Analyticity Requirement for Regge Poles and Backward Unequal-Mass Scattering

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We construct a Regge amplitude for the backward unequal-mass scattering which satisfies the Mandelstam representation. This amplitude is analytic at $s=0$ for large u. It is shown that the leading term gives the Regge behavior $u^{\alpha(s)}$ at $s=0$ and in its neighborhood, including the region in which the cosine of the backward scattering angle is bounded. We also obtain nonleading terms such as $u^{\alpha(0)-1}$ and $u^{\alpha(0)-1}\ln u$. We consider the possibility of writing these nonleading terms as a sum of fixed and moving poles with a singular coefficient. A comparison with experiment is made.

I. INTRODUCTION

THERE has been considerable interest in establishing the Regge asymptotic behavior $u^{\alpha(0)}$ for backward unequal-mass scattering.[1–4] Since the Regge amplitude has a singularity at $s=0$ and since the cosine of the backward scattering is bounded for large u in the region $0\leq s\leq r^2/u$, where $r^2=(m^2-\mu^2)^2$, the usual asymptotic expansion cannot be applied for this unequal-mass case.

The asymptotic behavior of the backward Regge amplitude depends on the order in which the two limits $u\to\infty$ and $s\to 0$ are taken. If the $u\to\infty$ limit is taken first, then the amplitude gives a $u^{\alpha(0)}$ behavior but the $s\to 0$ limit cannot be taken without introducing poles in the coefficients of $u^{\alpha(0)-1}$, $u^{\alpha(0)-2}$, $\cdots$ terms. If the $s\to 0$ limit is taken first, then there is no $u^{\alpha(0)}$ behavior.

In order to remedy this situation Goldberger and Jones[2] modified the Regge amplitude in such a way that the new amplitude will satisfy Mandelstam analyticity, and then derived a $u^{\alpha(0)}$ behavior.[5] They observed that the undesired $u^{\alpha(0)-1}$ term which persists for nonzero s can be swept into the background term if $\alpha(0)<\frac{1}{2}$. Freedman *et al.*[4] later concluded that the coefficient of the $u^{\alpha(0)-1}$ term has a pole at $s=0$, and therefore this must be canceled by a counteracting daughter pole.

The main difficulty in this program seems to be the fact that the Legendre function cannot be expanded in the conventional manner.[6] We shall use in this paper a mathematical method in which no such expansions are necessary. We shall first construct an amplitude which is analytic at $s=0$ and expand this amplitude in a Laurent series. Since the evaluation of the Laurent-series coefficients does not require the conventional expansion, we can completely avoid the main difficulty.

As in all previous papers on this subject, we shall specifically deal with the pion-nucleon backward scattering.

In Sec. II, we first construct a "modified Regge amplitude" satisfying Mandelstam analyticity. We then observe that this amplitude is analytic at $s=0$, and therefore can be expanded in a power series around this point. The Mandelstam representation assures us that the radius of convergence is independent of u. Therefore the modified amplitude is a well-defined quantity independent of the order in which the limits $u\to\infty$ and $s\to 0$ are taken. In Sec. III, we carry out the explicit evaluation of the first two coefficients in the Laurent-series expansion around $s=0$. From this, we conclude that the $u^{\alpha(s)}$ behavior is maintained at $s=0$ and in its neighborhood. We find the $u^{\alpha(0)-1}$ and $u^{\alpha(0)-1}\ln u$ terms which could be a manifestation of violation of unitarity. We then discuss the possibility of writing these nonleading terms as a sum of fixed and moving poles with singular coefficients. We discuss this point in connection with the support properties of the Mandelstam weight function. Our leading term $u^{\alpha(s)}$ comfortably gives a shrinkage of the backward peak which has been observed in pion-nucleon scattering.

In Sec. IV, we compare our results with available

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1 J. D. Stack, Phys. Rev. Letters **16**, 286 (1966); see also G. F. Chew and J. D. Stack, University of California Laboratory Report No. UCRL-16293 (unpublished).

