Similarly, $K_2(s,u)$ can be brought to the form

$$K_2(s,u)=\frac{\pi^{1/2}r^2\gamma'(0)\Gamma[\alpha(0)+\frac{1}{2}]}{2\Gamma[\alpha(0)+1]}\times\left\{-\frac{1}{2\pi i}\int_{C_\infty}dy\left(1-\frac{\epsilon+\alpha(0)}{y}\right)\right\}.$$

Thus

$$K_2(s,u)=-\frac{\pi^{1/2}\gamma'(0)\Gamma[\alpha(0)+\frac{1}{2}]}{\Gamma[\alpha(0)+1]}\times\{su^{\alpha(0)}-\tfrac{1}{2}r^2\alpha(0)u^{\alpha(0)-1}\}. \quad \text{(A6)}$$

Similarly,

$$K_3(s,u)=-\frac{\pi^{1/2}\alpha'(0)\gamma(0)r^2}{2u}\left\{\frac{1}{2\pi i}\int_{C_\infty}dy\left(1-\frac{\epsilon}{y}\right)\times\frac{\partial}{\partial\alpha}\left[\left(1-\frac{\alpha}{y}\right)\frac{u^\alpha\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}\right]\bigg|_{\alpha=\alpha(0)}\right\}$$

$$=\tfrac{1}{2}\pi^{1/2}\alpha'(0)\gamma(0)r^2\left\{\frac{\Gamma[\alpha(0)+\frac{1}{2}]}{\Gamma[\alpha(0)+1]}u^{\alpha(0)-1}+\left[\alpha(0)u^{-1}-\frac{2s}{r^2}\right]\frac{\partial}{\partial\alpha}\left[u^\alpha\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}\right]\bigg|_{\alpha=\alpha(0)}\right\}. \quad \text{(A7)}$$

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New Type of Dispersion-Theory Sum Rule*

R. Atkinson III
Department of Physics, Tufts University, Medford, Massachusetts 02155
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We show that a new class of sum rules can be obtained by comparing, for example, fixed-t and fixed-u dispersion relations. In particular, if an amplitude obeys both fixed-t and fixed-u dispersion relations at t^* and u^*, respectively, we obtain a sum rule by equating the two dispersion relations for the amplitude evaluated at t^* and u^*. Under special circumstances the no-subtraction requirement can be lifted. We apply our procedure to the $A^{(-)}$ and $B^{(-)}$ πN amplitudes to derive new sum rules, and we show that these sum rules are reasonably well satisfied with only ρ, N, and N^* contributions for the choice $t^*=u^*=0$.

It is our purpose in this paper to discuss a new type of dispersion-theory sum rule and to apply such sum rules to pion-nucleon scattering in the sharp-resonance approximation with N, N^*, and ρ states. Our main point is that an interesting type of sum rule can be derived by, for example, comparing fixed-t and fixed-u dispersion relations for a given amplitude. Here, and in the following, s, t, and u are the usual Mandelstam variables for a two-body reaction. Let us consider an amplitude $A(s,t,u)$, and let us suppose that at fixed $t=t^*$, $A(s,t^*,u)$ obeys an unsubtracted dispersion relation and that at fixed $u=u^*$, $A(s,t,u^*)$ obeys an unsubtracted dispersion relation. Then, recalling that

$$s+t+u=\Sigma\,, \quad (1)$$

where Σ is the sum of the squares of the external masses of the reacting particles, we have

$$A(s^*,t^*,u^*)=\frac{1}{\pi}\int\frac{a_s(s',t^*)ds'}{s'-s^*}+\frac{1}{\pi}\int\frac{a_u(u',t^*)du'}{u'-u^*} \quad (2)$$

$$=\frac{1}{\pi}\int\frac{a_s(s',u^*)ds'}{s'-s^*}+\frac{1}{\pi}\int\frac{a_t(t',u^*)dt'}{t'-t^*}, \quad (2')$$

where $s^*=\Sigma-t^*-u^*$ and $a_x(x',y^*)$ is the x-channel absorptive part at fixed $y=y^*$ as a function of $x'=(x$-channel c.m. energy$)^2$. The sum rule is given by Eq. (2′).

