$W_\lambda-1$	0.98	0.93	0.80	$N_E^u/\bar{N}_E^u$	$W_\lambda-1$	1.28	14.39	−1.82
	W_λ	0.98	0.95	0.87		W_λ	1.20	9.65	−0.56
	$W_\lambda+1$	0.98	0.96	0.89		$W_\lambda+1$	1.16	7.43	−0.07

contribution should be more justified. We have checked that variations of $W_1\approx W_\lambda$ of the order $W_1-W_\lambda=\pm 1$ lead only to insignificant changes (1%) in the region of the resonance.

Case (b). It is notable in this case that changes already appear in the region of the first resonance, particularly in the ratio $E_{1+}^{3/2}/M_{1+}^{3/2}$, if one compares again solutions with the same coupling constants g_λ (see, e.g., curve 5′ in Fig. 6). This emphasizes again that the prediction of $E_{1+}^{3/2}$ needs a particularly careful treatment of the high-energy contributions. We also found that the results are more sensitive to small changes in the zero W_2 of $P_{1/2,2}$ than in case (a) (see Table IV). Partly this may be due to the approximation (1.37). But this has to be used as long as it is impossible to improve the result (1.34) for $g_{\lambda,1}$ or as long as $g_{\lambda,1}$ is not treated as a further free parameter. Finally, we mention that the type 5′ of solution for $E_{1+}^{3/2}/M_{1+}^{3/2}$ in Fig. 6 is more favored by present experimental data.[13]

IV. CONCLUSION

Different types of N/D representations have been derived for the partial amplitudes of the first resonance. The results are suitable for a phenomenological treatment of the first resonance in pion photoproduction. Parameters are introduced which characterize uncertain or unknown high-energy contributions in the basic equations of the theory. With the help of these parameters, the asymptotic behavior of the approximations is also controlled.

A systematic treatment of the influence of the phenomenological parameters revealed their importancc for a correct prediction of the resonant multipoles. It follows with respect to the large magnetic dipole excitation $M_{1+}^{3/2}$ that a prediction of the height of the resonance $[\mathrm{Im}M_{1+}^{3/2}(W_R)]$ is only possible with a presumable error of 10%. For the small electric quadrupole excitation $E_{1+}^{3/2}$ the situation is worse; even the sign of $\mathrm{Im}E_{1+}^{3/2}(W_R)$ cannot be predicted, and this quantity has to be considered as a completely free parameter. But also with respect to functional behavior, high-energy contributions are more important for $E_{1+}^{3/2}$ than for $M_{1+}^{3/2}$ in the region of the resonance. Their uncertainty could make it necessary to introduce further parameters, primarily in $E_{1+}^{3/2}(W)$ to get the right energy dependence.