2 M. L. Goldberger and C. E. Jones, Phys. Rev. Letters **17**, 105 (1966); Phys. Rev. **150**, 1269 (1966).

3 D. Z. Freedman and J. M. Wang, Phys. Rev. Letters **17**, 569 (1966); Phys. Rev. **153**, 1645 (1967).

4 D. Z. Freedman, C. E. Jones, and J. M. Wang, Phys. Rev. **155**, 1645 (1967).

5 In Ref. 2 [Phys. Rev. **150**, 1269 (1966)], Eq. (3.5) is incorrect and inadequate for making an analytic continuation in u from $u<0$ to $u>0$, and Eq. (3.6) is therefore incorrect. One can see this very easily by writing the integral of Eq. (3.5) as a contour integral which encloses all the singularities generated by ν^α and those of the Q function, and then by investigating the movements of those singularities within the contour. We are aware of a typographical error in Eq. (3.5).

6 In Ref. 4, the expansion in their Eq. (5) is not justified because, in order to avoid the approaching pole at $s'=s\to 0$, the contour C (which does not enclose the pole $s'=s$) has to pass through the region in which the argument of the Legendre function is bounded. The subsequent arguments including their assertion of the existence of the second Regge trajectory are therefore invalid.

experimental data. For pion-nucleon scattering, our leading term $u^{\alpha(s)}$ comfortably confirms the shrinkage of the backward peak. The effect of the nonleading terms $u^{\alpha(0)-1}$ and $u^{\alpha(0)-1}\ln u$ cannot be tested with the experimental data available at present.

II. CONSTRUCTION OF A REGGE AMPLITUDE SATISFYING THE MANDELSTAM REPRESENTATION

It is well known that a single Regge-pole term does not have analyticity required for writing the Mandelstam representation.[7] In this section we construct a modified Regge amplitude satisfying the Mandelstam representation by systematically removing undesired singularities.

Let us start from the backward Regge amplitude

$$R(s,u)=-\pi\gamma(s)(-q^2)^{\alpha(s)}P_{\alpha(s)}\left(-1-\frac{u-r^2/s}{2q^2}\right), \quad (1)$$

where

$$r^2=(m^2-\mu^2)^2 \text{ and } q^2=[s-(m-\mu)^2][s-(m+\mu)^2]/4s,$$

with m and μ the masses of the nucleon and pion, respectively. $\alpha(s)$ and $\gamma(s)$ are the pole position and residue, respectively. It is widely accepted that both $\alpha(s)$ and $\gamma(s)$ are analytic except along the physical s cut.[8] In spite of this analyticity, the Regge amplitude of Eq. (1) has undesired singularities in both s and u planes.

This amplitude has a singularity at $s=0$, and the argument of the Legendre function is bounded for

$$0\leq s\leq r^2/u. \quad (2)$$

This makes it impossible to have a Regge behavior $u^{\alpha(s)}$ in the neighborhood of $s=0$. It is the aim of this paper to remove this difficulty by correcting the analyticity of the Regge term.

In order to get rid of the unwanted singularities in the s plane, we consider the amplitude $R'(s,u)$ defined as

$$R'(s,u)=\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'\,\mathrm{Im}R(s',u)}{s'-s}, \quad (3)$$

where $s_0=(m+\mu)^2$ and where $\mathrm{Im}R(s,u)$ is the discontinuity function along the physical cut. By construction, $R'(s,u)$ is analytic except along the physical cut and differs from $R(s,u)$ by contributions from other singularities. To our knowledge, the destiny of these other cut contributions is not completely known and warrants further investigation.

Let us next remove the unwanted singularities in the u plane. We first observe that $\mathrm{Im}R(s,u)$ has a cut extending from r^2/s to ∞. In order to remove the section of the cut extending from r^2/s to $(m+\mu)^2$, we construct the function $H(s,u)$ by

$$H(s,u)=\frac{1}{2\pi i}\int_{s_0}^{\infty}\frac{ds'}{s'-s}[G(s'+i\epsilon,u)-G(s'-i\epsilon,u)], \quad (4)$$

where

$$G(s,u)=-\sqrt{2}[2\alpha(s)+1]\beta(s)\times\int_0^{\xi}dx\left[\cosh x-1-\frac{u-r^2/s}{2q^2}\right]^{-1/2}\cosh[(\alpha+\tfrac{1}{2})x].$$

The upper limit ξ is determined by the u-channel threshold $[u_0=(m+\mu)^2]$ and is

$$\cosh^{-1}\left[1+\frac{u_0-r^2/s}{2q^2}\right]. \quad (5)$$

$G(s,u)$ is a modification of the background contribution originally derived by Khuri.[9]