As is the case for superconvergence sum rules,[1] a sum rule of the type of Eq. (2′) rests on very simple assumptions, the validity of dispersion relations and the validity of the no-subtraction requirement. As we will show later, under some circumstances we can lift the no-subtraction requirement for the fixed-variable dispersion relations. In general, we expect that if Eq. (2′) is valid for $t=t^*$ and $u=u^*$, it will be valid for a range of t and u values in the neighborhood of t^* and u^*, so that Eq. (2′) gives a family of sum rules. An interesting feature of Eq. (2′) is that it will relate parameters referring to different channels, as is also the case with some superconvergence relations.[2] When a sum rule of the type of Eq. (2′) is saturated with resonances in the sharp-resonance approximation, we will obtain relations among masses and coupling constants of the form familiar from superconvergence relations.[1] In the case

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[1] V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, Phys. Letters 21, 576 (1966); detailed discussion of applications is given by F. J. Gilman and H. Harari, Phys. Rev. 165, 1803 (1968).
[2] See, for example, the fixed-u sum rules for π-N scattering: D. S. Beder and J. Finkelstein, Phys. Rev. 160, 1363 (1967); D. Griffiths and W. Palmer, *ibid.* 161, 1606 (1967); R. Ramachandran, *ibid.* 166, 1528 (1968).

of saturation with a finite number of resonances, we will run into a problem well known from superconvergence, namely, that such saturation cannot hold over the whole range of validity of the sum rule. Attempting to saturate Eq. (2′) with a few resonances and in addition the high-energy Regge-pole contributions over a range of t^* and u^* may lead to useful information concerning Regge-pole parameters[3]; we hope to return to this point elsewhere.

As a concrete example, let us consider elastic pion-nucleon scattering, with the standard CGLN (Chew-Goldberger-Low-Nambu) notation for the invariant amplitudes.[4] In particular, we have for the $B^{(-)}$ amplitude

$$\frac{-g^2}{u^*-M^2}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\frac{b_s^{(-)}(s',t^*)}{s'-s^*}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} du'\frac{b_u^{(-)}(u',t^*)}{u'-u^*}=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\frac{b_s^{(-)}(s',u^*)}{s'-s^*}+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\frac{b_t^{(-)}(t',u^*)}{t'-t^*}, \quad (3)$$

where g is the usual $\pi-N$ coupling constant ($g^2/4\pi\cong 15$), M and μ are the nucleon, and pion masses, respectively, $s^*+t^*+u^*=\Sigma=2(M^2+\mu^2)$, and t^* and u^* are such that unsubtracted fixed-t and fixed-u dispersion relations are valid. We use a lower case letter to denote absorptive part following the notation used in Eq. (2). We take the point of view that the u-channel nucleon pole should not be an explicit factor in the fixed-u dispersion relation, but rather that this pole would appear only after the dispersion integrals are performed. If we accept the Regge-pole description of high-energy behavior,[5] we have for $s\rightarrow\infty$, at least for small t, $B^{(-)}\rightarrow s^{\alpha_\rho(t)}$ at fixed t, and, at least for small u, $B^{(-)}\rightarrow s^{\alpha_N(u)-\frac{1}{2}}$ at fixed u, where α_ρ and α_N are the ρ and nucleon trajectories. A particularly simple and symmetrical configuration is $t^*=u^*=0$, and $s^*=\Sigma$. According to analysis of experimental data $\alpha_\rho(0)\cong 0.5$ [6] and $\alpha_N(0)\cong -0.3$,[7] so that our sum rule should be valid for $t^*=u^*=0$. Thus we have

$$g^2/M^2=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\frac{b_s^{(-)}(s',u^*=0)}{s'-\Sigma}+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\frac{b_t^{(-)}(t',u^*=0)}{t'}-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\frac{b_s^{(-)}(s',t^*=0)}{s'-\Sigma}-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} du'\frac{b_u^{(-)}(u',t^*=0)}{u'}. \quad (4)$$

Before attempting an approximate evaluation of Eq. (4), let us derive a sum rule for the $A^{(-)}$ amplitude.

Under $s-u$ crossing, $A^{(-)}(s,t,u)=-A^{(-)}(u,t,s)$, hence $A^{(-)}(s,t,s)=0$, and $A^{(-)\prime}=(s-u)^{-1}A^{(-)}(s,t,u)$ is well behaved at $s=u$. According to Regge-pole theory, as $s\rightarrow\infty$, $A^{(-)}\rightarrow s^{\alpha_\rho(t)}$ at fixed t, and $A^{(-)}\rightarrow s^{\alpha_N(u)-\frac{1}{2}}$ at fixed u.[5] Thus at $t^*=u^*=0$, $A^{(-)\prime}$ will obey a sum rule of the type of Eq. (2′):

$$\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\frac{a_s^{(-)}(s',t^*=0)}{(2s'-\Sigma)(s'-\Sigma)}-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} du'\frac{a_u^{(-)}(u',t^*=0)}{(2u'-\Sigma)u'}=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\frac{a_s^{(-)}(s',u^*=0)}{s'(s'-\Sigma)}+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\frac{a_t^{(-)}(t',u^*=0)}{t'(\Sigma-t')}. \quad (5)$$

We note that in general we can derive an Eq. (2′)-type sum rule when both fixed-variable dispersion relations need a subtraction if the amplitude is odd under $s-u$ or $s-t$ crossing; for the sum rule we use the amplitude divided by $(s-u)$ or $(s-t)$ as the case requires.

With the assumption that the Regge description of high-energy behavior is correct, the sum rules Eqs. (4) and (5) are exact, and they clearly relate quantities referring to the channels $\pi+N\rightarrow\pi+N$ and $\pi+\pi\rightarrow N+\bar{N}$. The two sum rules we have written down certainly do not exhaust all the possibilities; Eq. (2′)-type sum rules for $B^{(-)}$ and $A^{(-)\prime}$ will exist for ranges of t^* and u^* such that $\alpha_\rho(t^*)<1$ and such that $\alpha_N(u^*)<\frac{1}{2}$ for $B^{(-)}$ and $\alpha_N(u^*)<\frac{3}{2}$ for $A^{(-)\prime}$. Note that because the nucleon pole does not contribute to $A^{(-)}$, Eqs. (4) and (5) have a slightly different structure. Equation (4) is a sum rule, with more or less the traditional form, for the $\pi-N$ coupling constant, while Eq. (5) has the form of what we may call a consistency sum rule, inasmuch as Eq. (5) involves only integrals. A detailed analysis of the $B^{(-)}$ and $A^{(-)\prime}$ sum rules is in progress and will be reported on elsewhere; here we will discuss an approximate evaluation of Eqs. (4) and (5).

It is interesting to see what happens when we attempt to saturate Eqs. (4) and (5) with N, N^*, and ρ states. We calculate the N^* and the ρ contributions with the sharp-resonance approximation, and we assume that

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[3] Information on Regge-pole parameters has been obtained from finite-energy sum rules by R. Dolen, D. Horn, and C. Schmid, Phys. Rev. Letters **19**, 402 (1967). See also A. Logunov, L. D. Soloviev, and A. N. Tabkhelidze, Phys. Letters **24B**, 181 (1967); K. Igi and S. Matsuda, Phys. Rev. Letters **18**, 625 (1967).

[4] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[5] See, for example, S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962). For a justification of the Regge asymptotic form at fixed $u=0$ see D. Z. Freedman and Jiunn-Ming Wang, *ibid.* **153**, 1596 (1967).

[6] F. Arbab and C. B. Chiu, Phys. Rev. **147**, 1045 (1966).

[7] C. B. Chiu and J. D. Stack, Phys. Rev. **153**, 1575 (1967).

the ρ couples universally to the conserved isovector, vector current.[8] Equation (4) yields

$$\frac{g^2}{M^2}-\frac{f_\rho^2(1+2Mf_V)}{M_\rho^2}+\frac{8\pi f^{*2}}{\mu^2 M^*(E^*+M)}\times\left[\frac{(E^*+M)^2}{3}-\frac{M^{*2}}{2}-q^{*2}\right]=0\,, \quad (6)$$

where M_ρ and M^* are the ρ and N^* masses, respectively, $E^{*2}=q^{*2}+M^2$, where q^* is the c.m. momentum at resonance, $f_V=3.69/2M$ is the anomalous isovector nucleon magnetic moment, $f^{*2}=\mu^2\Gamma^*/2q^{*3}$, where Γ^* is the N^* width, and f_ρ is the universal coupling constant. Equation (5) yields

$$f^{*2}=\left(\frac{f_\rho^2}{4\pi}\right)2Mf_V\frac{\mu^2 M^*}{4MM_\rho^2}\frac{E^*+M}{M^*+M}\,. \quad (7)$$