Now, both $G(s\pm i\epsilon,u)$ decrease like $u^{-1/2}$ uniformly for large u in the region of the above integration. Therefore we can borrow the $H(s,u)$ function of Eq. (4) from the background term and define a new function

$$K(s,u)\equiv H(s,u)+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'\,\mathrm{Im}R(s',u)}{s'-s}. \quad (6)$$

Again, by construction,

$$K(s,u)=\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\mathrm{Im}K(s',u)}{s'-s}ds'. \quad (7)$$

The spectral function $\mathrm{Im}K(s',u)$ can be written as

$$\mathrm{Im}K(s',u)=\mathrm{Im}[R(s',u)+H(s',u)]$$
$$=\frac{1}{2i}\{R(s'+i\epsilon,u)+G(s'+i\epsilon,u)-R(s'-i\epsilon,u)-G(s'-\epsilon,u)\}. \quad (8)$$

It was shown by Khuri[9] that both $[R(s\pm i\epsilon,u)+G(s'\pm i\epsilon)]$ are analytic in the cut u plane with the cut extending from u_0 to ∞. Thus the amplitude $K(s,u)$ satisfies the Mandelstam representation

$$K(s,u)=\frac{1}{\pi^2}\int_{s_0}^{\infty}\int_{u_0}^{\infty}\frac{\rho(s',u')ds'du'}{(s'-s)(u'-u)}, \quad (9)$$

which enables us to continue $K(s,u)$ analytically to arbitrary values of s and u, in particular to $s=0$ for large u.

We emphasize that $K(s,u)$ is analytic at $s=0$, and therefore can be expanded in a power series whose radius of convergence is s_0 and is independent of u. No

[7] See, for instance, Refs. 2 and 9.

[8] This is shown to be true in potential scattering by J. R. Taylor [Phys. Rev. 127, 2257 (1962)]. We thank Professor D. Z. Freedman, Professor M. L. Goldberger, and Professor C. E. Jones for pointing out the importance of the role of the kinematical factor $(-q^2)^{\alpha}$.

[9] N. N. Khuri, Phys. Rev. 130, 429 (1963).

attempts have been made to take into account the support property of the Mandelstam weight function $\rho(s',u')$ given by the unitarity condition. We would therefore expect a symptom showing this violation of unitarity at a later stage of this work and, further, that the effect of this violation will become small as $u\to\infty$ because the support curve is asymptotic to the normal threshold for large u. In the following section we shall discuss these points in detail.

III. ASYMPTOTIC BEHAVIOR OF THE MODIFIED REGGE AMPLITUDE

In the preceding section we have constructed a modified Regge amplitude satisfying the Mandelstam representation. Since the $H(s,u)$ function decreases as $u^{-1/2}$ for large u, this term can be dropped from our discussion. We have therefore

$$K(s,u)=\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\operatorname{Im}R(s',u)ds'}{s'-s}. \tag{10}$$

In this section we investigate the large-u behavior of the above amplitude for small s, using known properties of $\alpha(s)$ and $\gamma(s)$.

First of all, we note that $K(s,u)$ of Eq. (11) is analytic at $s=0$, and therefore can be expanded in a power series:

$$K(s,u)=a_0(u)+a_1(u)s+a_2(u)s^2+\cdots. \tag{11}$$

Although the coefficients a_0, a_1, $\cdots$ depend on the variable u, the above series converges uniformly within the circle of radius s_0. Using the linear approximation for $\alpha(s)$ and $\gamma(s)$,

$$\begin{aligned}\alpha(s)&=\alpha(0)+\alpha'(0)s,\\ \gamma(s)&=\gamma(0)+\gamma'(0)s,\end{aligned} \tag{12}$$

we shall compute $a_0(u)$ and $a_1(u)$ in this section.

We now evaluate the dispersion integral of Eq. (10) by replacing it by the contour integrals enclosing counterclockwise all the singularities of $R(s',u)/(s'-s)$ other than the physical cut. Then, for sufficiently small s $[s\ll(m+\mu)^2]$,

$$K(s,u)=-\pi\gamma(s)\left(\frac{-r^2}{4s}\right)^{\alpha(s)}P_{\alpha(s)}\left(1-\frac{2us}{r^2}\right)+\frac{1}{2\pi i}\int_C\frac{ds'}{s'-s}\gamma(s')\left(\frac{-r^2}{4s'}\right)^{\alpha(s')}P_{\alpha(s')}\left(1-\frac{2us'}{r^2}\right), \tag{13}$$

where the contour C encloses counterclockwise the cut of the integrand running from 0 to r^2/u. Using Eq. (12) for $\alpha(s)$ and $\gamma(s)$, we write

$$K(s,u)=K_1(s,u)+K_2(s,u)+K_3(s,u), \tag{14}$$

where

$$K_1(s,u)=-\pi\gamma(0)\left\{\left(\frac{-r^2}{4s}\right)^{\alpha(0)}P_{\alpha(0)}\left(1-\frac{2us}{r^2}\right)+\frac{1}{2\pi i}\int_C\frac{ds'}{s'-s}\left(\frac{-r^2}{4s'}\right)^{\alpha(0)}P_{\alpha(0)}\left(1-\frac{2us'}{r^2}\right)\right\},$$

$$K_2(s,u)=\tfrac{1}{4}\pi r^2\gamma'(0)\left\{\left(\frac{-r^2}{4s}\right)^{\alpha(0)-1}P_{\alpha(0)}\left(1-\frac{2us}{r^2}\right)+\frac{1}{2\pi i}\int_C\frac{ds'}{s'-s}\left(\frac{-r^2}{4s'}\right)^{\alpha(0)-1}P_{\alpha(0)}\left(1-\frac{2us'}{r^2}\right)\right\},$$

$$K_3(s,u)=-\pi\alpha'(0)\gamma(0)\times\left\{s\frac{\partial}{\partial\alpha}\left[\left(\frac{-r^2}{4s}\right)^{\alpha}P_\alpha\left(1-\frac{2us}{r^2}\right)\right]\Bigg|_{\alpha=\alpha(0)}+\frac{1}{2\pi i}\int_C\frac{ds'}{s'-s}s'\frac{\partial}{\partial\alpha}\left[\left(\frac{-r^2}{4s'}\right)^{\alpha}P_\alpha\left(1-\frac{2us}{r^2}\right)\right]\Bigg|_{\alpha=\alpha(0)}\right\}.$$

These integrals are evaluated in the Appendix. From the expressions for $K_1(s,u)$, $K_2(s,u)$, and $K_3(s,u)$ derived there, we finally obtain

$$\begin{aligned}a_0(u)=-\frac{\pi^{1/2}\Gamma[\alpha(0)+\frac{1}{2}]}{\Gamma[\alpha(0)+1]}\{&\gamma(0)u^{\alpha(0)}\\ &-\tfrac{1}{2}r^2\alpha(0)\alpha'(0)\gamma(0)u^{\alpha(0)-1}\ln u\\ &-\tfrac{1}{2}r^2[\alpha(0)\gamma'(0)+\alpha'(0)\gamma(0)\\ &\qquad+\alpha(0)\alpha'(0)\gamma(0)L]u^{\alpha(0)-1}\},\\ a_1(u)=-\frac{\pi^{1/2}\Gamma[\alpha(0)+\frac{1}{2}]}{\Gamma[\alpha(0)+1]}\{&[\gamma'(0)+\alpha'(0)\gamma(0)L]u^{\alpha(0)}\\ &+\alpha'(0)\gamma(0)u^{\alpha(0)}\ln u\},\end{aligned} \tag{15}$$

where

$$L=\frac{\partial}{\partial\alpha}\left[\ln\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}\right]\Bigg|_{\alpha=\alpha(0)}.$$

According to the above $a_0(u)$ and $a_1(u)$, the $K(s,u)$ function takes the form

$$K(s,u)=-\pi^{1/2}\frac{\Gamma[\alpha(s)+\frac{1}{2}]\gamma(s)}{\Gamma[\alpha(s)+1]}u^{\alpha(s)}+\frac{\pi^{1/2}r^2\alpha(0)\alpha'(0)\gamma(0)\Gamma[\alpha(0)+\frac{1}{2}]}{2\Gamma[\alpha(0)+1]}\times\left\{u^{\alpha(0)-1}\ln u+\left[L+\frac{\gamma'(0)}{\alpha'(0)\gamma(0)}+\frac{1}{\alpha(0)}\right]u^{\alpha(0)-1}\right\}$$

for sufficiently small s.