Using the relation between f_ρ and Γ_ρ, the $\rho\rightarrow\pi\pi$ width, $\Gamma_\rho=(f_\rho^2/4\pi)q_\rho^3/12M_\rho^2$ with $q_\rho^2=M_\rho^2-4\mu^2$, we find that Eq. (7) leads to

$$\Gamma_\rho/\Gamma^*=(q_\rho^3/12Mf_Vq^{*3})\times(M/M^*)(M+M^*)/(E^*+M)\approx1.2\,, \quad (8)$$

whereas the current experimental value is ≈1.1.[9] If we combine Eqs. (6) and (7) we find, after putting in the numbers,

$$f_\rho^2/4\pi\approx0.17g^2/4\pi\approx2.3\,,$$

with $g^2/4\pi=14.7$. This value of f_ρ gives $\Gamma_\rho\approx120$ MeV, in reasonable agreement with experiment,[9] and, with Eq. (8), $\Gamma^*\approx100$ MeV, compared to the experimental value of 120 MeV.[9] It must be pointed out that if we used values of t^* and u^* other than those used here and then saturated with ρ, N, and N^* the resulting relations among f_ρ, g, and f^* would not agree with Eqs. (6) and (7). More analysis is necessary before we can understand why ρ, N, and N^* saturation at $t^*=u^*=0$ works as well as it does. Assuming, however, that Eqs. (6) and (7) are not fortuitous, it is interesting to look briefly at a possible interpretation of these equations. We note that if we dropped the last term in Eq. (6) we would obtain $f_\rho^2/4\pi=(M_\rho^2/M^2)(g^2/4\pi)(1+2Mf_V)^{-1}\approx0.14g^2/4\pi$, and we see that the ρ contribution dominates the N^* contribution in Eq. (6). Now, the reciprocal $N-N^*$ bootstrap[10] suggests that the N and N^* are composite states primarily generated by N and N^* exchange, respectively, while the ρ-exchange force in the N and N^* channels is of secondary importance. A naive application of the reciprocal bootstrap point of view, then, would lead us to expect that we should get reasonably good results without the ρ when we attempt to relate $g^2/4\pi$ to f^{*2}; yet our two sum rules give $g^2/4\pi=f^{*2}=0$ when they are saturated with only N and N^*. Again with the assumption that the quite good agreement of Eqs. (6) and (7) with experiment is not fortuitous, we are tempted to suggest that perhaps the ρ-exchange force is more important for the dynamics of the N and N^* than has previously been realized.[11] We must certainly note that such a suggestion is rather speculative, inasmuch as we do not have any firm basis for a dynamical interpretation of the results of an approximate evaluation of a sum rule. In any event, we have found that the sum rules Eqs. (4) and (5) are quite well satisfied with only N, N^*, and ρ contributions.

In conclusion, we have shown that comparison of fixed-t and fixed-u dispersion relations for a given amplitude can lead to sum rules, and we have obtained new sum rules for the $\pi-N$ amplitudes $A^{(-)}$ and $B^{(-)}$. An important feature of the type of sum rule discussed here is that such sum rules rest on very simple assumptions; our sum rules are exact consequences of the assumption of the validity of dispersion relations and assumptions about high-energy behavior. Finally, we have shown that our $t^*=u^*=0$ sum rules are reasonably well satisfied with only ρ, N, and N^* contributions.

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[8] See, for example, J. J. Sakurai in *Theoretical Physics: Lectures Presented at the Seminar on Theoretical Physics, Trieste, 1962* (International Atomic Energy Agency, Vienna, 1963).

[9] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).

[10] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).

[11] The possibility exists, for example, that multi-ρ exchange dominates multi-N or multi-N^* exchange; however, present calculational techniques are not capable of exploring this possibility. We note that the standard working hypothesis of bootstrap calculations, that the nearby "left-hand" singularities dominate the dynamics of the low-energy region of a partial-wave amplitude, has no firm theoretical justification.