We have derived above a small-s behavior of $K(s,u)$ for large u. The leading term $u^{\alpha(s)}$ gives the desired

Regge behavior for $s=0$ and in its immediate neighborhood, including the region $0\leq s\leq r^2/u$.

Our result for nonleading terms of the modified amplitude can also be written as

$$K_n(s,u)=(1/s)[b(s)u^{\alpha(s)-1}-b(0)u^{\alpha(0)-1}] \quad (16)$$

for small s, where $b(s)$ is uniquely determined by the original Regge parameters $\alpha(s)$ and $\gamma(s)$:

$$b(s)=-\pi^{1/2}r^2\gamma(s)\alpha(s)\frac{\Gamma[\alpha(s)+\frac{1}{2}]}{\Gamma[\alpha(s)+1]}.$$

It is important to realize that the $(1/s)b(s)u^{\alpha(s)-1}$ term does not come entirely from the correction term, nor entirely from the original (unmodified) Regge amplitude. (Note the nonexpansion restriction on the Legendre function.) It comes from the combined effect of the original and correction terms and is therefore one of the nonleading terms of the modified amplitude. It does not come from the second Regge trajectory mentioned in Ref. 4.

According to Ref. 4, the nonleading term may be written as

$$K_F(s,u)=(1/s)[b_1(s)u^{\alpha_1(s)}-b(0)u^{\alpha(0)-1}],$$

where $(1/s)b_1(s)u^{\alpha_1(s)}$, the daughter trajectory, is inserted to cancel the singularity $(1/s)b(0)u^{\alpha(0)-1}$. Thus, it is required only that $b_1(0)=b(0)$ and $\alpha_1(0)=\alpha(0)-1$, with $b_1(s)$ and $\alpha_1(s)$ unknown for $s\neq 0$. For this reason, the coefficients a, b, and c of Eq. (7) of Ref. 4, which contain the first derivatives of $b_1(s)$ and $\alpha_1(s)$ at $s=0$ in addition to $b_1(0)$ and $\alpha_1(0)$, are left undetermined.

On the other hand, if we use our modified amplitude, the coefficients a, b, and c are uniquely determined as in Eq. (15) and may be compared with experiment as is done in the following section.

Let us finally add a few remarks on the unitarity and the fixed Regge pole. Our result in Eq. (16) indicates the existence of a fixed Regge pole at $J=[\alpha(0)-1]$, which implies a violation of unitarity. Since this value of J is never physical, one cannot automatically rule out such behavior. We note here that the support property of the Mandelstam weight function imposed by unitarity has not been properly taken into account. We note also that the origin of the difference between $R(s,u)$ and $K(s,u)$ remains undetermined. We have, however, been unable to find a general mechanism to account for this violation (analogous to Mandelstam cuts which permit fixed poles at integral wrong-signature nonsense points).

IV. COMPARISON WITH EXPERIMENT

According to Eq. (16), the leading term of the modified amplitude $K(s,u)$ is

$$u^{\alpha(s)}=\exp[\alpha(s)\ln u], \quad (17)$$

and the next largest term contains

$$u^{\alpha(0)-1}\ln u. \quad (18)$$

The coefficient of this nonleading term is completely determined by the parameters $\alpha(0)$, $\alpha'(0)$, $\gamma(0)$, $\gamma'(0)$, and r^2. $\gamma(s)$ is approximately constant.[10]

We now exchange the variables s and u for convenience. Then Eq. (17) implies shrinkage of the backward peak. It is important to mention here that this $s^{\alpha(u)}$ behavior has not been established in previous works on backward scattering.

This shrinkage has been observed experimentally for both π^+p and π^-p scattering by Brody *et al.*[11] for beam momenta 4 to 8 GeV/c, when their data are combined with those of Frisken *et al.*[12,13] This effect is also seen by Orear *et al.*,[14] and more recently, at higher beam momenta, from 6 to 17 GeV/c, by Ashmore *et al.*[15] This phenomenon is more pronounced for π^-p scattering, though the π^-p backward peak is much smaller than the π^+p peak.[16] We interpret this experimental evidence as a justification of our basic assumption—that the Regge amplitude satisfies the Mandelstam representation.

For $u=0$, the term next largest in order of magnitude is $Cs^{\alpha(0)-1}\ln s$, and not $s^{\alpha(0)-1}$ as is commonly believed. The coefficient C of this term is explicitly determined, $C=-\frac{1}{2}r^2\alpha(0)\alpha'(0)/s_0$, in dimensionless units. Note that for $m=\mu$, $r^2\to 0$, and this term vanishes; it is a manifestation of the unequal-mass case. If we compare the ratio of the leading term to this logarithm term at the beam momenta of 1.7 and 17 GeV/c, we have

	1.7 GeV/c	17 GeV/c	
$(s/s_0)/C\ln(s/s_0)=$	35	149	for π^-p
$=$	12	51	for π^+p

The scaling factor[17] s_0 is taken to be $s_0=0.4$ (BeV)^2, as in Ref. 10. Thus the logarithm term could contribute as much as 8% for π^+p scattering at $p_{\rm lab}=1.7$ GeV/c.

[10] V. Barger and D. Cline, Phys. Rev. **155**, 1792 (1967).

[11] H. Brody, R. Lanza, R. Marshall, J. Niederer, W. Selove, M. Shochet, and R. Van Berg, Phys. Rev. Letters **16**, 828 (1966); **16**, 968(E) (1966). See especially Fig. 3 where data of previous authors have been included.

[12] W. R. Frisken, A. L. Read, H. Ruderman, A. D. Krisch, J. Orear, H. Rubenstein, D. B. Scarl, and D. H. White, Phys. Rev. Letters **15**, 313 (1965).

[13] Results at 10 GeV/c further corroborate this shrinkage phenomenon. See R. Lanza, Ph.D. dissertation, University of Pennsylvania, 1966 (unpublished). We thank Professor W. Selove for helpful comments on these points.

[14] J. Orear, R. Rubinstein, D. B. Scarl, D. H. White, A. D. Krisch, W. R. Frisken, A. L. Read, and H. Ruderman, Phys. Rev. **152**, 1162 (1966). See Fig. 9.

[15] A. Ashmore, C. J. S. Damerell, W. R. Frisken, R. Rubinstein, J. Orear, D. P. Owen, F. C. Peterson, A. L. Read, D. G. Ryan, and D. H. White, Phys. Rev. Letters **19**, 460 (1967).

[16] N trajectory exchange is dominant for πp scattering and accounts for this phenomenon. See C. B. Chiu and J. D. Stack, Phys. Rev. **153**, 1575 (1967).

[17] There will be a small error in taking over this number from Ref. 10 due to the fact that we assume s^α behavior whereas Ref. 10 assumes $P_\alpha(z)$ goes to z^α and to $[s-(m^2+\mu^2)]^\alpha$. In the region of investigation, $0\leq u\leq r^2/s$, z is bounded between ± 1 and the expansion in z is unjustified.

Unfortunately the experimental error is still too large to measure this effect, and the theory is not sufficiently refined to consider low laboratory momenta. The logarithm term remains an interesting possibility.

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APPENDIX

In this Appendix we evaluate the contour integrals of Eq. (14). Let us first evaluate $K_1(s,u)$. By changing the variables by

$$\epsilon=-2us/r^2, \quad y=2us'/r^2, \tag{A1}$$

we write

$$K_1(s,u)=-\pi\gamma(0)(\tfrac{1}{2}u)^{\alpha(0)}\left\{(1/\epsilon)^{\alpha(0)}P_{\alpha(0)}(1+\epsilon)+\frac{1}{2\pi i}\int_{C'}\frac{dy}{y+\epsilon}\left(-\frac{1}{y}\right)^{\alpha(0)}P_{\alpha(0)}(1-y)\right\}, \tag{A2}$$

where the contour C' encloses the cut of the integrand running from $y=0$ to $y=2$. The integrand also has a pole at $y=-\epsilon$ outside the contour C'. It has no other singularities. Therefore the contour C' can be replaced by a circular counterclockwise contour with infinite radius (we call this C) and a clockwise contour around the pole at $y=-\epsilon$. See Fig. 1. But this pole term cancels the first term in the curly bracket, and $K_1(s,u)$ is simply

$$K_1(s,u)=-\pi\gamma(0)(\tfrac{1}{2}u)^{\alpha(0)}\times\left\{\frac{1}{2\pi i}\int_{C_\infty}\frac{dy}{y+\epsilon}\left(-\frac{1}{y}\right)^{\alpha(0)}P_{\alpha(0)}(1-y)\right\}. \tag{A3}$$

Along this circular contour,

$$\left(-\frac{1}{y}\right)^{\alpha(0)}P_{\alpha(0)}{}^{(1-y)}=\left[\frac{2^{\alpha(0)}}{(\pi)^{1/2}}\frac{\Gamma[\alpha(0)+\frac{1}{2}]}{\Gamma[\alpha(0)+1]}\right]\left(1-\frac{\alpha(0)}{y}\right)$$

and

$$1/(y+\epsilon)=(1/y)(1-\epsilon/y). \tag{A4}$$

Thus the contour integral can be trivially performed, and

$$K_1(s,u)=-\frac{\pi^{1/2}\Gamma[\alpha(0)+\frac{1}{2}]}{\Gamma[\alpha(0)+1]}\gamma(0)u^{\alpha(0)}. \tag{A5}$$

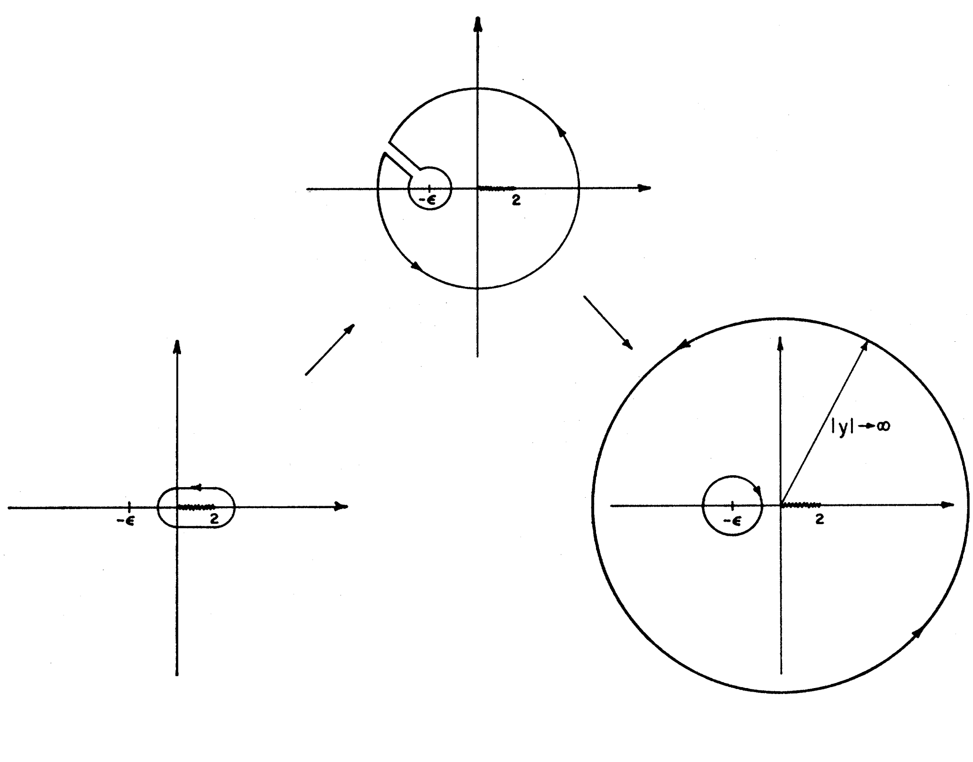

FIG. 1. Contours in the complex y plane.

Similarly, $K_2(s,u)$ can be brought to the form

$$K_2(s,u)=\frac{\pi^{1/2}r^2\gamma'(0)\Gamma[\alpha(0)+\frac{1}{2}]}{2\Gamma[\alpha(0)+1]}\times\left\{-\frac{1}{2\pi i}\int_{C_\infty}dy\left(1-\frac{\epsilon+\alpha(0)}{y}\right)\right\}.$$

Thus

$$K_2(s,u)=-\frac{\pi^{1/2}\gamma'(0)\Gamma[\alpha(0)+\frac{1}{2}]}{\Gamma[\alpha(0)+1]}\times\{su^{\alpha(0)}-\tfrac{1}{2}r^2\alpha(0)u^{\alpha(0)-1}\}. \quad \text{(A6)}$$

Similarly,

$$K_3(s,u)=-\frac{\pi^{1/2}\alpha'(0)\gamma(0)r^2}{2u}\left\{\frac{1}{2\pi i}\int_{C_\infty}dy\left(1-\frac{\epsilon}{y}\right)\times\frac{\partial}{\partial\alpha}\left[\left(1-\frac{\alpha}{y}\right)\frac{u^\alpha\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}\right]\Bigg|_{\alpha=\alpha(0)}\right\}$$

$$=\tfrac{1}{2}\pi^{1/2}\alpha'(0)\gamma(0)r^2\left\{\frac{\Gamma[\alpha(0)+\frac{1}{2}]}{\Gamma[\alpha(0)+1]}u^{\alpha(0)-1}+\left[\alpha(0)u^{-1}-\frac{2s}{r^2}\right]\frac{\partial}{\partial\alpha}\left[u^\alpha\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}\right]\Bigg|_{\alpha=\alpha(0)}\right\}. \quad \text{(A7)}$$

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New Type of Dispersion-Theory Sum Rule*

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We show that a new class of sum rules can be obtained by comparing, for example, fixed-t and fixed-u dispersion relations. In particular, if an amplitude obeys both fixed-t and fixed-u dispersion relations at t^* and u^*, respectively, we obtain a sum rule by equating the two dispersion relations for the amplitude evaluated at t^* and u^*. Under special circumstances the no-subtraction requirement can be lifted. We apply our procedure to the $A^{(-)}$ and $B^{(-)}$ πN amplitudes to derive new sum rules, and we show that these sum rules are reasonably well satisfied with only ρ, N, and N^* contributions for the choice $t^*=u^*=0$.

It is our purpose in this paper to discuss a new type of dispersion-theory sum rule and to apply such sum rules to pion-nucleon scattering in the sharp-resonance approximation with N, N^*, and ρ states. Our main point is that an interesting type of sum rule can be derived by, for example, comparing fixed-t and fixed-u dispersion relations for a given amplitude. Here, and in the following, s, t, and u are the usual Mandelstam variables for a two-body reaction. Let us consider an amplitude $A(s,t,u)$, and let us suppose that at fixed $t=t^*$, $A(s,t^*,u)$ obeys an unsubtracted dispersion relation and that at fixed $u=u^*$, $A(s,t,u^*)$ obeys an unsubtracted dispersion relation. Then, recalling that

$$s+t+u=\Sigma\,, \quad (1)$$

where Σ is the sum of the squares of the external masses of the reacting particles, we have

$$A(s^*,t^*,u^*)=\frac{1}{\pi}\int\frac{a_s(s',t^*)ds'}{s'-s^*}+\frac{1}{\pi}\int\frac{a_u(u',t^*)du'}{u'-u^*} \quad (2)$$

$$=\frac{1}{\pi}\int\frac{a_s(s',u^*)ds'}{s'-s^*}+\frac{1}{\pi}\int\frac{a_t(t',u^*)dt'}{t'-t^*}, \quad (2')$$

where $s^*=\Sigma-t^*-u^*$ and $a_x(x',y^*)$ is the x-channel absorptive part at fixed $y=y^*$ as a function of $x'=$(x-channel c.m. energy)^2. The sum rule is given by Eq. (2′).

As is the case for superconvergence sum rules,[1] a sum rule of the type of Eq. (2′) rests on very simple assumptions, the validity of dispersion relations and the validity of the no-subtraction requirement. As we will show later, under some circumstances we can lift the no-subtraction requirement for the fixed-variable dispersion relations. In general, we expect that if Eq. (2′) is valid for $t=t^*$ and $u=u^*$, it will be valid for a range of t and u values in the neighborhood of t^* and u^*, so that Eq. (2′) gives a family of sum rules. An interesting feature of Eq. (2′) is that it will relate parameters referring to different channels, as is also the case with some superconvergence relations.[2] When a sum rule of the type of Eq. (2′) is saturated with resonances in the sharp-resonance approximation, we will obtain relations among masses and coupling constants of the form familiar from superconvergence relations.[1] In the case

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[1] V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, Phys. Letters 21, 576 (1966); detailed discussion of applications is given by F. J. Gilman and H. Harari, Phys. Rev. 165, 1803 (1968).

[2] See, for example, the fixed-u sum rules for π-N scattering: D. S. Beder and J. Finkelstein, Phys. Rev. 160, 1363 (1967); D. Griffiths and W. Palmer, *ibid.* 161, 1606 (1967); R. Ramachandran, *ibid.* 166, 1528 (1968).